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On algebraic solutions of algebraic Pfaff equations

by

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Abstract. We give a new proof of Jouanolou's theorem about non-existence of algebraic solutions to the system $\dot{x} = z^s$, $\dot{y} = x^s$, $\dot{z} = y^s$. We also present some generalizations of the results of Darboux and Jouanolou about algebraic Pfaff forms with algebraic solutions.

Introduction. In [5] Jouanolou studied the Pfaff equations with polynomial coefficients. One of his main results states that the subset Z_m of the set V_m of Pfaff equations of degree $m > 2$ on $P^2_{\mathbb{C}}$ consisting of equations without algebraic solutions is dense in V_m in the usual topology (see [5, p. 158]). (Lins-Neto in [6] proved that Z_m is also open.) To show this he needs an example of a Pfaff equation without algebraic solutions and he chooses

$$(1)\qquad (x^{m-1}z - y^m)dx + (y^{m-1}x - z^m)dy + (z^{m-1}y - x^m)dz = 0,$$

where $m > 2$ is an integer. The whole Chapter 4 of [5] is devoted to the proof of non-algebraicity of the solutions of (1).

Below we present a new proof of this result based on the author's original generalization (Theorem 3 below) of a classical theorem of Darboux (Theorem 2 below) proved in the preprint [8].

Another generalization of the Darboux theorem, to higher dimensions, was given in [5] by Jouanolou (Theorem 4 below). In Theorem 5 below we present our generalization of Theorem 4. Our approach is different from the one developed by Darboux and Jouanolou. We are more interested in cases of few algebraic solutions of a Pfaff equation but in generic position whereas they consider situations with many but arbitrary solutions.

We treat Theorems 3 and 5 as the main results of this paper because the methods developed in them seem to be useful in applications (e.g. in the center-focus problem or in the problem of integrability).

The origin of the present work comes from the question of J.-M. Strelcyn at the seminar on dynamical systems in Warsaw in 1992. He stated the

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problem of finding a more analytic proof of the non-integrability of the system

$$\dot{x} = z^s, \quad \dot{y} = x^s, \quad \dot{z} = y^s. \tag{2}$$

J. Moulin-Ollagnier, A. Nowicki and J.-M. Strelcyn gave a detailed explanation of Jouanolou's proof in [7] (with some other applications of Jouanolou's ideas).

It turns out that there are two more proofs of Jouanolou's Theorem based on the analysis of the singular points of (2). They are given in [3] and [6].

The author thanks J.-M. Strelcyn for introducing him to the problem and for pointing out to him the book [5].

2. Definitions and the Jouanolou theorem. An *algebraic Pfaff equation* of degree m on $P^r_{\mathbb{C}}$ is an equation of the type

$$\omega = \sum_{i=1}^{r+1} P_i(x)dx_i = 0,$$

where the P_i are homogeneous polynomials of degree m and ω is *projective*, which means that $i_X\omega = \sum P_i x_i = 0$, where $X = \sum x_i\partial_{x_i}$.

The algebraic hypersurface $f = 0$, f homogeneous, is said to be a *solution* to the equation $\omega = 0$ iff

$$\omega \wedge df = f\alpha$$

with some 2-form α. (An equivalent definition is that $i^*\omega = 0$, where $i : \{f = 0\} \to P^r_{\mathbb{C}}$ is the natural embedding.)

On the 2-dimensional projective plane $P^2_{\mathbb{C}}$ there is a natural correspondence between homogeneous polynomial vector fields

$$V = X\partial_x + Y\partial_y + Z\partial_z$$

and projective algebraic Pfaff equations

$$\omega = Pdx + Qdy + Rdz = 0,$$

$$\begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \wedge \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} Yz - Zy \\ Zx - Xz \\ Xy - Yx \end{pmatrix}$$

(see [5, p. 5]).

The algebraic curve $f = 0$ forms an algebraic *particular solution* (or is *invariant*) for the vector field V iff

$$V(f) = f \cdot g$$

for some polynomial g. This agrees with the notion of a solution to the algebraic Pfaff equation.

One can easily see that the equation (1) and the system (2) correspond to each other for $s = m - 1$. The main result about them is the following (see [5, pp. 157, 158, 193]).

THEOREM 1 (Jouanolou). *If $s \geq 2$ is an integer then the vector field* (2) *does not have algebraic invariant curves in $P^2_{\mathbb{C}}$.*

The other results of this paper are given in Sections 5 and 7.

3. Symmetries of the vector field (2). From the qualitative theory point of view it is convenient to consider vector fields on $P^2_{\mathbb{C}} = \{(x : y : z)\}$ as polynomial vector fields on $\mathbb{C}^2 = \{(x : y : 1)\}$. The vector field (2) transforms to

$$\dot{x} = 1 - xy^s, \quad \dot{y} = x^s - y^{s+1}. \tag{3}$$

LEMMA 1. *The vector field* (3) *admits actions of the three groups $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/(s^2+s+1)\mathbb{Z}$.*

Proof. The generator of $\mathbb{Z}/2\mathbb{Z}$ is the usual conjugation $\varrho : (x,y) \to (\bar{x},\bar{y})$. The generator of $\mathbb{Z}/3\mathbb{Z}$ is obtained from the cyclic permutation (in reverse order) of the homogeneous coordinates in (2) and reads $(x,y) \to (y/x, 1/x)$. (After this transformation we get a rational vector field and we have to multiply it by some polynomial, but such an operation does not change the phase portrait.) The generator of $\mathbb{Z}/(s^2+s+1)\mathbb{Z}$ is given by

$$\sigma : (x,y) \to (\zeta^{-s}x, \zeta y), \quad \zeta = \exp\left(\frac{2\pi i}{s^2+s+1}\right). \ \blacksquare$$

Remark 1. In [5] the semidirect product of the groups $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/(s^2+s+1)\mathbb{Z}$ is studied but the conjugation operation is not taken into account.

Remark 2. Near the line at infinity $l_\infty = \{z = 0\}$ we choose other affine charts:

$(x,z) = (x : 1 : z)$ with the vector field $\dot{x} = z^s - x^{s+1}$, $\dot{z} = 1 - x^s z$ and

$(y,z) = (1 : y : z)$ with the vector field $\dot{y} = 1 - yz^s$, $\dot{z} = y^s - z^{s+1}$.

4. Some qualitative properties of the vector field (3)

LEMMA 2. (a) *The vector field* (3) *has s^2+s+1 singular points $p_0 = (1,1)$, $p_j = \sigma^j p_0$, $j = 1,\ldots,s^2+s$.*

(b) *The eigenvalues of the linear parts of the vector field* (3) *at the points p_j are equal to $\lambda_{1,2} = \frac{1}{2}(-s-2 \pm is\sqrt{3})\zeta^{sj}$, $i = \sqrt{-1}$.*

(c) *For any p_j there are only two locally analytic curves tangent to the vector field* (3) *and passing through p_j; they intersect transversally.*

(d) *The line at infinity l_∞ is not invariant for the vector field* (2). *The trajectories of* (2) *cross l_∞ transversally except the trajectory γ go-*

ing through the point $(1:0:0)$. *A part of* $\gamma^{\mathbb{R}}$ *(the real part of* γ*) lies in the first quadrant* $\Delta = \{x, y, z > 0\}$.

(e) *The only singular point of the vector field* (3) *in* $\mathbb{R}^2$, p_0, *is a stable focus and all trajectories starting in the first quadrant tend to* p_0. *None of these trajectories is algebraic. If* s *is even then no real* 1*-dimensional trajectory is algebraic.*

Proof. Statements (a), (b) and (c) are standard and were proved in [5].

(d) In the (x, z)-chart we have $\dot{z}|_{l_\infty} = 1 > 0$. In the (y, z)-chart we have $\dot{z}|_{l_\infty} = y^s$ and near the point $z = y = 0$ we have $dz/dy = y^s + \ldots$ So the equation of the trajectory γ is $z = y^{s+1}/(s+1) + \ldots$ and $\gamma^{\mathbb{R}} \cap \{y > 0\}$ is in Δ.

(e) The type of p_0 follows from (b). To prove that p_0 attracts all points of Δ we notice that at the boundary of Δ the vector field (3) is directed to the inside of Δ. Next, the divergence of this vector field is equal to $-(s+2)y^s < 0$ and any limit cycle in Δ is attracting (see [1]). By the Poincaré–Bendixson theorem (see [1]), a possible non-empty limit set in $\Delta \setminus \{p_0\}$ contains a limit cycle whose stability type contradicts the stability type of p_0. Therefore, the trajectories in Δ are spirals and cannot be algebraic.

If s is even then all this holds for any trajectory starting in $\mathbb{R}P^2 \setminus \{(1:1:1)\}$. ■

COROLLARY 1. *Any irreducible invariant algebraic curve* $S_i = \{f_i = 0\}$, $f_i(x, y)$ *a polynomial, of the vector field* (3) *has the following properties*:

(i) *its singular points are all double points (transversal self-intersections) and*

(ii) S_i *intersects* l_∞ *transversally. (We denote the projective compactification of* S_i *in* $P^2_{\mathbb{C}}$ *also by* S_i.)

If $S_1, \ldots, S_p$ *are irreducible algebraic invariant curves of the vector field* (3) *then*:

(iii) *they intersect transversally*,

(iv) *there are no triple intersections and*

(v) *they do not intersect at infinity.*

Proof. Assertions (i), (iii) and (iv) follow from Lemma 2(c). Statements (ii) and (v) follow from assertions (d) and (e) of that lemma. ■

5. The generalization of the Darboux theorem. In [4] Darboux proved the following result, which we have adapted to the case of vector fields.

THEOREM 2 (Darboux). *If a polynomial vector field* V *on* $\mathbb{R}^2$ *of degree* n *has at least* p *different algebraic invariant curves, where* $p > n(n+1)/2$, *then it has a Darboux first integral*

$$H = \prod f_i^{\alpha_i}.$$

Proof. Let $f_1 = 0, \ldots, f_p = 0$ be the algebraic invariant curves. We have $V(f_i) = f_i \cdot g_i$, $i = 1, \ldots, p$, where $\deg g_i = n - 1$. If $p > \dim\{$polynomials of degree $\leq n - 1\}$ then the functions g_i are dependent, $\sum \alpha_i g_i = 0$. This means that $0 = \sum \alpha_i \dot{f}_i / f_i = \frac{d}{dt} \ln H$. ■

We generalize this result in the following way (part A was proved in [8]).

THEOREM 3. A. *If* $S_1, \ldots, S_p$, *where* $S_i = \{f_i = 0\}$, *are irreducible algebraic curves in* $\mathbb{C}^2$ *satisfying the conditions* (ii)–(v) *from Corollary* 1 *and*

(i′) $S_i = 0$ *are smooth in* $\mathbb{C}^2$

then any polynomial vector field V *of degree* n *tangent to all* S_i *is of the form described below.*

(a) *If* $n > \sum \deg f_i - 1 = k - 1$ *then*

$$V = W \prod f_i + \sum h_i \Big(\prod_{j \neq i} f_j \Big) X_{f_i}, \tag{4}$$

where $X_F = (\partial F/\partial y, -\partial F/\partial x)$ *is a Hamiltonian vector field, the* h_i *are polynomials of degree* $\leq n - k + 1$ *and* W *is a polynomial vector field of degree* $\leq n - k$.

(b) *If* $n = k - 1$ *then* $V = \sum \alpha_i (\prod_{j \neq i} f_j) X_{f_i}$, $\alpha_i \in \mathbb{C}$. *In this case a Darboux first integral exists.*

(c) *If* $n < k - 1$ *then* $V \equiv 0$.

B. *If we replace the condition* (i′) *by* (i) *then the previous result requires the following modification. Let* $D(x, y)$ *be a polynomial such that the curve* $D = 0$ *goes through the singular points of* S_j*'s and* $\{D = 0\} \cap S_j \cap l_\infty = \emptyset$.

If $n > k - d - 1$, *where* $d = \deg D$, *then formula* (4) *holds but with rational* W *and* h_i,

$$W = W'/D, \qquad h_i = h_i'/D,$$

where W' *is a polynomial vector field and the* h_i' *are polynomials.*

If $n = k - d - 1$ *then* V *has a Darboux first integral.*

If $n < k - d - 1$ *then* $V \equiv 0$.

We see that for the integrability we do not need many particular algebraic integrals; it is enough that the sum of their degrees is large and that they are in generic position.

Remark 3. An analog of Theorem 3 holds even in the case when assumptions (i), (iii) and (iv) fail. W and h_i are rational with common denominator D depending on the resolutions of singular points and intersection points of S_j's. This denominator can be chosen in the form $D = \prod d_i(x, y)$, where $d_i = 0$ are the equations of exceptional divisors obtained in the resolutions. We do not formulate the precise result.

Notice also that the denominator D in Theorem 3.B is not fixed.

In Section 7 we generalize Theorem 3.A to Pfaff forms on $\mathbb{C}^r$ (Theorem 5). The proof of Theorem 3 follows the proof of Theorem 5 when we take into account Remark 4 below.

6. Proof of Theorem 1. Suppose that $S_0 = \{f = 0\}$ is an irreducible algebraic invariant curve of the vector field (3). Of course, the curves $\sigma^i(S_0) = \sigma^i S_0$ and $\varrho\sigma^i S_0$ are also invariant. (It may happen that $\sigma^i S_0 = S_0$ or $\varrho\sigma^i S_0 = \sigma^i S_0$.)

Let us estimate the total number of intersections of these curves with l_∞. If $a_0 \in S_0 \cap l_\infty$ then we get the points $\sigma^i a_0$ and $\varrho\sigma^i a_0$, $i = 1, \ldots, s^2+s$. If $a_0 = (x_0 : y_0 : 0)$, $x_0 y_0 \neq 0$, then $\sigma^i a_0 = \sigma^j a_0$ iff $\zeta^{(i-j)(s+1)} = 1$; otherwise $a_0 = (1:0:0)$ or $a_0 = (0:1:0)$. The first case happens when $(i-j)(s+1) = 0 \pmod{s^2+s+1}$, where $s+1$ and s^2+s+1 are relatively prime; so $i = j$ and the points $\sigma^i a_0$ are distinct. The remaining cases do not occur, otherwise the trajectory of a_0 would go through the first quadrant and could not be algebraic (see Lemma 2(e)). Next, it is possible that the set $\{\varrho\sigma^i a_0\}_{i=0,1,\ldots}$ intersects the set $\{\sigma^i a_0\}$. But then it includes a real point, which cannot occur for s even (see Lemma 2(e)).

Let $S_0, S_1, \ldots$ be all irreducible algebraic invariant curves for the vector field (3), $\bigcup \sigma^i S_0 \cup \bigcup \varrho\sigma^i S_0 \subset \bigcup S_i$. The sum of their degrees is bounded from below by the number of their intersections with the line at infinity.

Therefore $k = \sum \deg S_i \geq s^2+s+1$ for s odd and $k \geq 2s^2+2s+2$ for s even.

By Corollary 1 the assumptions of Theorem 3.B hold. We apply it with $D = a(1-xy^s) + b(x^s - y^{s+1})$ for suitable constants a and b (a combination of components of V). We get $d = \deg D = s+1$, $n = \deg V = s+1$ and hence $n < k-d-1$ for $s \geq 2$. Therefore $V \equiv 0$.

This contradiction completes the proof of Theorem 1. ∎

7. Generalization of Jouanolou's generalization of the Darboux theorem. In Chapter 2 of his book [5] Jouanolou investigated Pfaff equations with algebraic solutions. Let us formulate his main result (see [5, p. 115]).

Let $\omega = \sum_{i=1}^{r+1} P_i dx_i \in \Omega^1_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_{r+1}])$ be a projective (i.e. $\sum P_i x_i = 0$) 1-form of degree m (i.e. the P_i are homogeneous polynomials of degree m) on $P^r_{\mathbb{C}}$.

THEOREM 4 (Jouanolou). *If the algebraic Pfaff equation $\omega = 0$ only has a finite number p of irreducible algebraic solutions $f_i = 0$, $i = 1, \ldots, p$, then this number is bounded by*

$$q = \frac{1}{2}m(m-1)\binom{m+r-1}{r-2} + 2.$$

If $p \geq q-1$ then

$$\left(\sum \alpha_i \frac{df_i}{f_i}\right) \wedge \omega = 0 \tag{5}$$

for some constants α_i. If this equation has an infinity of algebraic solutions then it has a first integral which is a rational function.

The identity (5) means that $\prod f_i^{\alpha_i}$ is a first integral of the Pfaff equation $\omega = 0$.

Notice also that for $r = 2$ we have the situation from Theorem 2 (where $n = m-1$).

Before formulating our generalization of Theorem 4 we make some assumptions about algebraic solutions. Let $S_i = \{f_i = 0\}$, $i = 1, \ldots, p$, be algebraic hypersurfaces in $\mathbb{C}^r = \{(x_1 : \ldots : x_r : 1)\} \subset P^r_{\mathbb{C}}$ given by irreducible polynomials f_i. We assume that:

(i) the S_i have no singular points;

(ii) all intersections $S_{i_1} \cap \ldots \cap S_{i_t}$, $t \leq r$, are transversal (i.e. the forms $df_{i_1}, \ldots, df_{i_t}$ are independent along these intersections and hence there are no $(r+1)$-ple intersections);

(iii) the intersections at infinity $S_\infty \cap S_{i_1} \cap \ldots \cap S_{i_t}$ are transversal, $1 \leq t \leq r-1$. (Here $S_\infty = \{x_{r+1} = 0\}$ is the hyperplane at infinity.)

THEOREM 5. *Let $S_i = \{f_i = 0\}$, $i = 1, \ldots, p$, be irreducible algebraic hypersurfaces in $\mathbb{C}^r$ satisfying the conditions* (i), (ii) *and* (iii) *above. Then any algebraic 1-form ω of degree m with the property that the hypersurfaces S_i are solutions of the Pfaff equation $\omega = 0$ is of the form described below.*

(a) *If $m > \sum \deg f_i - 1 = k-1$ then*

$$\omega = \eta \prod f_i + \sum_i g_i \Big(\prod_{j \neq i} f_j\Big) df_i, \tag{6}$$

where η is a 1-form of degree $m-k$ and the g_i are polynomials of degree $m-k+1$.

(b) *If $m = k-1$ then $\omega = \sum \alpha_i (\prod_{j\neq i} f_j)\, df_i$, $\alpha_i \in \mathbb{C}$. In this case a Darboux first integral exists.*

(c) *If $m < k-1$ then $\omega \equiv 0$.*

Proof. First, we prove the representation (6) using induction with respect to p, the number of algebraic solutions.

Let $p = 1$ and the surface $S_1 = \{f_1 = 0\}$ be a solution of the equation $\omega = 0$. The forms $\omega|_{S_1} \in \Gamma(S_1, \Omega^1(\mathbb{C}^r))$ and $df_1|_{S_1} \neq 0$ have the same kernel and hence they are proportional. Therefore $\widetilde{g} = (\omega|_{S_1})/(df|_{S_1})$ is a regular function on S_1. The space of such functions forms the quotient ring $\mathcal{O}_{S_1} = \mathbb{C}[x_1, \ldots, x_r]/(f_1)$, where (f_1) is the ideal generated by f_1. From this the representation

$$\omega = f_1\eta_1 + g_1 df_1 \tag{7}$$

follows.

R e m a r k 4. The representation (7) with rational η and g_1 appears in [3] and [6]. When η and g_1 are polynomials then we say that ω has the *property of relative division* (see [6]).

Let us look what changes in (7) when S_1 has a non-degenerate (Morse) critical point z. Then df_1 vanishes at z and the coefficients of df_1 form a regular system of coordinates near z. Since the kernel of ω at z contains all lines which are limits of the lines from $\operatorname{Ker}\omega(z')$ as $z' \to z$ and these lines span the whole space we have $\operatorname{Ker}\omega(z) = \mathbb{C}^r$ and $\omega(z) = 0$. So, the function $\widetilde{g} = (\omega/df)|_{S_1}$ is regular on each local component of S_1 near z.

However, the function $\widetilde{g}$ can take different values at different local components and hence $\widetilde{g}$ may not be regular on the whole curve S_1.

In the proof of Theorem 3.B we allow the Morse singularities for the curve S_1 (and S_i). (There we worked with vector fields V but the case of 1-forms ω is the same.) We encounter the problem of different values of the function $\widetilde{g}$ on different local components of S_1 (or S_i). We solve this problem by multiplying ω (or V) by a function D, where $D = 0$ is some curve going through z. Then the function $\widetilde{g}$ is replaced with the function $\widetilde{\widetilde{g}} = D\widetilde{g}$. The latter is zero at singular points of S_1 and thus is regular on S_1. We obtain the representation (7) (and then also (6)) for the form $D\omega$. In the representation (7) for ω we get the form η and the function g_1 rational.

Let $p = 2$ and the hypersurfaces S_1 and $S_2 = \{f_2 = 0\}$ be algebraic solutions to $\omega = 0$. By assumption they intersect transversally. Take a point of such intersection, where we can assume that in some local analytic coordinates $f_1 = x$, $f_2 = y$. From the case $p = 1$ we have

$$\omega = x\eta + g\,dx = y\varrho + h\,dy,$$

where η does not contain dx and ϱ does not contain dy. Writing $\eta = \phi dy + \eta_1$, $g = g_1 y + g_2$, $\varrho = \psi dx + \varrho_1$, $h = h_1 x + h_2$, $\partial g_2/\partial y = \partial h_2/\partial x = 0$, we get $x\phi = h_1 x + h_2$, $g_1 y + g_2 = y\psi$, $x\eta_1 = y\varrho_1$. So, $h_2 = 0$, $g_2 = 0$, $\phi = h_1$, $\psi = g_1$, $\varrho_1 = x\xi$, $\eta_1 = y\xi$ and

$$\omega = xh_1 dy + yg_1 dx + xy\xi.$$

Therefore, the representation (6) holds near the intersection $S_1 \cap S_2$. To prove it globally we reason as follows. Let $\omega = f_1\eta + g df_1 = f_2\varrho + h df_2$. From the local analysis it follows that $g|_{S_2} = 0$ and $h|_{S_1} = 0$ (see $g_2 = 0, h_2 = 0$), or $g = f_2 g_1$, $h = f_1 h_1$ globally. So, $\omega = f_1\eta + f_2 g_1 df_1$ and restricting it to S_2 we see that η has the surface $f_2 = 0$ as a solution or that $\eta = \phi df_2 + \eta_2 f_2$. Therefore

$$\omega = \phi f_1 df_2 + g_1 f_2 df_1 + \eta_2 f_1 f_2.$$

If $\omega = 0$ has solutions $S_1, \ldots, S_{p+1}$, $p \geq 2$, then by the induction assumption applied to $S_1, \ldots, S_p$ and $S_2, \ldots, S_{p+1}$ we get

$$\begin{aligned}\omega &= \Big(\prod_{j=1}^{p} f_j\Big)\Big[\eta + \sum_{i=1}^{p} h_i \frac{df_i}{f_i}\Big] = \Big(\prod_{j=2}^{p+1} f_j\Big)\Big[\varrho + \sum_{i=2}^{p+1} g_i \frac{df_i}{f_i}\Big] \\ &= \Big(\prod_{j=2}^{p} f_j\Big)\Big[f_1\eta + h_1 df_1 + \sum_{i=1}^{p} f_1 h_i \frac{df_i}{f_i}\Big] \\ &= \Big(\prod_{j=2}^{p} f_j\Big)\Big[f_{p+1}\varrho + g_{p+1} df_{p+1} + \sum_{i=1}^{p} f_{p+1} g_i \frac{df_i}{f_i}\Big].\end{aligned} \tag{8}$$

We get two representations (6) for $S_2, \ldots, S_p$. The difference between them is described in the following.

LEMMA 3. *The general solution of the equation*

$$\Big(\prod f_i\Big)\Big[\eta + \sum g_i \frac{df_i}{f_i}\Big] = 0 \tag{9}$$

is $g_i = f_i g_i'$, $\eta = -\sum g_i' df_i$.

P r o o f. The restriction of the equation (9) to S_i gives $(g_i \prod_{j\neq i} f_j)|_{S_i} = 0$, where the equations $f_i = 0$ have only regular solutions on S_i. So $g_i = f_i g_i'$. Now the formula for the 1-form η is obvious. ∎

Therefore $f_1 h_i = f_{p+1} g_i \ (\operatorname{mod} f_i)$, or $h_i = k_i f_{p+1} + \phi_i f_i$, $g_i = k_i f_1 + \psi_i f_i$ (here the regularity of mutual intersections of S_j is used). The same holds for h_1 and g_{p+1}, $h_1 = k_1 f_{p+1} + \phi_1 f_1$, $g_{p+1} = k_{p+1} f_1 + \psi_{p+1} f_{p+1}$ (because of symmetry between f_j). We obtain the identity

$$f_1\eta' + k_1 f_{p+1} df_1 = f_{p+1}\varrho' + k_{p+1} f_1 df_{p+1},$$

where $\eta' = \eta + \sum \phi_i df_i$ and $\varrho' = \varrho + \sum \psi_i df_i$ (see (8)). Restricting this to S_{p+1} we find that S_{p+1} is a solution to η' and hence $\eta' = f_{p+1}\eta'' + \phi df_{p+1}$. From this the representation (6)

$$\omega = \Big(\prod_{j=2}^{p} f_j\Big)\Big[f_1 f_{p+1}\eta'' + k_1 f_{p+1} df_1 + \phi f_1 df_{p+1} + \sum_{i=2}^{p} k_i f_1 f_{p+1} \frac{df_i}{f_i}\Big]$$

for p replaced with $p+1$ follows.

It remains to estimate the degrees of g_i and η in (6). We have $\omega|_{f_i=0} = [g_i(\prod_{j\neq i} f_j) df_i]|_{f_i=0}$. Because S_i intersects S_∞ transversally we have $|df_i|(x) \sim |x|^{\deg f_i - 1}$ as $x \in S_i$ and $x \to S_\infty$ along a generic ray. Similarly, from the assumption (iii) we get $|f_j|(x) \sim |x|^{\deg f_j}$. Since $\omega|_{S_i}$ is of order m we see that $g_i|_{S_i}$ is of order $m - \sum \deg f_j + 1$. The latter means that g_i can be written as $g_i = f_i g_i' + g_i''$, $\deg g_i'' \le m - k + 1$.

Indeed, as the highest order homogeneous part $\widehat{f}_i$ of f_i is irreducible (for $p > 2$) or has only simple zeroes (on $S_\infty = P^1_{\mathbb{C}}$ for $p = 2$) the highest order homogeneous part $\widehat{g}_i$ of g_i vanishes on $\widehat{f}_i = 0$. So, $\widehat{f}_i$ divides $\widehat{g}_i$, $\widehat{g}_i = \kappa \widehat{f}_i$ and $g_i = \kappa f_i + g_{i1}$ with g_{i1} of smaller degree than $\deg g_i$. Then we apply the same to g_{i1} etc.

Hence,

$$\omega = \Big(\prod f_j\Big)\Big[\eta + \sum g_i' df_i\Big] + \omega', \quad \omega' = \sum g_i''\Big(\prod_{j\ne i} f_j\Big) df_i,$$

ω' is of degree $\le m$ and the form $\eta + \sum g_i' df_i$ has degree $m - k$.

If $m = k - 1$ then $\deg g_i = 0$, $g_i = \alpha_i =$const. Because $\deg \prod f_j \ge m + 1$ we have $\eta = 0$. Therefore $\omega = M^{-1} dH$, where $H = \prod f_j^{\alpha_j}$ is a Darboux first integral and $M = \prod f_j^{\alpha_j - 1}$ is an integrating factor.

If $m < k - 1$ then we find $g_i = 0$ and $\eta = 0$. Thus $\omega = 0$. ■

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Averages of unitary representations and weak mixing of random walks

by

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Abstract. Let $\mathcal{S}$ be a locally compact (σ-compact) group or semigroup, and let $T(t)$ be a continuous representation of $\mathcal{S}$ by contractions in a Banach space X. For a regular probability μ on $\mathcal{S}$, we study the convergence of the powers of the μ-average $Ux = \int T(t)x\,d\mu(t)$. Our main results for random walks on a group G are:

(i) The following are equivalent for an adapted regular probability on G: μ is strictly aperiodic; U^n converges weakly for every continuous unitary representation of G; U is weakly mixing for any ergodic group action in a probability space.

(ii) If μ is ergodic on G metrizable, and U^n converges strongly for every unitary representation, then the random walk is weakly mixing: $n^{-1}\sum_{k=1}^n |\langle \mu^k * f, g\rangle| \to 0$ for $g \in L_\infty(G)$ and $f \in L_1(G)$ with $\int f\,d\lambda = 0$.

(iii) Let G be metrizable, and assume that it is nilpotent, or that it has equivalent left and right uniform structures. Then μ is ergodic and strictly aperiodic if and only if the random walk is weakly mixing.

(iv) Weak mixing is characterized by the asymptotic behaviour of μ^n on $UCB_l(G)$.

1. Introduction. Let $\mathcal{S}$ be a locally compact (σ-compact) semigroup (always assumed Hausdorff). For a regular probability μ on $\mathcal{S}$, the convolution operator $\mu * f(t) = \int f(ts)\,d\mu(s)$ is a Markov operator on $C(\mathcal{S})$, which is the average of the translation operators $\delta_s * f(t) = f(ts)$. When $\mathcal{S} = G$ is a locally compact group with right Haar measure λ, the *regular* representation $s \to \delta_s$ is continuous in $L_p(G,\lambda)$, $1 \le p < \infty$, $C_0(G)$ and $UCB_l(G)$.

Let X be a Banach space, and let $T : \mathcal{S} \to B(X)$ be a bounded operator representation of $\mathcal{S}$ (i.e., $T(st) = T(s)T(t)$, and $\sup_s \|T(s)\| < \infty$). The representation is called *continuous* if $t \to T(t)x$ is continuous for every $x \in X$, and *weakly continuous* if $f(t) = \langle x^*, T(t)x\rangle$ is continuous for $x \in X^*$ and $x \in X$. For groups, this implies (strong) continuity [HRo, p. 340]. For a regular probability μ on $\mathcal{S}$, the μ-average $U_\mu x = \int T(t)x\,d\mu$ is defined in the

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