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Boundedness of certain oscillatory singular integrals

by

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Abstract. We prove the L^p and H^1 boundedness of oscillatory singular integral operators defined by $Tf = \text{p.v.}\,\Omega * f$, where $\Omega(x) = e^{i\Phi(x)}K(x)$, $K(x)$ is a Calderón–Zygmund kernel, and Φ satisfies certain growth conditions.

1. Introduction. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\Phi(x)$ be a real-valued function. Consider the oscillatory singular integral operator T defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{i\Phi(x-y)} K(x-y) f(y)\, dy, \tag{1}$$

where $K(x)$ is a Calderón–Zygmund kernel. Operators of this type, with various kinds of phase functions, have been studied by many authors. For example, operators with polynomial phases were considered in [2], [9], [12], [14], [15], [20]. Among the results obtained in the papers cited above are the L^p and weak (1,1) boundedness of such operators, as well as their boundedness on Hardy spaces. Results concerning oscillatory singular integrals with smooth phase functions can be found in [11], [13].

In this paper, we are interested in operators with phase functions of another type. A typical example of the operators under our investigation is the following hypersingular integral operator:

$$f \to \text{p.v.} \int_{\mathbb{R}} e^{i\frac{1}{|x-y|}} \frac{1}{x-y} f(y)\, dy. \tag{2}$$

Fefferman [7] and Fefferman–Stein [8] showed that, among other things, this operator has weak-type (1,1) and is bounded on L^p and H^1. This particular operator belongs to a class of operators which are given by (1) with $\Phi(x) = |x|^a$. These operators were studied extensively and their boundedness properties were established in Sjölin [17], Jurkat–Sampson [10], Chanillo *et al.* [3], in addition to [7] and [8] mentioned earlier (see also [1], [16], [18], [21]).

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We shall consider operators with phase functions Φ satisfying (3) and (4), with $\Phi(x) = |x|^a$ as our model case. The main results we obtain here are the boundedness of such operators on L^p $(1 < p < \infty)$ and H^1. A similar problem in the context of the Besov space $\dot{B}_0^{1,0}(\mathbb{R}^n)$ was studied earlier by one of the authors ([6]).

We shall use the following notation. For $\alpha = (\alpha_1, \ldots, \alpha_n)$ we write

$$|\alpha| = \alpha_1 + \ldots + \alpha_n, \qquad D^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}.$$

Sometimes we use D_z^α to emphasize the differential operator D^α acting on the z-variable.

Let $\Phi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a real-valued function which satisfies

$$(3) \qquad |D^\alpha \Phi(x)| \le C|x|^{a-|\alpha|} \quad \text{for } |\alpha| \le 3,$$

$$(4) \qquad \sum_{|\alpha|=2} |D^\alpha \Phi(x)| \ge C'|x|^{a-2},$$

where a is some fixed real number, and C and C' are constants independent of $x \in \mathbb{R}^n \setminus \{0\}$. We now state our results.

THEOREM 1. *Let T be given as in* (1). *Suppose Φ satisfies* (3) *and* (4) *with some $a \neq 0$. Then, for $1 < p < \infty$, there is a $C_p > 0$ such that*

$$\|Tf\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)}.$$

THEOREM 2. *Let T be given as in* (1). *Suppose Φ satisfies* (3) *and* (4) *with some $a \neq 0, 1$. Then the operator T extends to a bounded operator on the Hardy space $H^1(\mathbb{R}^n)$.*

When $\Phi(x) = |x|^a$, by simple calculation we find that

$$\Big(\sum_{|\alpha|=2} |D^\alpha \Phi(x)|^2\Big)^{1/2} = |a|[(a-1)^2 + (n-1)]^{1/2}|x|^{a-2}.$$

Therefore, Φ satisfies (3) and (4) unless $a = 0$ or $a = n = 1$. When $a = 0$, T is reduced to the usual Calderón–Zygmund singular integral operator, whose boundedness is well known. For $a = n = 1$, T is known to be unbounded on both L^p and H^1. Other examples of Φ satisfying (3) and (4) are easily available, such as $\Phi(x) = x_1^2 \cos(x_1/(4|x|))$, $x \in \mathbb{R}^2$.

In light of Theorem 1, a question that arises naturally is whether L^p boundedness holds for operators whose phase functions satisfy (3) and (4) with $a = 0$. The answer to this question is, in general, no. An example will be given in Section 3. It was known previously that H^1 boundedness does not hold when $\Phi(x) = |x|$, $x \in \mathbb{R}^n$ ([17]), which explains why the condition $a \neq 1$ is imposed in Theorem 2.

This paper is organized as follows. In the second section we recall some definitions and state a few lemmas. Theorems 1 and 2 are proved in Sections 3 and 4, respectively. In this paper, the letter C will denote (possibly different) constants that are independent of the essential variables.

2. Notations and lemmas. First we recall the definition of the Hardy space $H^1(\mathbb{R}^n)$. Let Ψ be a Schwartz testing function which satisfies $\int_{\mathbb{R}^n} \Psi(x)\,dx \neq 0$. For each $f \in \mathcal{S}'(\mathbb{R}^n)$, we set

$$f^*(x) = \sup_{t>0} |f * \Psi_t(x)|,$$

where $\Psi_t(x) = t^{-n}\Psi(x/t)$.

DEFINITION 1 ([8]). A locally integrable function f is in the space $H^1(\mathbb{R}^n)$ if and only if

$$\|f^*\|_1 = \int_{\mathbb{R}^n} f^*(x)\,dx < \infty,$$

and we define $\|f\|_{H^1} = \|f^*\|_1$.

In order to prove Theorem 2, we need the atomic decomposition of H^1 functions. Let us recall the definition of atoms ([4]).

DEFINITION 2. A real-valued function $a(x)$ is a $(1, \infty)$ *atom* if

(1) $a(x)$ is supported in a cube $Q \subset \mathbb{R}^n$,
(2) $\int_Q a(x)\,dx = 0$,
(3) $\|a\|_\infty \le |Q|^{-1}$, where $|Q|$ is the volume of Q.

The following result can be found in [4] or [5].

LEMMA 1. *For each $f \in H^1(\mathbb{R}^n)$, there exist $(1, \infty)$ atoms $\{a_k\}$ and coefficients $\{c_k\}$ such that*

$$(5) \qquad f = \sum_k c_k a_k,$$

and $\sum_k |c_k| \simeq \|f\|_{H^1}$, where the sum in (5) *is both in the sense of distributions and in the H^1 norm.*

DEFINITION 3. $K \in C^1(\mathbb{R}^n \setminus \{0\})$ is said to be a *Calderón–Zygmund kernel* if there is an $A > 0$ such that

$$(6) \qquad |K(x)| \le A|x|^{-n}, \qquad |\nabla K(x)| \le A|x|^{-n-1},$$

$$(7) \qquad \int_{a<|x|<b} K(x)\,dx = 0 \quad \text{for } 0 < a < b.$$

LEMMA 2. *Suppose that $\psi \in C_0^1(\mathbb{R}^n)$, ϕ is real-valued and for some $k \ge 1$,*

$$(8) \qquad \sum_{|\alpha|=k} |D^\alpha \phi(x)| \ge 1$$

on the support of ψ. Then

(9) $$\Big|\int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x)\,dx\Big| \le c_k(\phi)\lambda^{-1/k}(\|\psi\|_\infty + \|\nabla\psi\|_1),$$

and the constant $c_k(\phi)$ is independent of λ and ψ and remains bounded as long as the C^{k+1} norm of ϕ remains bounded.

Lemma 2 is a slightly stronger version of Proposition 5 in [20]. The proof given by Stein in [20] can be used here with little modification. Our next lemma is a variation of Lemma 3.2 of [13] whose proof can be given by using essentially the same argument as presented in [13]. We omit the details.

LEMMA 3. *Let $\Psi \in C^\infty(\mathbb{R}^n)$, $\varphi \in C_0^\infty$. Let k be a positive integer. Assume that*

(10) $$\sum_{|\alpha|=k} |D^\alpha\Psi(x)|^2 \le B \le M$$

for $x \in \operatorname{supp}(\varphi)$. Let

$$V = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \operatorname{supp}(\varphi)) \le B\}.$$

Assume also that $|D^\beta\Psi(x)| \le A$ for all $|\beta| = k+1$ and $x \in V$. Then there exists a constant C which depends only on A, M and φ such that

$$\Big|\int_{\mathbb{R}^n} e^{i\lambda\Psi(x)}\varphi(x)\,dx\Big| \le C\lambda^{-\varepsilon/k}\int_V \Big(\sum_{|\alpha|=k}|D^\alpha\Psi(x)|^2\Big)^{-\varepsilon(1+1/(2k))}dx$$

for $\varepsilon \in [0,1]$.

LEMMA 4. *Let $F : (-1/2, 1/2)^{n+1} \to \mathbb{R}$ be C^2 and satisfy*

(i) $F(0,0) = 0$,
(ii) $|\partial F(0,0)/\partial t| \ge m > 0$,
(iii) $|\partial^{\alpha+\beta}F(x,t)/\partial x^\alpha \partial t^\beta| \le M$,

for $(x,t) \in (-1/2,1/2)^{n+1}$ and $1 \le |\alpha| + \beta \le 2$, where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Then there are constants h, d, with $0 < d < h \le 1$, and a function $g : (-d,d)^n \to (-h,h)$ such that for each $x \in (-d,d)^n$ we have $F(x, g(x)) = 0$ and $|F(x,t)| \ge (m/2)|t - g(x)|$ for $(x,t) \in (-d,d)^n \times (-h,h)$, where the constants d and h depend on M and m only.

Lemma 4 is a quantitative version of the implicit function theorem and can be proved by using the usual proof of the implicit function theorem (see, for instance, [19]).

3. Boundedness on $L^p(\mathbb{R}^n)$. We now prove Theorem 1. Let ψ be a non-negative C^∞ radial function which satisfies

$$\operatorname{supp}(\psi) \subset \{1/2 \le |x| \le 2\} \quad \text{and} \quad \sum_{j=-\infty}^{\infty} \psi(2^{-j}x) \equiv 1 \quad \text{for } x \ne 0.$$

First we prove the case $a > 0$. Let $\psi_j(x) = \psi(2^{-j}x)$ and $\eta(x) = 1 - \sum_{j=1}^\infty \psi_j(x)$. For $j = 0, 1, \ldots$, define T_j by $T_j f = f * \Omega_j$, where

(11) $$\Omega_0(x) = e^{i\Phi(x)}K(x)\eta(x),$$

(12) $$\Omega_j(x) = e^{i\Phi(x)}K(x)\psi_j(x) \quad \text{for } j = 1, 2, \ldots$$

Clearly we have $Tf = T_0 f + \sum_{j=1}^\infty T_j f$. Since $\eta(x) \equiv 1$ for $|x| < 1$, we may further decompose Ω_0 as

$$\Omega_0(x) = K(x)\chi_{\{|x|<1\}}(x) + K(x)(e^{i\Phi(x)} - 1)\chi_{\{|x|<1\}}(x) + e^{i\Phi(x)}K(x)\eta(x)\chi_{\{|x|\ge 1\}}(x).$$

By (3) and (6) we find that

$$\int_{|x|\le 1} |K(x)||e^{i\Phi(x)} - 1|\,dx \le C\int_{|x|\le 1} |x|^{-n+a}\,dx \le C,$$

and

$$\int_{|x|\ge 1} |e^{i\Phi(x)}K(x)\eta(x)|\,dx \le C\int_{1\le|x|\le 4} |x|^{-n}\,dx \le C.$$

Thus we have proved that $\|T_0 f\|_{L^p} \le C_p\|f\|_{L^p}$.

For $j \ge 1$, let $\widehat{T_j f}(\xi) = m_j(\xi)\widehat{f}(\xi)$, where

$$m_j(\xi) = \int_{\mathbb{R}^n} e^{i[\Phi(x) - \xi\cdot x]}K(x)\psi(2^{-j}x)\,dx = \int_{\mathbb{R}^n} e^{i[\Phi(2^j x) - 2^j\xi\cdot x]}2^{jn}K(2^j x)\psi(x)\,dx.$$

Let $\phi(x) = 2^{-ja}[\Phi(2^j x) + 2^j\xi\cdot x]$. If $|\alpha| = 2$, then $D^\alpha\phi(x) = 2^{j(2-a)}(D^\alpha\Phi)(2^j x)$. Thus by (4), for $1/2 < |x| \le 2$ we have

$$\sum_{|\alpha|=2} |D^\alpha\phi(x)| = 2^{j(2-a)}\sum_{|\alpha|=2} |(D^\alpha\Phi)(2^j x)| \ge C2^{j(2-a)}|2^j x|^{a-2} \ge C.$$

For $|x| \le 2$ and $|\alpha| = 3$, by (3) we have

$$|D^\alpha\phi(x)| \le C2^{j(3-a)}|(D^\alpha\Phi)(2^j x)| \le C.$$

Invoking Lemma 2, we obtain $\|m_j\|_\infty \le C2^{-ja/2}$, which implies that $\|T_j\|_{2,2} \le C2^{-ja/2}$. By the uniform boundedness of $\|T_j\|_{1,1}$ and interpolation we have $\|T_j\|_{p,p} \le C2^{-ja/p'}$, for $j = 1, 2, \ldots$, $1 < p \le 2$ and $1/p + 1/p' = 1$. Thus we have proved that for $1 < p \le 2$,

$$\|T\|_{p,p} \le \sum_{j\ge 0} \|T_j\|_{p,p} \le C.$$

Finally, by using duality, we obtain the boundedness for T on L^p when $p > 2$.

The proof for the case $a < 0$ is similar. We let $\widetilde{\eta}(x) = 1 - \sum_{j=-\infty}^{-1} \psi_j(x)$ and write

$$e^{i\Phi(x)}K(x) = \widetilde{\Omega}_0(x) + \sum_{j=-\infty}^{-1} \Omega_j(x),$$

where the Ω_j are given by (12) with $j \le -1$ and $\widetilde{\Omega}_0(x) = e^{i\Phi(x)}K(x)\widetilde{\eta}(x)$. We have

$$\|f * \widetilde{\Omega}_0\|_p \le C\|f\|_p, \quad \|f * \Omega_j\|_p \le C2^{-ja}\|f\|_p, \quad j \le -1.$$

Since $a < 0$, we obtain

$$\|Tf\|_p \le C\Big(1 + \sum_{j=-\infty}^{-1} 2^{-ja}\Big)\|f\|_p \le C\|f\|_p.$$

The proof of Theorem 1 is now complete.

By inspecting the proof of Theorem 1, it is clear that the following theorem holds.

THEOREM 3. *Let Φ be a real-valued function satisfying*

$$\sum_{|\alpha|=k} |D^\alpha \Phi(x)| \ge C|x|^{a-k},$$

and

$$|D^\alpha \Phi(x)| \le C|x|^{a-|\alpha|} \quad \textit{for } |\alpha| = 0 \textit{ and } k+1,$$

for some fixed $k \ge 2$ and $a \ne 0$. Let K be a Calderón–Zygmund kernel and $\Omega(x) = e^{i\Phi(x)}K(x)$. Then the operator $Tf = \text{p.v. } \Omega * f$ is bounded on L^p for $1 < p < \infty$.

A special case which is worth noting is when $K(x)$ is an odd and homogeneous Calderón–Zygmund kernel and $\Phi(x)$ is even and homogeneous of degree a with $a \ne 1$. An application of the method of rotation ([22]) would reduce the operator T to one with phase function $|t|^a$ ($t \in \mathbb{R}$) and yield the L^p ($1 < p < \infty$) boundedness of T. In this case, there is no need to impose any smoothness conditions on $\Phi|_{S^{n-1}}$. However, such a method is not applicable to problems concerning endpoint estimates.

We now present an example for the case when $a = 0$. Let $x \in \mathbb{R}^2$ and

$$\Phi(x) = x_2/|x| \in C^\infty(\mathbb{R}^2\backslash\{0\}).$$

Then we have

$$|D^\alpha \Phi(x)| \le C|x|^{-|\alpha|} \quad \text{for } |\alpha| \le 3, \quad \sum_{|\alpha|=2} |D^\alpha(x)|^2 \ge |x|^{-2}.$$

Let $K(x) = x_2/|x|^3$. Then K is C^∞ away from the origin, homogeneous of degree -2 and satisfies

$$\int_{x'\in S^1} K(x')\,d\sigma(x') = 0.$$

Let $\Omega(x) = e^{i\Phi(x)}K(x)$, $\Omega_{\varepsilon,N}(x) = \Omega(x)\chi_{\{\varepsilon\le|x|\le N\}}(x)$. We find that

(i) $\Omega \in C^\infty(\mathbb{R}^2\backslash\{0\})$,

(ii) Ω is homogeneous of degree -2.

Since

$$\int_{S^1} \operatorname{Im} \Omega(x')\,d\sigma(x') = 2\int_0^\pi \sin(\sin\theta)\sin\theta\,d\theta > 0,$$

we see that $\lim_{\varepsilon\to 0, N\to\infty} \Omega_{\varepsilon,N}$ does not exist (in the sense of distributions). This example shows that Theorem 1 cannot hold when $a = 0$.

4. Boundedness on $H^1(\mathbb{R}^n)$. By a standard procedure which uses the atomic decomposition (Lemma 1) and the Riesz transform characterization of H^1, the proof of Theorem 2 can be reduced to the proof of the following proposition (see, for instance, [13]).

PROPOSITION 1. *Let T be given as in Theorem 2. If $b(x)$ is a $(1,\infty)$ atom then $\|Tb\|_1 \le C$.*

Proof. Let $b(x)$ be a $(1,\infty)$ atom which is supported in a cube $Q \subset \mathbb{R}^n$. Let x_0 be the center of Q, and δ be its sidelength. Set $a(x) = \delta^n b(x_0 + \delta x)$. It is easy to see that $a(x)$ is a $(1,\infty)$ atom with support contained in $Q_0 = (-1/2, 1/2)^n$ and satisfies $\|a\|_\infty \le 1$, $\int_{Q_0} a(x)\,dx = 0$. Let $K_\delta(x) = \delta^n K(\delta x)$ and

$$T_\delta f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{i\Phi(\delta x - \delta y)} K_\delta(x-y) f(y)\,dy.$$

Hence we have $(Tb)(x_0 + \delta x) = \delta^{-n}(T_\delta a)(x)$, which leads to $\|Tb\|_{L^1} = \|T_\delta a\|_{L^1}$. To prove our proposition, it suffices to show that

$$\|T_\delta a\|_1 \le C, \tag{13}$$

for some constant C which is independent of $\delta > 0$. We observe that the function K_δ satisfies (6) and (7) with a constant A that is independent of δ. For the sake of simplicity, we shall denote K_δ by K.

First we consider the case $a > 1$. By Theorem 1 we have $\|T_\delta\|_{L^2,L^2} = \|T\|_{L^2,L^2} \le C$. Thus

$$\int_{|x|\le 4} |(T_\delta a)(x)|\,dx \le C\|T_\delta a\|_2 \le C.$$

Choose $b = \max\{4, \delta^{-a/(a-1)}\}$, hence $b^{a-1}\delta^a \ge 1$. To estimate the integral $\int_{4\le|x|\le b} |T_\delta a(x)|\,dx$, we may assume $b > 4$, hence $b^{a-1}\delta^a = 1$. We have

$$\int_{4\le|x|\le b} |(T_\delta a)(x)|\,dx \le \int_{4\le|x|\le b}\int_{Q_0} |e^{i\phi(\delta x-\delta y)} - e^{i\phi(\delta x)}||K(x-y)a(y)|\,dy\,dx$$
$$+ \int_{4\le|x|\le b} \Big|\int_{Q_0} K(x-y)a(y)\,dy\Big|\,dx = I_1 + I_2.$$

Because the Calderón–Zygmund operator is bounded from H^1 to L^1, we have

$$I_2 \le C\|a\|_{H^1} \le C.$$

By the mean value theorem and (3), we also have

$$I_1 \le C \int_{|x|\le b} \delta|\delta x|^{a-1}|x|^{-n}dx \le C.$$

Next we estimate $\int_{|x|\ge b} |(T_\delta a)(x)|\,dx$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\psi(x) = 1$ if $1 \le |x| \le 2$ and $\psi(x) = 0$ if $|x| < 3/4$ or $|x| > 3$. Let $Q \subset Q_0$ be a cube with $\operatorname{diam}(Q) = d > 0$, which will be chosen later. Define S_j^Q by

$$S_j^Q f(x) = \psi(2^{-j}x) \int_Q e^{i\Phi(\delta x-\delta y)} f(y)\,dy.$$

We shall first obtain an estimate for the L^2-norm of each operator S_j. Let

$$L_j(x,y) = \chi_Q(x)\chi_Q(y) \int_{\mathbb{R}^n} e^{i[\Phi(\delta z-\delta y)-\Phi(\delta z-\delta x)]}\psi^2(2^{-j}z)\,dz$$
$$= 2^{jn}\chi_Q(x)\chi_Q(y) \int_{\mathbb{R}^n} e^{i[\Phi(2^j\delta z-\delta y)-\Phi(2^j\delta z-\delta)]}\psi^2(z)\,dz.$$

Then

$$(S_j^Q)^* S_j^Q f(x) = \int_{\mathbb{R}^n} L_j(x,y)f(y)\,dy.$$

We fix j such that $2^j \ge b$. Then $j \ge 2$. Define

$$H_\delta(x,y,z) = (2^j\delta)^{1-a}\delta^{-1}[\Phi(2^j\delta z-\delta y) - \Phi(2^j\delta z - \delta x)].$$

By (3) and (4), we find

(14) $$|\nabla_z H_\delta(x,y,z)| \le Cd,$$

and

(15) $$\sum_{|\alpha|=2} |D_z^\alpha H_\delta(x,y,z)| \le C,$$

for $x,y \in Q$, $z \in \operatorname{supp}(\psi)$, where C is a constant independent of δ and j. Invoking Lemma 3, we get (by letting $k = 1$, and d be small if necessary)

$$|L_j(x,y)| \le 2^{jn}\chi_Q(x)\chi_Q(y)[(2^j\delta)^{a-1}\delta]^{-\varepsilon} \int_{1/2\le|z|\le 4} |\nabla_z H_\delta(x,y,z)|^{-3\varepsilon}\,dz.$$

For fixed x,

$$\int_{\mathbb{R}^n} |L_j(x,y)|\,dy$$
$$\le 2^{jn}[(2^j\delta)^{a-1}\delta]^{-\varepsilon}\chi_Q(x) \int_{1/2\le|z|\le 4}\int_Q |\nabla_z H_\delta(x,y,z)|^{-3\varepsilon}\,dy\,dz.$$

For fixed $x \in Q$ and $z \in \{1/2 \le |z| \le 4\}$, by (4) there are $\mu,\nu \in \{1,\dots,n\}$ such that

$$\left|\frac{\partial^2\Phi}{\partial x_\mu\partial x_\nu}(2^j\delta z - \delta x)\right| \ge C|2^j\delta|^{a-2}.$$

Let $\varepsilon = 1/6$, $h(y) = (2^j\delta)^{2-a}\delta^{-1}[D_\mu\Phi(2^j\delta z-\delta y) - D_\mu\Phi(2^j\delta z-\delta x)]$. Then clearly $h(x) = 0$, $|(\partial h/\partial y_\nu)(x)| \ge C_0$, and $|D^\alpha h(y)| \le C_1$ for $1 \le |\alpha| \le 2$, $y \in Q_0$, where C_0 and C_1 are constants independent of δ. By Lemma 4, if we choose $d = \operatorname{diam}(Q)$ to be sufficiently small, then for $y \in x + (2d)Q_0 = Q_x$,

$$|h(y)| \ge (C_0/2)|y_\nu - g(y')|,$$

where $y' = (y_1,\dots,y_{\nu-1},y_{\nu+1},\dots,y_n)$, and $g(y')$ may depend on x and z. Therefore we have

$$\int_Q |\nabla_z H_\delta(x,y,z)|^{-3\varepsilon}\,dy \le \int_Q \left|\frac{\partial H_\delta}{\partial z_\mu}(x,y,z)\right|^{-1/2} dy \le \int_{Q_x} |h(y)|^{-1/2}\,dy$$
$$\le C \int_{|y'-x'|\le 2d}\Big(\int_{|y_\nu-x_\nu|\le 2d} |y_\nu - g(y')|^{-1/2}\,dy_\nu\Big)\,dy' \le C.$$

This implies that

(16) $$\sup_x \int_{\mathbb{R}^n} |L_j(x,y)|\,dy \le C2^{jn}[(2^j\delta)^{a-1}\delta]^{-1/6}.$$

In a similar manner, we have

(17) $$\sup_y \int_{\mathbb{R}^n} |L_j(x,y)|\,dx \le C2^{jn}[(2^j\delta)^{a-1}\delta]^{-1/6}.$$

From (16) and (17), we now obtain

$$\|S_j^Q\|_{L^2,L^2} \le C2^{jn/2}[(2^j\delta)^{a-1}\delta]^{-1/12}. \tag{18}$$

Write $Q_0 = \bigcup_{m=1}^N Q_m$, where $Q_m \subset Q_0$ $(1 \le m \le N)$ are disjoint cubes with $\operatorname{diam}(Q_m) < d$. By (18), we have

$$\begin{aligned}\int_{|x|>b} |T_\delta a(x)|\,dx &\le \int_{|x|>b}\int_Q |K(x-y)-K(x)||a(y)|\,dy\,dx \\ &\quad + \sum_{2^j\ge b}\int_{2^j\le|x|\le 2^{j+1}} |x|^{-n}\Big|\int_{Q_0} e^{i\Phi(\delta x-\delta y)}a(y)\,dy\Big|\,dx \\ &\le C + \sum_{2^j\ge b}\int_{2^j\le|x|\le 2^{j+1}} |x|^{-n}\Big|\sum_{m=1}^N (S_j^{Q_m}a)(x)\Big|\,dx \\ &\le C + \sum_{m=1}^N\sum_{2^j\ge b} 2^{-jn/2}\|S_j^{Q_m}a\|_{L^2(\mathbb{R}^n)} \\ &\le C + C\sum_{m=1}^N\sum_{2^j\ge b}[(2^j\delta)^{a-1}\delta]^{-1/12} \\ &\le C(1+[\delta^a b^{a-1}]^{-1/12}) \le C.\end{aligned}$$

This finishes the proof of Proposition 1 in the case when $a > 1$.

Finally, we prove the case $a < 1$, $a \ne 0$. First we assume that $b = \delta^{a/(1-a)} > 4$. Since $a < 1$, we have

$$\begin{aligned}\int_{|x|>b} |T_\delta a(x)|\,dx &\le \int_{|x|>b}\Big|\int_{\mathbb{R}^n} K(x-y)a(y)\,dy\Big|\,dx \\ &\quad + \int_{|x|>b}\int_{Q_0} |e^{i\Phi(\delta x-\delta y)} - e^{i\Phi(\delta x)}||K(x-y)a(y)|\,dy\,dx \\ &\le C + \int_{|x|>b} \delta|\delta x|^{a-1}|x|^{-n}\,dx \le C.\end{aligned}$$

By mimicking the proof for the case $a > 1$, we find

$$\begin{aligned}\int_{4\le|x|\le b} |T_\delta a(x)|\,dx &\le C + C\sum_{4\le 2^j\le b} 2^{j(1-a)/12}\delta^{-a/12} \\ &\le C + Cb^{(1-a)/12}\delta^{-a/12} \le C,\end{aligned}$$

since $1-a > 0$. If $\delta^{a/(1-a)} \le 4$, then the estimate is even simpler:

$$\begin{aligned}\int_{|x|\ge 4} |T_\delta a(x)|\,dx &\le C + C\int_{|x|\ge 4} \delta|\delta x|^{a-1}|x|^{-n}\,dx \\ &\le C(1+[4\delta^{a/(a-1)}]^{a-1}) \le C.\end{aligned}$$

The proof of Theorem 2 is thus complete.

We have the following extension of Theorem 2.

THEOREM 4. *Let Φ be a real-valued function satisfying*

$$\sum_{|\alpha|=k} |D^\alpha\Phi(x)| \ge C|x|^{a-k},$$

and

$$|D^\alpha\Phi(x)| \le C|x|^{a-|\alpha|} \quad \text{for } |\alpha| = 0, 1 \text{ and } k+1,$$

for some fixed $k \ge 2$, $a \ne 0, 1$. Let K be a Calderón–Zygmund kernel and $\Omega(x) = e^{i\Phi(x)}K(x)$. Then the operator $Tf = \text{p.v. } \Omega * f$ extends to a bounded operator on $H^1(\mathbb{R}^n)$.

Remark. Theorems 1–4 remain valid if L^p and H^1 spaces are replaced by weighted L^p and H^1 spaces with weights in A_p $(p > 1)$ and A_1, respectively. We omit the details.

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On algebraic solutions of algebraic Pfaff equations

by

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Abstract. We give a new proof of Jouanolou's theorem about non-existence of algebraic solutions to the system $\dot{x} = z^s$, $\dot{y} = x^s$, $\dot{z} = y^s$. We also present some generalizations of the results of Darboux and Jouanolou about algebraic Pfaff forms with algebraic solutions.

Introduction. In [5] Jouanolou studied the Pfaff equations with polynomial coefficients. One of his main results states that the subset Z_m of the set V_m of Pfaff equations of degree $m > 2$ on $P^2_{\mathbb{C}}$ consisting of equations without algebraic solutions is dense in V_m in the usual topology (see [5, p. 158]). (Lins-Neto in [6] proved that Z_m is also open.) To show this he needs an example of a Pfaff equation without algebraic solutions and he chooses

$$(1)\qquad (x^{m-1}z - y^m)dx + (y^{m-1}x - z^m)dy + (z^{m-1}y - x^m)dz = 0,$$

where $m > 2$ is an integer. The whole Chapter 4 of [5] is devoted to the proof of non-algebraicity of the solutions of (1).

Below we present a new proof of this result based on the author's original generalization (Theorem 3 below) of a classical theorem of Darboux (Theorem 2 below) proved in the preprint [8].

Another generalization of the Darboux theorem, to higher dimensions, was given in [5] by Jouanolou (Theorem 4 below). In Theorem 5 below we present our generalization of Theorem 4. Our approach is different from the one developed by Darboux and Jouanolou. We are more interested in cases of few algebraic solutions of a Pfaff equation but in generic position whereas they consider situations with many but arbitrary solutions.

We treat Theorems 3 and 5 as the main results of this paper because the methods developed in them seem to be useful in applications (e.g. in the center-focus problem or in the problem of integrability).

The origin of the present work comes from the question of J.-M. Strelcyn at the seminar on dynamical systems in Warsaw in 1992. He stated the

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