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On reduction of two-parameter prediction problems

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Abstract. We present a general method for the extension of results about linear prediction for q-variate weakly stationary processes on a separable locally compact abelian group G_2 (whose dual is a Polish space) with known values of the processes on a separable subset $S_2 \subseteq G_2$ to results for weakly stationary processes on $G_1 \times G_2$ with observed values on $G_1 \times S_2$. In particular, the method is applied to obtain new proofs of some well-known results of Ze Pei Jiang.

1. Introduction. Consider a multivariate weakly stationary stochastic process X on a locally compact abelian group G with values in a Cartesian power of a Hilbert space $\mathcal{H}$. Central problems in prediction theory are the search for methods to determine the orthogonal projection of a value of the process onto the closed matrix linear span $\mathcal{W}_S$ of the values of the process on the set $S \subseteq G$ as well as to find criteria for the $\mathbf{S}$-regularity and $\mathbf{S}$-singularity of the process, where $\mathbf{S}$ denotes a certain family of subsets of G. The values of the process on S (resp. on the sets of $\mathbf{S}$) may be interpreted as observed values of the process.

If G is the group $\mathbb{Z}$ or $\mathbb{R}$ of integers or reals, resp., and if the subset S is the set of negative numbers in G, the above mentioned problems are known as extrapolation problems of Kolmogorov. If S is an interval $(-a, a)$, we have Krein's extrapolation problem. If S is the exterior of a finite interval $(-a, a) \subset G$, we have an interpolation problem. If the system $\mathbf{S}$ comprises all sets $(-\infty, a) \subset G$, $a \in G$, then $\mathbf{S}$-regularity and $\mathbf{S}$-singularity coincide with the usual notions of regularity and singularity. These specializations were extensively discussed in the literature.

It is natural to search generalizations of the corresponding results for the following situation: Suppose that the process X is defined on the group $G = G_1 \times \mathbb{Z}$ or $G = G_1 \times \mathbb{R}$, where S or the sets from $\mathbf{S}$ have the form $S = G_1 \times S_2$, $S_2 \subseteq \mathbb{Z}$ or $S_2 \subseteq \mathbb{R}$. In this case G_1 denotes an arbitrary locally compact abelian group. Papers dealing with such problems are [4] and

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[5], see also [6] and [7]. These papers treat generalizations of Kolmogorov's extrapolation problem in the cases $G_1 = \mathbb{Z}$ and $G_1 = \mathbb{R}$.

Here we present a general method for the extension of results about linear prediction for a q-variate weakly stationary process on a *separable* locally compact abelian group G_2 whose dual group is a Polish space to appropriate results for processes on the group $G = G_1 \times G_2$, where G_1 is an arbitrary locally compact abelian group. We write g_1, g_2, and $g = (g_1, g_2)$ for the generic element of G_1, G_2, and the direct product $G := G_1 \times G_2$, resp. Similarly, the dual groups are denoted by Γ_1, Γ_2, and Γ. It is well known that $\Gamma = \Gamma_1 \times \Gamma_2$ in a natural way. The action of the character $\gamma \in \Gamma$ on $g \in G$ is denoted by $[\gamma, g]$ and we use similar notations for the other groups. In particular, if $\gamma = (\gamma_1, \gamma_2)$ and $g = (g_1, g_2)$, we have $[\gamma, g] = [\gamma_1, g_1][\gamma_2, g_2]$. Thus, throughout the paper, G_1 will denote an arbitrary locally compact abelian group and G_2 a separable locally compact abelian group whose dual group Γ_2 is a Polish space, that is, a topological space whose topology is metrizable, such that Γ_2 is separable and complete.

The main tools are direct integrals of Hilbert spaces. For the reader's convenience we state some appropriate results in the second section. Let $\mathcal{V}_G$ be the image of the space spanned by the values of the process under Kolmogorov's isomorphism. We obtain

$$\mathcal{V}_G = \int_{\Gamma_1}^{\oplus} H(\gamma_1)\,\mu_1(d\gamma_1)$$

as a direct integral of Hilbert spaces. The Hilbert spaces $H(\gamma_1), \gamma_1 \in \Gamma_1$, are spaces of square-integrable functions on Γ_2. We choose a non-negative σ-finite measure μ on Γ such that all entries of the spectral measure of the process X are absolutely continuous w.r.t. μ. The measure μ_1 appearing in the direct integral is the marginal measure of μ, i.e. it is given by $\mu_1(\Delta_1) = \mu(\Delta_1 \times \Gamma_2)$ for all Borel subsets $\Delta_1 \subseteq \Gamma_1$.

The use of direct integral seems to be artificial, but it has the advantage that the appropriate theory yields immediately the measurability of some objects, for which measurability is essential and not obvious.

Our results allow us to reduce questions for the process X, e.g. **S**-regularity, to corresponding questions in the "smaller" spaces $H(\gamma_1)$. Thus the above mentioned results of Ze Pei Jiang are plain from ours and we get them in a much more general setting. Since the way in which the special results are deduced from our results is obvious, we will state only some applications in the last section.

2. Direct integral Hilbert spaces. For the reader's convenience, we collect some facts about direct integral Hilbert spaces. Suppose that $(\Omega, \mathcal{B}, \nu)$ is a positive measure space. We assume that we are given Hilbert spaces $H(\omega)$ for each $\omega \in \Omega$ with inner product $\langle \cdot, \cdot \rangle_\omega$ and norm $\|\cdot\|_\omega$. The collection $(H(\omega))_{\omega \in \Omega}$ is referred to as a *field of Hilbert spaces*. A system $x := (x(\omega))_{\omega \in \Omega}$, $x(\omega) \in H(\omega)$, is called a *vector field* on Ω. Let $\mathcal{L}$ denote the vector space of all such vector fields. The field of Hilbert spaces is called *measurable* if there is a linear subspace $\widetilde{\mathcal{L}} \subseteq \mathcal{L}$ with the following properties:

(i) For each $x \in \widetilde{\mathcal{L}}$, the mapping $\Omega \ni \omega \to \|x(\omega)\|_\omega$ is measurable.

(ii) If the mapping $\Omega \ni \omega \to \langle x(\omega), y(\omega) \rangle_\omega$ is measurable for all $x \in \widetilde{\mathcal{L}}$ and some $y \in \mathcal{L}$, then $y \in \widetilde{\mathcal{L}}$.

(iii) There is a countable system $\{x_1, x_2, \ldots\} \subseteq \widetilde{\mathcal{L}}$ such that the system $\{x_1(\omega), x_2(\omega), \ldots\}$ is total in $H(\omega)$ for all ω.

The vector fields belonging to $\widetilde{\mathcal{L}}$ are called *measurable*. The countable system of vector fields in (iii) is called a *fundamental system of measurable vector fields*. A measurable vector field x is called *square-integrable w.r.t.* ν if $\int_\Omega \|x(\omega)\|_\omega^2\,\nu(d\omega) < \infty$. One introduces a semi-scalar product $\langle x, y \rangle := \int_\Omega \langle x(\omega), y(\omega) \rangle_\omega\,\nu(d\omega)$ on the linear space of all square-integrable vector fields. The corresponding quotient space H can be shown to be a Hilbert space (cf. [2], Prop. II.1.5). The Hilbert space H is said to be the *direct integral* of the Hilbert spaces $H(\omega)$, $\omega \in \Omega$, and it is denoted by

$$H = \int_{\Omega}^{\oplus} H(\omega)\,\nu(d\omega).$$

LEMMA 1 ([2], Prop. II.1.9). *Suppose that $(H(\omega))_{\omega \in \Omega}$ is a measurable field of complex Hilbert spaces and that we have closed subspaces $K(\omega)$ of $H(\omega)$ for all $\omega \in \Omega$. Let $P(\omega)$ denote the corresponding orthogonal projections. Let $\mathcal{K}$ be the linear space of all measurable vector fields x with $x(\omega) \in K(\omega)$ for all $\omega \in \Omega$. Then the following conditions are equivalent*:

(i) *The field $(K(\omega))_{\omega \in \Omega}$ is measurable w.r.t. the system of vector fields $\{(x(\omega))_{\omega \in \Omega} \in \widetilde{\mathcal{L}} : x(\omega) \in K(\omega)\}$.*

(ii) *There is a countable system $\{x_1, x_2, \ldots\}$ of measurable vector fields such that $\{x_1(\omega), x_2(\omega), \ldots\}$ is total in $K(\omega)$ for all $\omega \in \Omega$.*

(iii) *For each measurable vector field x, the field $(P(\omega)x(\omega))_{\omega \in \Omega}$ is measurable.*

Consider a measurable field $(H(\omega))_{\omega \in \Omega}$ of Hilbert spaces. A field $(T(\omega))_{\omega \in \Omega}$ of bounded linear operators $T(\omega) \in B(H(\omega))$ is said to be *measurable* if the vector field $(T(\omega)x(\omega))_{\omega \in \Omega}$ is measurable for each measurable vector field x. If $\operatorname{ess\,sup} \|T(\omega)\|$ is finite, then the vector field $(T(\omega))_{\omega \in \Omega}$ is called *essentially bounded*. In this case, the vector field $(T(\omega)x(\omega))_{\omega \in \Omega}$ is square-integrable for each square-integrable vector field x, and

$$T : H \ni x \to (T(\omega)x(\omega))_{\omega \in \Omega} \in H \tag{1}$$

defines a bounded operator T in H with $\|T\| = \operatorname{ess\,sup} \|T(\omega)\|$. Two such fields of operators define the same operator in H if and only if they coincide ν-a.e. ([2], Prop. II.2.2 and Corollary). A bounded operator T defined by (1) is called *decomposable* and we write

$$T = \int_{\Omega}^{\oplus} T(\omega)\,\nu(d\omega).$$

LEMMA 2 ([2], Prop. II.2.3). *If $T_j = \int_{\Omega}^{\oplus} T_j(\omega)\,\nu(d\omega)$, $j = 1, 2$, are decomposable operators, then*

$$\lambda T_1 + T_2 = \int_{\Omega}^{\oplus} (\lambda T_1(\omega) + T_2(\omega))\,\nu(d\omega), \qquad \lambda \in \mathbb{C},$$

$$T_1 T_2 = \int_{\Omega}^{\oplus} T_1(\omega) T_2(\omega)\,\nu(d\omega) \quad \text{and} \quad T_1^* = \int_{\Omega}^{\oplus} T_1^*(\omega)\,\nu(d\omega).$$

If almost all operators $T(\omega)$ are multiples of the identity $I(\omega)$ in $H(\omega)$, i.e. if $T = \int_{\Omega}^{\oplus} \varphi(\omega) I(\omega)\,\nu(d\omega)$ for some bounded measurable function φ on Ω, then T is called *diagonalizable*. The next property of direct integral operators is essential for the following.

PROPOSITION 3 ([2], Theorem II.2.1). *A bounded linear operator T in the direct integral $H = \int_{\Omega}^{\oplus} H(\omega)\,\nu(d\omega)$ that commutes with all diagonalizable operators is decomposable.*

Finally, we quote the following fact for sequences of decomposable operators from [2] (Prop. II.2.4).

PROPOSITION 4. *Consider decomposable operators $T_n = \int_{\Omega}^{\oplus} T_n(\omega)\,\nu(d\omega)$ for $n = 1, 2, \ldots$ and $T = \int_{\Omega}^{\oplus} T(\omega)\,\nu(d\omega)$.*

(i) *If T_n converges strongly to T, then there is a subsequence T_{n_k} such that $T_{n_k}(\omega)$ converges strongly to $T(\omega)$ ν-a.e.*

(ii) *If $T_n(\omega)$ converges strongly to $T(\omega)$ ν-a.e. and if $\sup_n \|T_n\|$ is finite, then T_n converges strongly to T.*

3. Weakly stationary processes. First we recall some facts about q-variate weakly stationary processes on locally compact abelian groups (cf. also [9]).

For a positive integer q, let $\mathcal{M}_q$ and $\mathcal{M}_q^{(\mathrm{rat})}$ denote the sets of all $q \times q$-matrices with complex and complex rational entries, respectively.

A *q-variate weakly stationary process* X on G is a mapping $X : G \to \mathcal{H}^q$, where $\mathcal{H}$ denotes a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, having the following properties:

(i) The $\mathcal{M}_q$-valued correlation function

$$G \times G \ni (g, h) \to (\langle X(g)_j, X(h)_k \rangle)_{j,k=1}^q =: K(g, h)$$

depends on the difference $g - h$ only. Accordingly, we write $K(g - h)$ instead of $K(g, h)$.

(ii) The function K is continuous on G.

The correlation function K has an integral representation

$$K(g) = \int_{\Gamma} [\gamma, g]\,F(d\gamma), \qquad g \in G,$$

with a regular non-negative hermitian $\mathcal{M}_q$-valued Borel measure F, the so-called *spectral measure* of the process X.

For a subset $S \subseteq G$, we write $\mathcal{W}_S$ for the closed $\mathcal{M}_q$-linear hull of all $X(g)$, $g \in S$. In particular, $\mathcal{W} := \mathcal{W}_G$ is the space spanned by all values of the process.

According to Kolmogorov's isomorphism theorem,

$$V : X(g) \to [\cdot, g] I_q$$

defines an isometric $\mathcal{M}_q$-linear mapping from $\mathcal{W}$ onto the space $L^2(F)$ of square-integrable (w.r.t. F) $\mathcal{M}_q$-valued functions on Γ. Here I_q denotes the $q \times q$-unit matrix.

DEFINITION. Fix a system $\mathbf{S}$ of subsets of G. The process X is called $\mathbf{S}$-*regular* if $\bigcap_{S \in \mathbf{S}} \mathcal{W}_S = \{0\}$. It is called $\mathbf{S}$-*singular* if $\mathcal{W}_S = \mathcal{W}$ for all $S \in \mathbf{S}$.

The notions of $\mathbf{S}$-regularity and $\mathbf{S}$-singularity as well as the problem of finding the gramian-orthogonal projection of an element from $\{X(g) : g \in G\}$ onto some $\mathcal{W}_S$ play an important role in the theory of linear prediction of stationary processes. Using Kolmogorov's isomorphism theorem, these problems are usually transferred to $L^2(F)$. Let $\mathcal{V}_S := V\mathcal{W}_S$ denote the closed $\mathcal{M}_q$-linear hull of all $\mathcal{M}_q$-valued functions $[\cdot, g] I_q$, $g \in S$, in $L^2(F)$. Then the following assertion holds.

LEMMA 5. *The process X is $\mathbf{S}$-regular* (*resp.* $\mathbf{S}$-*singular*) *if and only if $\bigcap_{S \in \mathbf{S}} \mathcal{V}_S = \{0\}$* (*resp. $\mathcal{V}_S = L^2(F)$ for all $S \in \mathbf{S}$*).

The gramian-orthogonal projection $Y_{g,S}$ of $X(g)$ on the subspace $\mathcal{W}_S$ is $V^{-1}\Phi_{g,S}$, where $\Phi_{g,S}$ denotes the gramian-orthogonal projection of $[\cdot, g] I_q$ on $\mathcal{V}_S$ in $L^2(F)$.

We note that the gramian-orthogonal projection coincides with the usual orthogonal projection.

Suppose that $\mathbf{S}_2$ is a system of subsets of G_2 and that the family $\mathbf{S}$ of subsets of G is given by

$$\mathbf{S} := \{G_1 \times S_2 : S_2 \in \mathbf{S}_2\}. \tag{2}$$

It is the aim of this paper to show that for a system of the special form (2), the investigation of $\mathbf{S}$-regularity, $\mathbf{S}$-singularity, or of the gramian-orthogonal projection onto $\mathcal{W}_S$, $S \in \mathbf{S}$, reduces to the corresponding questions for the system $\mathbf{S}_2$.

4. Decomposition of processes. Consider a non-negative finite regular Borel measure μ on Γ such that all entries of F are absolutely continuous w.r.t. μ. Let $\mathcal{F} := dF/d\mu$ denote the matrix of the Radon–Nikodym derivatives of the entries of F. Hence we can express the matrix-valued inner product of $\Phi, \Psi \in L^2(F)$ as

$$\langle \Phi, \Psi \rangle_F = \int_\Gamma \Phi(\gamma)\mathcal{F}(\gamma)\Psi(\gamma)^* \, \mu(d\gamma).$$

Note that $L^2(F)$ is a Hilbert space w.r.t. the scalar product $\tau\langle \Phi, \Psi\rangle_F$, $\Phi, \Psi \in L^2(F)$. Here τ denotes the trace of a matrix. For more information on $L^2(F)$ we refer the reader to [10]. Let $\mathcal{B}_j$ be the σ-algebra of Borel subsets of Γ_j, $j = 1, 2$, and let μ_1 denote the marginal measure of μ on Γ_1, i.e. $\mu_1(\Delta_1) := \mu(\Delta_1 \times \Gamma_2)$, $\Delta_1 \in \mathcal{B}_1$. Since Γ_2 is a Polish space, there exists a function $w : \Gamma_1 \times \mathcal{B}_2 \ni (\gamma_1, \Delta_2) \to w(\gamma_1, \Delta_2) \in [0, \infty)$ with the following properties (cf. [3], Theorem and Corollary 2 of Section 21.2):

(i) For each $\Delta_2 \in \mathcal{B}_2$, the function $\gamma_1 \to w(\gamma_1, \Delta_2)$ is $\mathcal{B}_1$-measurable,

(ii) for μ_1-a.a. $\gamma_1 \in \Gamma_1$ the function $\Delta_2 \to w(\gamma_1, \Delta_2)$ is a non-negative finite Borel measure on Γ_2,

(iii) for $\Delta_1 \in \mathcal{B}_1$ and $\Delta_2 \in \mathcal{B}_2$ we have $\mu(\Delta_1 \times \Delta_2) = \int_{\Delta_1} w(\gamma_1, \Delta_2)\, \mu_1(d\gamma_1)$.

For $\varphi \in L^\infty(\mu_1)$, let M_φ denote the operator of multiplication by φ in $L^2(F)$. If $\varphi = [\cdot, g_1]$, $g_1 \in G_1$, we write M_{g_1} instead of M_φ.

LEMMA 6. *The von Neumann algebra $\{M_\varphi : \varphi \in L^\infty(\mu_1)\}$ is generated by $\{M_{g_1} : g_1 \in G_1\}$.*

Proof. Otherwise we could find some $\Phi, \Psi \in L^2(F)$ and $\varphi \in L^\infty(\mu_1)$ such that $\langle M_{g_1}\Phi, \Psi\rangle_F = 0$, $g_1 \in G_1$, and $\langle M_\varphi \Phi, \Psi\rangle_F \neq 0$. Since

$$0 = \langle M_{g_1}\Phi, \Psi\rangle_F = \int_\Gamma [\gamma_1, g_1]\Phi(\gamma)\mathcal{F}(\gamma)\Psi(\gamma)^* \, \mu(d\gamma)$$

$$= \int_{\Gamma_1} [\gamma_1, g_1] \int_{\Gamma_2} \Phi(\gamma_1, \gamma_2)\mathcal{F}(\gamma_1, \gamma_2)\Psi(\gamma_1, \gamma_2)^* \, w(\gamma_1, d\gamma_2)\, \mu_1(d\gamma_1)$$

for $g_1 \in G_1$, and since the Fourier transform is one-to-one, the measure

$$\lambda(d\gamma_1) := \int_{\Gamma_2} \Phi(\gamma_1, \gamma_2)\mathcal{F}(\gamma_1, \gamma_2)\Psi(\gamma_1, \gamma_2)^* \, w(\gamma_1, d\gamma_2)\, \mu_1(d\gamma_1)$$

has to be zero. Hence

$$\langle M_\varphi \Phi, \Psi\rangle_F$$

$$= \int_{\Gamma_1} \varphi(\gamma_1) \int_{\Gamma_2} \Phi(\gamma_1, \gamma_2)\mathcal{F}(\gamma_1, \gamma_2)\Psi(\gamma_1, \gamma_2)^* \, w(\gamma_1, d\gamma_2)\, \mu_1(d\gamma_1) = 0$$

contrary to our assumption, which proves the assertion.

Consider a separable subset $S_2 \subseteq G_2$ and $S := G_1 \times S_2$.

LEMMA 7. *The orthogonal projection P in $L^2(F)$ on the subspace $\mathcal{V}_S$ commutes with all operators M_φ, $\varphi \in L^\infty(\mu_1)$.*

Proof. Since the subspace $\mathcal{V}_S$ is invariant w.r.t. all operators M_{g_1}, $g_1 \in G_1$, P commutes with all M_{g_1}. Hence the assertion follows from Lemma 6.

For $\gamma_1 \in \Gamma_1$ we define a non-negative hermitian $\mathcal{M}_q$-valued measure F_{γ_1} by $F_{\gamma_1}(\Delta_2) := \int_{\Delta_2} \mathcal{F}(\gamma_1, \gamma_2)\, w(\gamma_1, d\gamma_2)$, where Δ_2 runs through all Borel subsets of Γ_2. Let $H(\gamma_1) := L^2(F_{\gamma_1})$ and $H := \int_{\Gamma_1}^{\oplus} H(\gamma_1)\, \mu_1(d\gamma_1)$, the direct integral of these Hilbert spaces with the fundamental system of measurable vector fields $\{[\cdot, g_2]A : g_2 \in D_2,\ A \in \mathcal{M}_q^{(\mathrm{rat})}\}$, where D_2 is a countable dense subset of G_2. We define a linear mapping $\mathbf{i}$ on $L^2(F)$ assigning to $\Phi \in L^2(F)$ the vector field $\mathbf{i}\Phi = (\Phi(\gamma_1, \cdot))_{\gamma_1 \in \Gamma_1}$.

LEMMA 8. *The mapping $\mathbf{i}$ is an isometric $\mathcal{M}_q$-linear mapping from $L^2(F)$ onto H. In particular, it is correctly defined, i.e. for different representatives of the class of Φ the images coincide μ_1-a.e.*

Proof. The $\mathcal{M}_q$-linearity is obvious. Since

$$\int_\Gamma \Phi(\gamma)\mathcal{F}(\gamma)\Phi(\gamma)^* \, \mu(d\gamma)$$

$$= \int_{\Gamma_1} \Big[\int_{\Gamma_2} \Phi(\gamma_1, \gamma_2)\mathcal{F}(\gamma_1, \gamma_2)\Phi(\gamma_1, \gamma_2)^* \, w(\gamma_1, d\gamma_2)\Big] \mu_1(d\gamma_1)$$

$$= \int_{\Gamma_1} \|\Phi(\gamma_1, \cdot)\|_{\gamma_1}^2 \, \mu_1(d\gamma_1),$$

the mapping $\mathbf{i}$ is a correctly defined isometry from $L^2(F)$ into H. To prove that it is onto, we consider a vector field $(\Psi(\gamma_1, \cdot))_{\gamma_1 \in \Gamma_1}$ orthogonal to $\mathbf{i}L^2(F)$. Then, in particular,

$$\int_{\Gamma_1} [\gamma_1, g_1] \int_{\Gamma_2} [\gamma_2, g_2]\mathcal{F}(\gamma_1, \gamma_2)\Psi(\gamma_1, \gamma_2)^* \, w(\gamma_1, d\gamma_2)\, \mu_1(d\gamma_1) = 0$$

for all $g_1 \in G_1$, $g_2 \in G_2$. Since the Fourier transform is one-to-one, it follows that

$$\int_{\Gamma_2} [\gamma_2, g_2]\mathcal{F}(\gamma_1,\gamma_2)\Psi(\gamma_1,\gamma_2)^* \, w(\gamma_1, d\gamma_2) = 0$$

for all $g_2 \in G_2$ and μ_1-a.a. $\gamma_1 \in \Gamma_1$. This implies $\Psi(\gamma_1,\cdot) = 0$ in $H(\gamma_1)$ for μ_1-a.a. $\gamma_1 \in \Gamma_1$. Hence $\Psi = 0$ in H, which completes the proof.

According to Lemma 8, we may (and do) identify H and $L^2(F)$.

Then the multiplication operators $M_\varphi, \varphi \in L^\infty(\mu_1)$, are diagonalizable and according to Lemma 7 and Proposition 3 the orthogonal projection P on $\mathcal{V}_S$ is decomposable:

$$P = \int_{\Gamma_1}^{\oplus} P(\gamma_1)\, \mu_1(d\gamma_1).$$

Let $\mathcal{V}_{\gamma_1,S_2}$ denote the closed $\mathcal{M}_q$-linear span of all functions $[\cdot, g_2]I_q$, $g_2 \in S_2$, in $H(\gamma_1)$, and $Q(\gamma_1)$ the corresponding orthoprojection.

LEMMA 9. $P = \int_{\Gamma_1}^{\oplus} Q(\gamma_1)\, \mu_1(d\gamma_1)$.

Proof. Let D_2 denote a countable dense subset of S_2. All (constant) vector fields $([\cdot, g_2]A)_{\gamma_1\in\Gamma_1}$, $g_2 \in D_2$, $A \in \mathcal{M}_q^{(\mathrm{rat})}$, are measurable and constitute a countable set. Moreover, the vectors $[\cdot, g_2]A$, $g_2 \in D_2$, $A \in \mathcal{M}_q^{(\mathrm{rat})}$, considered as elements of $H(\gamma_1)$ are total in $\mathcal{V}_{\gamma_1,S_2}$. According to Lemma 1, $(\mathcal{V}_{\gamma_1,S_2})_{\gamma_1\in\Gamma_1}$ is a measurable field of subspaces and $Q := \int_{\Gamma_1}^{\oplus} Q(\gamma_1)\, \mu_1(d\gamma_1)$ is correctly defined. According to Lemma 2, Q is an orthoprojection in $H = L^2(F)$. Obviously, QH is a subspace of $\mathcal{V}_S$. Suppose now that there is some $\Phi \in PH = \mathcal{V}_S$ orthogonal to QH. Then

$$\begin{aligned} 0 &= \int_{\Gamma} [\gamma_1, g_1][\gamma_2, g_2]\mathcal{F}(\gamma)\Phi(\gamma)^*\, \mu(d\gamma) \\ &= \int_{\Gamma_1} [\gamma_1, g_1] \int_{\Gamma_2} [\gamma_2, g_2]\mathcal{F}(\gamma_1,\gamma_2)\Phi(\gamma_1,\gamma_2)^*\, w(\gamma_1, d\gamma_2)\, \mu_1(d\gamma_1) \end{aligned}$$

for all $g_1 \in G_1$, $g_2 \in S_2$. Using a similar argument to that in Lemma 8 one concludes that $\Phi = 0$ in H, which completes the proof.

Lemmas 8 and 9 yield immediately the following assertion for processes on $G = G_1 \times G_2$, where G_2 is separable and Γ_2 is a Polish space. Recall that $S = G_1 \times S_2$, $S_2 \subseteq G_2$ separable.

PROPOSITION 10. *The gramian-orthogonal projection of an element* $X(g)$, $g \in G$, $g = (g_1, g_2)$, *on the space* $\mathcal{W}_S$ *is* $V^{-1}\int_{\Gamma_1}^{\oplus} Q(\gamma_1)([\cdot, g_2]I_q)\, \mu_1(d\gamma_1)$.

If $\mathbf{S}$ is a countable system of the form (2), where all S_2 are assumed to be *separable*, then the following results concerning $\mathbf{S}$-singularity and $\mathbf{S}$-regularity can be derived.

PROPOSITION 11. *The process X is* $\mathbf{S}$*-singular if and only if for* μ_1*-a.a.* $\gamma_1 \in \Gamma_1$,

$$\mathcal{V}_{\gamma_1,S_2} = H(\gamma_1) \quad \forall S_2 \in \mathbf{S}_2.$$

Proof. The assertion follows imediately from Lemmas 9 and 5.

PROPOSITION 12. *The process X is* $\mathbf{S}$*-regular if and only if* $\bigcap_{S_2\in\mathbf{S}_2} \mathcal{V}_{\gamma_1,S_2} = \{0\}$ *for* μ_1*-a.e.* $\gamma_1 \in \Gamma_1$.

Proof. If K_1 and K_2 are two subspaces of a Hilbert space K, and P_1 and P_2 the corresponding orthogonal projections, then the orthogonal projection on $K_1 \cap K_2$ is the strong limit of the sequence of operators $((P_1P_2)^k)_{k\in\mathbb{N}}$. Together with Lemma 2 and Proposition 4, we deduce that the orthoprojection $\widetilde{P}$ on $\bigcap_{S\in\mathbf{S}} \mathcal{V}_S$ is decomposable, i.e.

$$\widetilde{P} = \int_{\Gamma_1}^{\oplus} \widetilde{P}(\gamma_1)\, \mu_1(d\gamma_1),$$

where $\widetilde{P}(\gamma_1)$ is the orthoprojection in $H(\gamma_1)$ on $\bigcap_{S_2\in\mathbf{S}_2} \mathcal{V}_{\gamma_1,S_2}$. Lemma 5 now yields our assertion.

5. Applications. We illustrate the results of the preceding section by three examples. We use the notation $\nu \prec \lambda$ if the measure ν is absolutely continuous w.r.t. the measure λ. As a preliminary we prove the following lemma:

LEMMA 13. *Let* $\mu(\Delta_1 \times \Delta_2) = \int_{\Delta_1} w(\gamma_1, \Delta_2)\, \mu_1(d\gamma_1)$ *for all* $\Delta_1 \in \mathbf{B}_1$, $\Delta_2 \in \mathbf{B}_2$ *and let ν be some Borel measure on* Γ_2. *Then* $w(\gamma_1,\cdot) \prec \nu(\cdot)$ *for* μ_1*-a.a.* $\gamma_1 \in \Gamma_1$ *if and only if* $\mu \prec \mu_1 \otimes \nu$. *In this case*

$$\frac{d\mu}{d(\mu_1\otimes\nu)} = \frac{dw(\gamma_1,\cdot)}{d\nu(\cdot)} \quad \mu_1\otimes\nu\text{-a.e.} \tag{3}$$

Proof. Let $N_{1,1} \in \mathcal{B}_1$ be such that $\mu_1(N_{1,1}) = 0$ and $w(\gamma_1,\cdot) \prec \nu(\cdot)$ for $\gamma_1 \in \Gamma_1 \setminus N_{1,1}$. Further, let $\Delta \in \mathcal{B} := \mathcal{B}_1 \otimes \mathcal{B}_2$ and assume that $(\mu_1\otimes\nu)(\Delta) = 0$. It follows that

$$\int_{\Gamma} I_\Delta(\gamma)\,(\mu_1\otimes\nu)(d\gamma) = \int_{\Gamma_1} \nu(\Delta_{\gamma_1})\, \mu_1(d\gamma_1) = 0, \tag{4}$$

where I_Δ denotes the indicator function of Δ and $\Delta_{\gamma_1} := \{\gamma_2 \in \Gamma_2 : (\gamma_1,\gamma_2) \in \Delta\}$. Relation (4) implies the existence of a set $N_{1,2} \in \mathcal{B}_1$ such that $\mu_1(N_{1,2}) = 0$ and $\nu(\Delta_{\gamma_1}) = 0$ for all $\gamma_1 \in \Gamma_1 \setminus N_{1,2}$. Set $N_1 := N_{1,1} \cup N_{1,2}$. Then we have $\mu_1(N_1) = 0$ and $w(\gamma_1, \Delta_{\gamma_1}) = 0$ for all $\gamma_1 \in \Gamma_1 \setminus N_1$. But then

$$\mu(\Delta) = \int_{\Gamma_1}\int_{\Gamma_2} I_\Delta(\gamma_1,\gamma_2)\, w(\gamma_1, d\gamma_2)\, \mu_1(d\gamma_1) = \int_{\Gamma_1} w(\gamma_1, \Delta_{\gamma_1})\, \mu_1(d\gamma_1) = 0,$$

hence $\mu \prec \mu_1 \otimes \nu$.

Conversely, let $\mu \prec \mu_1 \otimes \nu$. Note that μ is a σ-finite measure. Consequently, the Radon–Nikodym derivative $d\mu/d(\mu_1 \otimes \nu)$ exists; we will denote it by g. For $\Delta_1 \in \Gamma_1$, $\Delta_2 \in \Gamma_2$, we get

$$\mu(\Delta_1 \times \Delta_2) = \int_{\Delta_1} \int_{\Delta_2} g(\gamma_1, \gamma_2)\,\nu(d\gamma_2)\,\mu_1(d\gamma_1) = \int_{\Delta_1} w(\gamma_1, \Delta_2)\,\mu_1(d\gamma_1).$$

Thus, there exists a set $N_{\Delta_2} \in \mathcal{B}_1$ such that $\mu_1(N_{\Delta_2}) = 0$ and

$$w(\gamma_1, \Delta_2) = \int_{\Delta_2} g(\gamma_1,\gamma_2)\,\nu(d\gamma_2) \qquad \text{for all } \gamma_1 \in \Gamma_1 \setminus N_{\Delta_2}.$$

Since Γ_2 is a Polish space, there exists a sequence $\{\Delta_2^{(j)}\}_{j=1}^{\infty} \subseteq \mathcal{B}_2$ generating $\mathcal{B}_2$. Let $N_1 := \bigcup_{j=1}^{\infty} N_{\Delta_2^{(j)}}$. We obtain $\mu_1(N_1) = 0$ and $w(\gamma_1, \Delta_2) = \int_{\Delta_2} g(\gamma_1,\gamma_2)\,\nu(d\gamma_2)$ for all $\gamma_1 \in \Gamma_1 \setminus N_1$ and all $\Delta_2 \in \mathcal{B}_2$, hence $w(\gamma_1, \cdot) \prec \nu(\cdot)$ and the relation (3) is established.

5.1. We generalize some results of [4]. Let $G_2 = \mathbb{R}$ and let m denote the Lebesgue measure on $\mathbb{R}$. Consider the system $\mathbf{S}_2 := \{(-\infty, t] : t \in \mathbb{R}\}$ and a univariate (i.e. $q = 1$) weakly stationary process X on $G := G_1 \times \mathbb{R}$. Since the system $\mathbf{S}$ may be replaced by the countable system $\{G_1 \times (-\infty, n] : n \in \mathbb{Z}\}$, Propositions 11 and 12 may be applied. The spectral measure F is a nonnegative finite (scalar) measure. We choose $\mu := F$ and hence $\mathcal{F} \equiv 1$. For $\gamma_1 \in \Gamma_1$, set $\mu_{\gamma_1}(\Delta_2) := \int_{\Delta_2} w(\gamma_1, d\gamma_2)$, $\Delta_2 \in \mathcal{B}$. Let $\mu_{\gamma_1}^{\mathrm{a}}$ and μ^{a} be the absolutely continuous parts of μ_{γ_1} and μ w.r.t. m and $\mu_1 \otimes m$, resp. Lemma 12 implies that $\mu_{\gamma_1} \prec m$ for μ_1-a.a. $\gamma_1 \in \Gamma_1$ if and only if $\mu \prec \mu_1 \otimes m$ and

$$\frac{\mu_{\gamma_1}^{\mathrm{a}}}{dm} = \frac{\mu^{\mathrm{a}}}{d(\mu_1 \otimes m)} \qquad \mu_1 \otimes m\text{-a.e.}$$

Using Proposition 11 and the known criteria for $\mathbf{S}_2$-singularity of processes on $\mathbb{R}$ (cf. e.g. [11], pp. 161–162) we immediately see that X is $\mathbf{S}$-singular if and only if

$$\int_{\mathbb{R}} (1 + \gamma_2^2)^{-1} \log \frac{d\mu^{\mathrm{a}}}{d(\mu_1 \otimes m)}(\gamma_1, \gamma_2)\,m(d\gamma_2) = -\infty \qquad \mu_1\text{-a.e.}$$

(cf. [4], Theorem 3.5).

From Lemma 13, Proposition 12 and known results about $\mathbf{S}_2$-regularity of processes on $\mathbb{R}$ (cf. e.g. [11], p. 161) it follows that X is $\mathbf{S}$-regular if and only if $\mu \prec \mu_1 \otimes m$ and

$$\int_{\mathbb{R}} (1 + \gamma_2^2)^{-1} \log \frac{d\mu}{d(\mu_1 \otimes m)}(\gamma_1, \gamma_2)\,m(d\gamma_2) > -\infty \qquad \mu_1\text{-a.e.}$$

(cf. [4], Theorems 3.2 and 3.3).

5.2. Let G_2 be a discrete abelian group with counting measure m and put $\mathbf{S}_2 := \{G_2 \setminus \{g_2\} : g_2 \in G_2\}$. In ([8], Theorem 4.6) and ([9], Theorem 5.3) some results on $\mathbf{S}_2$-singularity and $\mathbf{S}_2$-regularity, resp., of multivariate processes on G_2 were proved. Suppose now additionally that G_2 is countable and note that in this case Γ_2 is a Polish space. From the cited theorems we can derive results on $\mathbf{S}_2$-singularity and $\mathbf{S}_2$-regularity of multivariate processes on $G_1 \times G_2$. The argument is similar to that in 5.1. In the case of a univariate process X on $G_1 \times G_2$ the result reads as follows. Suppose that the measures $\mu_1, \mu^{\mathrm{a}}, \mu_{\gamma_1}, \gamma_1 \in \Gamma_1$, are defined similarly to 5.1. Then X is $\mathbf{S}$-singular if and only if

$$\int_{\Gamma_2} \left(\frac{d\mu^{\mathrm{a}}}{d(\mu_1 \otimes m)}(\gamma_1, \gamma_2) \right)^{-1} m(d\gamma_2) = \infty \qquad \mu_1\text{-a.e.}$$

It is $\mathbf{S}$-regular if and only if $\mu \prec \mu_1 \otimes m$ and

$$\int_{\Gamma_2} \left(\frac{d\mu}{d(\mu_1 \otimes m)}(\gamma_1, \gamma_2) \right)^{-1} m(d\gamma_2) < \infty \qquad \mu_1\text{-a.e.}$$

5.3. From Proposition 10 it follows that for $\Phi \in L^2(F)$ the prediction error $\|\Phi - P\Phi\|$ is equal to

$$\|\Phi - P\Phi\| = \Big(\int_{\Gamma_1} \|\Phi(\gamma_1, \cdot) - P(\gamma_1)\Phi(\gamma_1, \cdot)\|_{\gamma_1}^2\,\mu_1(d\gamma_1) \Big)^{1/2}.$$

Using the well known formulas for the prediction error in Kolmogorov's extrapolation problem for univariate processes on $\mathbb{Z}$ or $\mathbb{R}$, we immediately obtain prediction error formulas of the appropriate problem for univariate processes on $G_1 \times \mathbb{Z}$ or $G_1 \times \mathbb{R}$. For example, we can get Theorem 4.5 of [1].

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Banach space properties of strongly tight uniform algebras

by

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Abstract. We use the work of J. Bourgain to show that some uniform algebras of analytic functions have certain Banach space properties. If X is a Banach space, we say X is *strong* if X and X^* have the Dunford-Pettis property, X has the Pełczyński property, and X^* is weakly sequentially complete. Bourgain has shown that the ball-algebras and the polydisk-algebras are strong Banach spaces. Using Bourgain's methods, Cima and Timoney have shown that if K is a compact planar set and A is $R(K)$ or $A(K)$, then A and A^* have the Dunford–Pettis property. Prior to the work of Bourgain, it was shown independently by Wojtaszczyk and Delbaen that $R(K)$ and $A(K)$ have the Pełczyński property for special classes of sets K. We show that if A is $R(K)$ or $A(K)$, where K is arbitrary, or if A is $A(D)$ where D is a strictly pseudoconvex domain with smooth C^2 boundary in $\mathbb{C}^n$, then A is a strong Banach space. More generally, if A is a uniform algebra on a compact space K, we say A is *strongly tight* if the Hankel-type operator $S_g : A \to C/A$ defined by $f \mapsto fg + A$ is compact for every $g \in C(K)$. Cole and Gamelin have shown that $R(K)$ and $A(K)$ are strongly tight when K is arbitrary, and their ideas can be used to show $A(D)$ is strongly tight for the domains D considered above. We show strongly tight uniform algebras are strong Banach spaces.

Introduction. Let X and Y be Banach spaces. If $T : X \to Y$ is a bounded linear operator, we say T is *completely continuous* (or is a *Dunford–Pettis operator*) if T takes weakly null sequences in X to norm null sequences in Y. We say X has the *Dunford–Pettis property* if every weakly compact operator $T : X \to Y$ is completely continuous. It may be shown that this is equivalent to saying weakly null sequences in X tend to zero uniformly on weakly compact subsets of X^*. It follows easily from the latter characterization that X has the Dunford–Pettis property whenever X^* does.

Let $\{x_n\}$ be a sequence in X. We say $\{x_n\}$ is a *weakly unconditionally Cauchy (w.u.C.) series* if $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for every $x^* \in X^*$, and a bounded linear operator $T : X \to Y$ is an *unconditionally converging operator* if T takes w.u.C. series to series that converge unconditionally in norm. This is equivalent to saying T is never an isomorphic embedding

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