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Relatively perfect σ-algebras for flows

by

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Abstract. We show that for every ergodic flow, given any factor σ-algebra $\mathcal{F}$, there exists a σ-algebra which is relatively perfect with respect to $\mathcal{F}$. Using this result and Ornstein's isomorphism theorem for flows, we give a functorial definition of the entropy of flows.

Introduction. Perfect σ-algebras play an important role in ergodic theory and statistical mechanics, especially in the spectral theory of dynamical systems with discrete time (measure preserving $\mathbb{Z}^d$-actions). The existence of these σ-algebras in the case $d \ge 2$ has been proved by the use of their relative versions (for $\mathbb{Z}^{d-1}$-actions), the so-called relatively perfect σ-algebras ([K1]). In [K2] the relatively perfect σ-algebras have been used to give a functorial definition of entropy of a $\mathbb{Z}^d$-action.

Blanchard in [B1] and Gurevič in [G2] have shown that for every ergodic flow there exists a perfect σ-algebra. The main purpose of the present paper is to prove a relative version of this result (Theorem B). The motivations of this theorem are, on the one hand, expected applications of relatively perfect σ-algebras to the investigation of the spectral structure of multidimensional flows and, on the other hand, an application to an axiomatic, i.e. functorial definition of entropy of one-dimensional flows. Such definitions have been given for $\mathbb{Z}^d$-action by Rokhlin ([Ro]) in the case $d = 1$ and by Kamiński in [K2] for $d \ge 2$, but it was not known whether such a characterization exists for flows. Section 1 contains definitions and auxiliary results needed in the sequel. In Section 2 we prove a relative version of the Abramov formula for the entropy of a special flow. Section 3 is devoted to relatively excellent σ-algebras. Results of these sections together with a relative version of the Ambrose–Kakutani–Rudolph theorem allow us to prove in Section 4 the existence of relatively perfect σ-algebras.

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In Section 5 we first introduce a concept of a principal factor for a flow in terms of increasing σ-algebras. Next, using Theorem B we describe principal factors in terms of entropy. It follows from the Ornstein isomorphism theorem for Bernoulli flows and the result above that, as in the case of $\mathbb{Z}^d$-actions, the behaviour of the entropy of flows under factor and principal factor maps and direct products determines it uniquely up to normalization.

1. Preliminaries. Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space and let $\mathcal{N}_X$ be the trivial sub-σ-algebra of $\mathcal{B}$.

With every measurable partition P of X we associate the σ-algebra $\widehat{P}$ of P-sets, i.e. $\widehat{P}$ consists of all measurable sums of elements of P. It is well known that for every sub-σ-algebra $\mathcal{A} \subset \mathcal{B}$ there is a unique (mod μ) measurable partition $\widehat{\mathcal{A}}$ such that $\mathcal{A}$ is the σ-algebra of $\widehat{\mathcal{A}}$-sets.

The symbol $\mathcal{Z}$ stands for the set of all countable measurable partitions with finite entropy.

Let $P = \{P_i\} \in \mathcal{Z}$ and let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{B}$. The information of P given $\mathcal{A}$ is

$$I(P \mid \mathcal{A}) = -\sum_i \chi_{P_i} \cdot \log \mu(P_i \mid \mathcal{A}).$$

For a given function $f \in L^1(X, \mu)$ we put

$$E(f) = \int_X f \, d\mu.$$

Let now τ be an automorphism of $(X, \mathcal{B}, \mu)$. With every partition $P \in \mathcal{Z}$ we associate the two σ-algebras

$$P_\tau^- = \bigvee_{i=1}^{\infty} \tau^{-i} P, \qquad P_\tau = \bigvee_{i=-\infty}^{+\infty} \tau^i P.$$

Let $T = (T^t)$ be a measurable flow on $(X, \mathcal{B}, \mu)$.

A sub-σ-algebra $\mathcal{A} \subset \mathcal{B}$ is said to be *increasing* if $T^t \mathcal{A} \supset \mathcal{A}$ for all $t > 0$. It is called a *factor σ-algebra* if $T^t \mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{R}$.

We denote by $h(T^t)$ and $\pi(T^t)$ the entropy and the Pinsker σ-algebra of the automorphism T^t respectively, $t \in \mathbb{R}$. It is well known ([A2], [G1]) that

$$h(T^t) = |t| \cdot h(T^1), \quad \pi(T^t) = \pi(T^1), \quad t \in \mathbb{R}.$$

Recall that the entropy $h(T)$ and the Pinsker σ-algebra $\pi(T)$ of the flow T are defined as follows:

$$h(T) = h(T^1), \qquad \pi(T) = \pi(T^1).$$

Let now $\mathcal{H}$ be a fixed factor sub-σ-algebra of T and let $T_{\mathcal{H}}$ be the factor flow induced by $\mathcal{H}$. We denote by $h(T^t \mid \mathcal{H})$ and $\pi(T^t \mid \mathcal{H})$ the relative entropy and the relative Pinsker σ-algebra of T^t with respect to $\mathcal{H}$ respectively, $t \in \mathbb{R}$. Proceeding similarly to [A1] and [G1] one easily obtains

$$h(T^t \mid \mathcal{H}) = |t| \cdot h(T^1 \mid \mathcal{H}), \quad \pi(T^t \mid \mathcal{H}) = \pi(T^1 \mid \mathcal{H}), \quad t \in \mathbb{R}.$$

We define the relative entropy and the relative Pinsker σ-algebra of T with respect to $\mathcal{H}$ as

$$h(T \mid \mathcal{H}) = h(T^1 \mid \mathcal{H}), \qquad \pi(T \mid \mathcal{H}) = \pi(T^1 \mid \mathcal{H})$$

respectively. Clearly

$$h(T) = h(T \mid \mathcal{N}) \quad \text{and} \quad \pi(T) = \pi(T \mid \mathcal{N}).$$

We shall use in the sequel the following equalities:

$$h(T) = h(T_{\mathcal{H}}) + h(T \mid \mathcal{H}), \tag{1}$$

$$\pi(T \mid \pi(T)) = \pi(T). \tag{2}$$

They have been shown for $\mathbb{Z}$-actions in [K2] and [K1] respectively.

Now we recall the concept of a special flow built under a function.

Let $(Y, \mathcal{C}, \nu)$ be a Lebesgue probability space, τ be an automorphism of Y and $f : Y \to \mathbb{R}^+$ a measurable function such that $\inf\{f(y) : y \in Y\} > 0$ and $f \in L^1(Y, \nu)$. Let $Y_f = \{(y, u) \in Y \times \mathbb{R}^+ : u < f(y)\}$ and let $\mathcal{C}_f$ be the restriction of the product σ-algebra $\mathcal{C} \otimes \mathcal{L}$ to Y_f, where $\mathcal{L}$ denotes the σ-algebra of Lebesgue sets of $\mathbb{R}^+$. We denote by ν_f the measure on $\mathcal{C}_f$ defined by

$$\nu_f = (E(f))^{-1} \cdot (\nu \times \lambda),$$

where λ stands for Lebesgue measure. Let τ_f be the measurable flow on $(Y_f, \mathcal{C}_f, \nu_f)$ defined as follows. For $0 \le t < \inf\{f(y), y \in Y\}$ we put

$$\tau_f^t(y, u) = \begin{cases} (y, u + t) & \text{if } u + t < f(y), \\ (\tau y, u + t - f(y)) & \text{if } u + t \ge f(y). \end{cases}$$

For other values of t the automorphism τ_f^t is uniquely determined by the condition that (τ_f^t) is a one-parameter group of automorphisms.

The flow τ_f is called the *special flow built under the function* f, the automorphism τ is the *base automorphism* and f is the *ceiling function* of τ_f.

Since Bernoulli flows will play an essential role in Section 5 we recall their definition and the Ornstein result which we will need.

A flow T is said to be a *Bernoulli flow* if for every $t \neq 0$ the automorphism T^t is a Bernoulli shift.

The existence of Bernoulli flows has been proved by Ornstein in [O1]. The following "Sinai type" theorem may be easily deduced from the corollary to the Main Lemma from [O2].

THEOREM A. *For every ergodic flow T with $h(T) > 0$ and every $a \in (0, h(T)]$ there exists a Bernoulli flow S which is isomorphic to a factor of T, with $h(S) = a$.*

2. Relative version of the Abramov formula for flows. Let $(Y, \mathcal{C}, \nu)$ be a Lebesgue probability space, τ an automorphism of Y and $\mathcal{F} \subset \mathcal{C}$ a factor σ-algebra of τ. Let $A \in \mathcal{C}$ be a set of positive measure such that $\bigcup_{n=0}^{\infty} \tau^n A = Y$. It is well known that if τ is ergodic then every set A of positive measure satisfies this condition. The automorphism of A induced by τ is denoted by τ_A.

Let $\mathcal{F}^A$ be the sub-σ-algebra of A consisting of the sets of the form $F \cap A$ where $F \in \mathcal{F}$.

Remark 1. If $A \in \mathcal{F}$ then $\mathcal{F}^A$ is a factor σ-algebra of τ_A. Indeed, the Poincaré recurrence theorem implies that $A = \bigcup_{m=1}^{\infty} A_m$, where A_m denotes the set of the mth return time. The desired property follows at once from the equality

$$\tau_A(F \cap A) = \bigcup_{m=1}^{\infty} \tau^m(F \cap A \cap \tau^{-1}A^c \cap \ldots \cap \tau^{-(m-1)}A^c) \cap A,$$

where $F \in \mathcal{F}$.

LEMMA 1. *For every $A \in \mathcal{F}$ with $\nu(A) > 0$ we have*

$$h(\tau_A \mid \mathcal{F}^A) = (\nu(A))^{-1} \cdot h(\tau \mid \mathcal{F}).$$

We omit the proof because it may be easily obtained from the proof in the absolute case (see [A1]).

Let now σ be an automorphism of $(Y, \mathcal{C}, \nu)$ and let $\mathcal{F} \subset \mathcal{C}$ be a factor σ-algebra of σ. Let $\mathcal{L}^1$ denote the σ-algebra of Lebesgue sets of the interval $[0, 1)$, λ the Lebesgue measure on $[0, 1)$ and $\varphi : Y \to [0, 1)$ a $\mathcal{C}$-measurable function.

We consider the product measure space

$$(X, \mathcal{B}, \mu) = (Y, \mathcal{C}, \nu) \times ([0, 1), \mathcal{L}^1, \lambda)$$

and the automorphism $\tau = \tau_\varphi$ of $(X, \mathcal{B}, \mu)$ defined by

$$\tau(y, u) = (\sigma y, u + \varphi(y)),$$

where $+$ means addition mod 1.

We put $\mathcal{H} = \mathcal{F} \otimes \mathcal{L}^1$. One easily checks the following

Remark 2. If φ is $\mathcal{F}$-measurable then $\mathcal{H}$ is a factor σ-algebra of τ_φ.

LEMMA 2. *If φ is $\mathcal{F}$-measurable, then*

$$h(\tau \mid \mathcal{H}) = h(\sigma \mid \mathcal{F}).$$

We omit the proof for the same reason as in the case of Lemma 1 (see the Lemma in [A2]).

Let now $(Y_f, \mathcal{C}_f, \nu_f, \tau_f)$ be the special flow over $(Y, \mathcal{C}, \nu, \tau)$ under a function $f : Y \to \mathbb{R}^+$.

For a σ-algebra $\mathcal{F} \subset \mathcal{C}$ we denote by $\mathcal{F}_f$ the restriction of the product σ-algebra $\mathcal{F} \otimes \mathcal{L}$ to Y_f.

Remark 3 ([B2]). If $\mathcal{F}$ is a factor σ-algebra for τ and f is $\mathcal{F}$-measurable then $\mathcal{F}_f$ is a factor σ-algebra for τ_f.

LEMMA 3. *If f is $\mathcal{F}$-measurable then*

$$h(\tau_f^t \mid \mathcal{F}_f) = |t| \cdot (E(f))^{-1} \cdot h(\tau \mid \mathcal{F}), \quad t \in \mathbb{R}.$$

Proof. It is enough to show the equality for $0 < t < \inf\{f(y) : y \in Y\}$. We consider, as in the absolute case ([A2]), the product space $\widetilde{Y}_t = Y \times [0, t)$ equipped with the product σ-algebra $\widetilde{\mathcal{F}}_t = \mathcal{F} \otimes \mathcal{L}^t$, where $\mathcal{L}^t$ denotes the σ-algebra of Lebesgue sets of $[0, t)$, and the natural product measure.

Let $\widetilde{\tau}^t$ be the automorphism of $\widetilde{Y}_t$ defined by

$$\widetilde{\tau}^t(y, u) = (\tau y, u + \varphi_t(y)),$$

where $+$ means addition mod t and

$$\varphi_t(y) = t - f(y) + \left[\frac{1}{t} f(y)\right].$$

As Abramov observed, $\widetilde{\tau}^t$ is the automorphism induced by τ_f^t on $\widetilde{Y}_t$. Therefore, by Remark 1, $\widetilde{\tau}^t \widetilde{\mathcal{F}}_t = \widetilde{\mathcal{F}}_t$. It follows from Lemma 1 that

$$h(\widetilde{\tau}^t \mid \widetilde{\mathcal{F}}_t) = (\nu_f(Y_t))^{-1} \cdot h(\tau_f^t \mid \mathcal{F}_f) = t^{-1} \cdot E(f) \cdot h(\tau_f^t \mid \mathcal{F}_f). \tag{3}$$

Applying Lemma 2 we have

$$h(\widetilde{\tau}^t \mid \widetilde{\mathcal{F}}_t) = h(\tau \mid \mathcal{F}). \tag{4}$$

Combining (3) and (4) finishes the proof.

3. Relatively excellent σ-algebras. Let τ be an automorphism of a Lebesgue probability space $(Y, \mathcal{C}, \nu)$ and let $\mathcal{F}$ be a factor σ-algebra of τ.

DEFINITION 1. A sub-σ-algebra $\mathcal{A} \subset \mathcal{C}$ is said to be *relatively excellent for τ with respect to $\mathcal{F}$* if

$$\mathcal{F} \subset \mathcal{A}, \quad \tau\mathcal{A} \supset \mathcal{A}, \tag{5}$$

$$\bigvee_{n=-\infty}^{+\infty} \tau^n \mathcal{A} = \mathcal{C}, \tag{6}$$

(7) there exists a sequence $(P_n) \subset \mathcal{Z}$ with $\widehat{P_n} \nearrow \mathcal{A}$ such that

$$\lim_{n \to \infty} (h(P_n, \tau \mid \mathcal{F}) - H(P_n \mid \mathcal{A}^-)) = 0.$$

The proof of Theorem 1 of [K2] yields

LEMMA 4. *For every factor σ-algebra $\mathcal{F}$ of τ there exists a relatively excellent σ-algebra $\mathcal{A}$ with respect to $\mathcal{F}$. Every such σ-algebra is relatively* *perfect with respect to* $\mathcal{F}$, *i.e. it also satisfies the following two equalities*:

$$\bigcap_{n=-\infty}^{+\infty} \tau^n \mathcal{A} = \pi(\tau \mid \mathcal{F}), \tag{8}$$

$$h(\tau \mid \mathcal{F}) = H(\tau\mathcal{A} \mid \mathcal{A}). \tag{9}$$

If $\mathcal{F} = \mathcal{N}$, then $\mathcal{A}$ is simply an excellent σ-algebra as defined in [B1].

In the sequel we shall use the relative Pinsker formula ([K2]):

$$\text{for } P, Q \in \mathcal{Z},\ h(P \vee Q, \tau \mid \mathcal{F}) = h(P, \tau \mid \mathcal{F}) + h(Q, \tau \mid P_\tau \vee \mathcal{F}). \tag{10}$$

LEMMA 5. *If* $\mathcal{A} \subset \mathcal{C}$ *is relatively excellent with respect to* $\mathcal{F}$ *and* $Q \in \mathcal{Z}$ *then the* σ*-algebra* $\mathcal{A} \vee \tau Q^-$ *is also relatively excellent with respect to* $\mathcal{F}$.

Proof. It is clear that the σ-algebra $\mathcal{A} \vee \tau Q^-$ satisfies (5) and (6). Let a sequence $(P_n) \subset \mathcal{Z}$ satisfy (7) and let $Q_n = \bigvee_{i=0}^{n} \tau^{-i} Q$, $n \geq 0$. We claim that the sequence $(P_n \vee Q_n)$ also satisfies (7) (for the σ-algebra $\mathcal{A} \vee \tau Q^-$).

In the sequel we shall use some ideas from the proof of Proposition 1.3 of [B1].

For any natural numbers n, m, formula (10) gives

$$\begin{aligned} h(P_n \vee Q_m, \tau \mid \mathcal{F}) &= h(P_n, \tau \mid \mathcal{F}) + h(Q_m, \tau \mid (P_n)_\tau \vee \mathcal{F}) \\ &= h(P_n, \tau \mid \mathcal{F}) + h(Q, \tau \mid (P_n)_\tau \vee \mathcal{F}). \end{aligned} \tag{11}$$

Assume $n < N$. Since $H(P_N \mid P_n) < \infty$ there exists a partition P_N^n with finite entropy such that $P_n \vee P_N^n = P_N$. Simple properties of the conditional entropy give

$$H(P_N \mid P_N^- \vee \mathcal{F}) = H(P_n \mid P_N^- \vee \mathcal{F}) + H(P_N^n \mid P_N^- \vee P_n \vee \mathcal{F}). \tag{12}$$

It follows from (11) with $n = N$, $m = n$ and from (12) that

$$\begin{aligned} &H(P_n \vee Q_n \mid P_N^- \vee Q^- \vee \mathcal{F}) \\ &\quad = H(P_n \vee Q_n \mid P_N^- \vee Q_n^- \vee \mathcal{F}) \\ &\quad = H(P_N \vee Q_n \mid P_N^- \vee Q_n^- \vee \mathcal{F}) - H(P_N^n \mid P_n \vee P_N^- \vee Q_n \vee Q_n^- \vee \mathcal{F}) \\ &\quad = H(P_n \mid P_N^- \vee \mathcal{F}) + H(P_N^n \mid P_N^- \vee P_n \vee \mathcal{F}) \\ &\qquad + H(Q \mid Q^- \vee (P_N)_\tau \vee \mathcal{F}) - H(P_N^n \mid P_n \vee P_N^- \vee Q_n \vee Q_n^- \vee \mathcal{F}) \\ &\quad \geq H(P_n \mid P_N^- \vee \mathcal{F}) + H(Q \mid Q^- \vee (P_N)_\tau \vee \mathcal{F}). \end{aligned}$$

Hence, in view of (11) for $m = n$, we get

$$\begin{aligned} 0 &\leq H(P_n \vee Q_n \mid P_n^- \vee Q_n^- \vee \mathcal{F}) - H(P_n \vee Q_n \mid P_N^- \vee Q^- \vee \mathcal{F}) \\ &\leq H(P_n \mid P_n^- \vee \mathcal{F}) - H(P_n \mid P_N^- \vee \mathcal{F}) \\ &\quad + H(Q \mid Q^- \vee (P_n)_\tau \vee \mathcal{F}) - H(Q \mid Q^- \vee (P_N)_\tau \vee \mathcal{F}), \quad n < N. \end{aligned}$$

Since $\mathcal{A}$ is generating and (P_n) satisfies (7), it follows that taking the limit, first as $N \to \infty$, and then as $n \to \infty$, we obtain the desired result.

COROLLARY. *If* $f : Y \to \mathbb{R}$ *is measurable with an a.e. finite set of values then for every factor* σ*-algebra* $\mathcal{F}$ *there exists a* σ*-algebra* $\mathcal{A}$ *relatively excellent with respect to* $\mathcal{F}$ *such that* f *is* $\mathcal{A}$*-measurable.*

Proof. Let $\mathcal{D}$ be an arbitrary relatively excellent σ-algebra with respect to $\mathcal{F}$. Lemma 4 assures that such a σ-algebra exists. Consider the partition $Q = \{Q_1, \ldots, Q_m\}$ of Y into sets where f is constant. It follows from Lemma 5 that the σ-algebra $\mathcal{A} = \mathcal{D} \vee \tau Q^-$ is also relatively excellent with respect to $\mathcal{F}$. It is clear that f is $\mathcal{A}$-measurable.

Now suppose g is an integrable function on Y with values in $\mathbb{N}$. Let $(Y^g, \mathcal{C}^g, \nu^g, \tau^g)$ be the integral dynamical system over $(Y, \mathcal{C}, \nu, \tau)$ under the function g (cf. [CFS]). We denote by Q^g the partition of Y generated by g, i.e.

$$Q^g = \{g^{-1}(\{k\}) : k \in \mathbb{N}\}.$$

It follows from Lemma 1.1 of [B1] that $Q^g \in \mathcal{Z}$.

For a given sub-σ-algebra $\mathcal{F} \subset \mathcal{C}$ we denote by $\mathcal{F}^g$ the sub-σ-algebra of $\mathcal{C}^g$ defined in the same way as $\mathcal{C}^g$, i.e.

$$A \in \mathcal{F}^g, \quad A = \bigcup_{i=1}^{\infty} A_i \times \{i\} \quad \text{iff} \quad A_i \in \mathcal{F},\ i \in \mathbb{N}.$$

One easily checks the following

Remark 4. If the function g is $\mathcal{F}$-measurable and $\mathcal{F}$ is a factor σ-algebra of τ then $\mathcal{F}^g$ is a factor σ-algebra of τ^g.

Let $P = (P_k)$ be a countable measurable partition of Y. We associate with it the partition $\overline{P}$ of Y^g as follows. The atoms of $\overline{P}$ are all the sets $P_k \times \{1\}$, $k \in \mathbb{N}$, and the set $Y^g \setminus (Y \times \{1\})$.

LEMMA 6. *If* $\mathcal{F}$ *is a factor* σ*-algebra of* τ *such that* g *is* $\mathcal{F}$*-measurable then for every* $P \in \mathcal{Z}$ *we have*

$$h(\overline{P \vee \tau Q^g}, \tau^g \mid \mathcal{F}^g) = (E(g))^{-1} \cdot h(P \vee \tau Q^g, \tau \mid \mathcal{F}).$$

Proof. Let $Y_0 = Y \times \{1\}$, $R = \overline{P \vee \tau Q^g}$ and $T = \tau^g$. It follows easily from the definition of T that

$$Y_0 \in R_T^- \tag{13}$$

and

$$(R_T^- \vee \mathcal{F}^g) \cap Y_0 = [(P \vee \tau Q^g)_\tau^- \vee \mathcal{F}] \times \{1\}. \tag{14}$$

Hence

$$\nu^g(Y_0 \mid R_T^- \vee \mathcal{F}^g) = \chi_{Y_0} \tag{15}$$

and

(16) $$\nu^g(A\times\{1\}\mid R_T^-\vee\mathcal{F}^g)(y,1)=\nu(A\mid(P\vee\tau Q^g)_\tau^-\vee\mathcal{F})(y),\quad A\in\mathcal{C},\ y\in Y.$$

From (16) it follows that

(17) $$\begin{aligned}\int_{Y_0} I(R\mid R_T^-\vee\mathcal{F}^g)\,d\nu^g &= -\int_{Y_0}\sum_{A\in P\vee\tau Q^g}\chi_{A\times\{1\}}(y,1)\cdot\log\nu^g(A\times\{1\}\mid R_T^-\vee\mathcal{F}^g)(y,1)\,d\nu^g\\ &=(E(g))^{-1}\cdot\int_Y I(P\vee\tau Q^g\mid(P\vee\tau Q^g)_\tau^-\vee\mathcal{F})(y)\,d\nu\\ &=(E(g))^{-1}\cdot h(P\vee\tau Q^g,\tau\mid\mathcal{F}).\end{aligned}$$

The equality (15) implies

(18) $$\int_{Y_0^c} I(R\mid R_T^-\vee\mathcal{F}^g)\,d\nu^g = -\int_{Y_0^c}\chi_{Y_0^c}(y,i)\cdot\log\nu^g(Y_0^c\mid R_T^-\vee\mathcal{F}^g)(y,i)\,d\nu^g=0.$$

Comparing (17) and (18) one gets

$$h(R,T\mid\mathcal{F}^g)=\int_{Y^g} I(R\mid R_T^-\vee\mathcal{F}^g)\,d\nu^g=(E(g))^{-1}\cdot h(P\vee\tau Q^g,\tau\mid\mathcal{F}),$$

which completes the proof.

Now suppose $f:Y\to\mathbb{R}^+$ is an integrable function such that

$$\inf\{f(y):y\in Y\}=\alpha>0.$$

Let τ_f be the special flow on the space $(Y_f,\mathcal{C}_f,\nu_f)$, built under f and over τ.

For a given sub-σ-algebra $\mathcal{A}\subset\mathcal{C}$ we denote by $\mathcal{A}_f$ the sub-σ-algebra of $\mathcal{C}_f$ defined by

$$\mathcal{A}_f=\{A\cap Y_f: A\in\mathcal{A}\otimes\mathcal{L}\}.$$

Remark 5 ([B2]). If $\mathcal{A}\subset\mathcal{C}$ is an increasing sub-σ-algebra for τ and f is $\mathcal{A}$-measurable then $\mathcal{A}_f$ is increasing for the flow τ_f.

Let $Y_1=Y\times[0,1)$. With any measurable partition P of Y we associate the partition $P^1=P\times[0,1)$ of Y_1. For a sub-σ-algebra $\mathcal{A}\subset\mathcal{C}$ put $\mathcal{A}^1=\mathcal{A}\otimes\mathcal{L}^1$.

Let $R_k=\{R_{k,i}:0\le i<2^{k-1}\}$ be the partition of Y_1 defined by

$$R_{k,i}=\{(y,u)\in Y_1; i\cdot 2^{-k}\le u<(i+1)\cdot 2^{-k}\},\quad 0\le i<2^k-1,\ k\ge 1.$$

It is clear that the smallest σ-algebra $\mathcal{R}$ containing all R_k, $k\ge1$, coincides with the σ-algebra $\mathcal{N}_Y\otimes\mathcal{L}^1$.

Let τ_1 be the automorphism of Y_1 induced by $T^1=\tau_f^1$. It follows from Abramov's remark (see the proof of Lemma 3) that

$$\tau_1(y,u)=(\tau u,u+\varphi(y)),$$

where $\varphi(y)=1-\{f(y)\}$. The σ-algebra $\mathcal{C}\otimes\mathcal{N}_{[0,1)}$ is a principal factor σ-algebra of τ_1 and the corresponding factor automorphism is isomorphic to τ.

Since τ_1 is induced by T^1 on the set Y_1, T^1 is an integral automorphism over τ_1. Let g be the corresponding ceiling function on Y_1, i.e. the Poincaré cocycle for τ_1. If the flow τ_f is ergodic then the well-known Katz theorem implies $\int_{Y_1} g\,d\nu_f=1$, i.e.

$$E(g)=\int_{Y_1} g(y,u)\,d\nu\,du=E(f).$$

For a given measurable partition Q of Y_1 we denote by $\overline{Q}$ the partition of Y^f which consists of all atoms of Q and the set Y_1^c. If $\mathcal{D}$ is a sub-σ-algebra of Y_1, $\overline{\mathcal{D}}$ stands for the σ-algebra $\overline{\mathcal{D}}=\{A\cup Y_1^c: A\in\mathcal{D}\}$.

Let $\mathcal{F}$ be a factor σ-algebra of τ.

LEMMA 7. *If $\mathcal{A}\subset\mathcal{C}$ is a relatively excellent for τ with respect to $\mathcal{F}$ and f is $\mathcal{A}$-measurable then $\mathcal{A}_f$ is*

(19) *increasing for the flow τ_f,*

(20) *relatively excellent for τ_f^α with respect to $\mathcal{F}_f$.*

Proof. Since $\mathcal{A}\supset\mathcal{F}$ and $\mathcal{A}$ is generating, $\mathcal{A}_f$ is of course generating and $\mathcal{A}_f\supset\mathcal{F}_f$.

We may assume $\alpha=1$. Put $T^1=\tau_f^1$ as above. It is easy to check that $T^1\mathcal{A}_f=\overline{(\tau\mathcal{A})^1}\vee\overline{\mathcal{R}}$. Since, by Remark 5, $\mathcal{A}_f$ is increasing with respect to T^1 the above equality implies

(21) $$\mathcal{A}_f=(\overline{(\tau\mathcal{A})^1}\vee\overline{\mathcal{R}})_{T^1}^-.$$

Let (P_n) be a sequence of partitions of Y with finite entropy such that $\widehat{P}_n\nearrow\mathcal{A}$ and

$$\lim_{n\to\infty}(h(P_n,\tau\mid\mathcal{F})-H(P_n\mid\mathcal{A}^-\vee\mathcal{F}))=0.$$

Let $Q_{n,k}=(\tau P_n)^1\vee R_k$, $n,k\ge1$. We shall show that there exists an increasing sequence (n_k) of natural numbers such that

$$h(\overline{Q}_{n_k,k},T^1\mid\mathcal{F}_f)-H(\overline{Q}_{n_k,k}\mid\mathcal{A}_f\vee\mathcal{F}_f)\to0$$

as $k\to\infty$. Since $\overline{Q}_{n_k,k}\nearrow\overline{(\tau\mathcal{A})^1\vee\mathcal{R}}$ the equality (21) implies that $\mathcal{A}_f$ is relatively excellent for T^1 with respect to $\mathcal{F}_f$. One easily checks the following equalities:

$$\nu_f(\tau A \times [0,1) \mid \mathcal{A}_f \vee \mathcal{F}_f)(y,u) = \nu(\tau A \mid \mathcal{A} \vee \mathcal{F})(y)$$

and

$$\nu_f(Y_1 \mid \mathcal{A}_f \vee \mathcal{F}_f) = \chi_{Y_1}, \quad (y,u) \in Y_1,\ A \in P_n,\ n \geq 1.$$

From these equalities a straightforward computation yields

$$H((\tau P_n)^1 \mid \mathcal{A}_f \vee \mathcal{F}_f) = (E(f))^{-1} \cdot H(\tau P_n \mid \mathcal{A} \vee \mathcal{F}). \tag{22}$$

For every set $A \in \mathcal{C}$ we have

$$\tau_1^{-1}(A \times [0,1)) = \tau^{-1}(A) \times [0,1),$$
$$(\nu \times \lambda)(A \times [0,1) \mid \mathcal{D} \otimes \mathcal{L}^1) = \nu(A \mid \mathcal{D}),$$

where $\mathcal{D}$ is an arbitrary sub-σ-algebra of $\mathcal{C}$. Therefore we get

$$h((\tau P_n)^1, \tau_1 \mid \mathcal{F} \otimes \mathcal{L}^1) = h(P_n, \tau \mid \mathcal{F}) \tag{23}$$

and

$$H((\tau P_n)^1 \mid [(\tau \mathcal{A})^1]_{\tau_1}^- \vee \mathcal{F} \otimes \mathcal{L}^1) = H(P_n \mid \mathcal{A}_\tau^- \vee \mathcal{F}). \tag{24}$$

We have

$$h(\overline{Q}_{n,k}, T^1 \mid \mathcal{F}_f) - H(\overline{Q}_{n,k} \mid \mathcal{A}_f \vee \mathcal{F}_f) = a_{n,k} + b_{n,k} + c_{n,k},$$

where

$$a_{n,k} = h(\overline{Q}_{n,k}, T^1 \mid \mathcal{F}_f) - (E(g))^{-1} \cdot h(Q_{n,k} \vee \tau_1 Q^g, \tau_1 \mid \mathcal{F} \otimes \mathcal{L}^1),$$
$$b_{n,k} = (E(g))^{-1} \cdot h(Q_{n,k} \vee \tau_1 Q^g, \tau_1 \mid \mathcal{F} \otimes \mathcal{L}^1) - (E(f))^{-1} \cdot h((\tau P_n)^1, \tau_1 \mid \mathcal{F} \otimes \mathcal{L}^1),$$
$$c_{n,k} = (E(f))^{-1} \cdot h((\tau P_n)^1, \tau_1 \mid \mathcal{F} \otimes \mathcal{L}^1) - H(\overline{Q}_{n,k} \mid \mathcal{A}_f \vee \mathcal{F}_f),$$

for $n, k \geq 1$. By Lemma 6 we have

$$a_{n,k} = h(\overline{Q}_{n,k}, T^1 \mid \mathcal{F}_f) - h(\overline{Q_{n,k} \vee \tau_1 Q^g}, T^1 \mid \mathcal{F}_f) \leq 0.$$

It follows from the relative Pinsker formula (10) and the equality $Ef = Eg$ that

$$b_{n,k} = (E(f))^{-1} \cdot h(R_k \vee \tau_1 Q^g, \tau_1 \mid (\tau P_n)_{\tau_1}^1).$$

Since $\overline{R}_k \subset \mathcal{A}_f$ we have, by (22)–(24),

$$\begin{aligned} c_{n,k} &= (E(f))^{-1} \cdot h(P_n, \tau \mid \mathcal{F}) - H(\overline{(\tau P_n)^1} \mid \mathcal{A}_f \vee \mathcal{F}_f) \\ &= (E(f))^{-1} \cdot (h(P_n, \tau \mid \mathcal{F}) - H(\tau P_n \mid \mathcal{A} \vee \mathcal{F})) \\ &= (E(f))^{-1} \cdot (h(P_n, \tau \mid \mathcal{F}) - H(P_n \mid \mathcal{A}^- \vee \mathcal{F})), \quad n, k \geq 1. \end{aligned}$$

Since $\widehat{P}_n \nearrow \mathcal{A}$ and $\mathcal{A}$ is generating we have $[(\tau P_n)^1]_{\tau_1} = [(P_n)_\tau]^1 \nearrow \mathcal{C} \otimes \mathcal{L}^1$. But $\mathcal{C}^1$ is a principal factor σ-algebra for τ_1 so $\lim_{n\to\infty} b_{n,k} = 0$ for every $k \geq 1$. Therefore there exists an increasing sequence (n_k) of natural numbers such that $\lim_{k\to\infty} b_{n_k,k} = 0$. Thus

$$\lim_{k\to\infty} (h(\overline{Q}_{n_k,k}, T^1 \mid \mathcal{F}_f) - H(\overline{Q}_{n_k,k} \mid \mathcal{A}_f \vee \mathcal{F}_f)) = 0,$$

i.e. $\mathcal{A}_f$ is relatively excellent for T^1 with respect to $\mathcal{F}_f$, which completes the proof.

4. Relatively perfect σ-algebras. Our proof of Theorem B below requires a relative version of the well-known Ambrose–Kakutani–Rudolph (AKR) theorem ([AK], [Ru]).

LEMMA 8. *For every ergodic flow T on a Lebesgue probability space $(X, \mathcal{B}, \mu)$, given a nonatomic factor σ-algebra $\mathcal{H}$ of T and two positive real numbers p and q with p/q irrational, there exists a special flow $(Y_f, \mathcal{C}_f, \nu_f, \tau_f)$, where f is a measurable function with values p and q, a factor σ-algebra $\mathcal{F}$ for τ such that f is $\mathcal{F}$-measurable and an isomorphism $\varphi : X \to Y_f$ of the flows T and τ_f such that $\varphi(\mathcal{H}) = \mathcal{F}_f$.*

Proof. Let ξ be a measurable partition of X associated with $\mathcal{H}$, i.e. $\xi = \widehat{\mathcal{H}}$. It is clear that $T^t\xi = \xi$, $t \in \mathbb{R}$. We consider the quotient Lebesgue space $(X/\xi, \mathcal{B}_\xi, \mu_\xi)$ equipped with the quotient flow T_ξ. We denote by $H_\xi : X \to X/\xi$ the natural homomorphism. It follows from the proof of the AKR theorem (cf. [CFS]) that there exists a measurable partition ζ_ξ of X/ξ which is regular for T_ξ, i.e.

(25) ζ_ξ is a measurable partition of X/ξ into intervals of trajectories with lengths p and q, i.e. sets of the form $\{T_\xi^t C : 0 \leq t \leq \widetilde{f}(C)\}$, where $\widetilde{f}(C) = p$ or $\widetilde{f}(C) = q$,

(26) the functions F, G defined by $F(D) = \widetilde{f}(C)$, $G(D) = t$, where $D = T_\xi^t(C)$, are $\mathcal{B}_\xi$-measurable.

Now we define a measurable partition ζ of X which is regular for T. Let E be an element of ζ_ξ and let $C_{\rm b}(E) \in X/\xi$, $C_{\rm b}(E) \subset E$ denoting the beginning of the trajectory of T_ξ included in E. We denote by ζ the partition of X consisting of the following intervals of trajectories of T:

$$\{T^t x : x \in H_\xi^{-1} C_{\rm b}(E)\},$$

where $E \in \zeta_\xi$.

It is easy to check that ζ is regular for T with the same lengths p and q of trajectories.

Now we construct the desired probability space $(Y, \mathcal{C}, \nu)$ and the automorphism τ of Y in the same way as in the proof of the AKR theorem. Recall that Y is the set of left ends of elements of ζ, i.e. the points belonging to $H_\xi^{-1} C_{\rm b}(E)$, $E \in \zeta_\xi$.

Let η be the measurable partition of Y whose elements are the sets $H_\xi^{-1}C_b(E)$, $E \in \zeta_\xi$, and let $\mathcal{F}$ be the σ-algebra of η-sets.

It is clear that $\mathcal{F}$ is a factor σ-algebra of τ. For every $y \in Y$, $y \in H_\xi^{-1}(C)$, $C = C_b(E)$, $E \in \zeta_\xi$, the length $f(y)$ of the trajectory of y is equal to $\widetilde{f}(C)$. Hence f is $\mathcal{F}$-measurable.

Denoting by φ the isomorphism between X and Y defined in [CFS] we obtain the equality $\varphi(\mathcal{H}) = \mathcal{F}_f$.

Let T be a measurable flow on a Lebesgue space $(X, \mathcal{B}, \mu)$ and let $\mathcal{H}$ be a factor σ-algebra of T.

DEFINITION 2. A sub-σ-algebra $\mathcal{A} \subset \mathcal{B}$ is said to be *relatively perfect with respect to* $\mathcal{H}$ if

(i) $\mathcal{A} \supset \mathcal{H}$, $T^t\mathcal{H} \supset \mathcal{H}$, $t > 0$,
(ii) $\bigvee_{t\in R} T^t\mathcal{A} = \mathcal{B}$,
(iii) $\bigcap_{t\in R} T^t\mathcal{A} = \pi(T \mid \mathcal{H})$,
(iv) $h(T^t \mid \mathcal{H}) = H(T^t\mathcal{A} \mid \mathcal{A})$, $t > 0$.

In the case $\mathcal{H} = \mathcal{N}$ the concept of a relatively perfect σ-algebra reduces to the concept of a perfect σ-algebra ([B1], [G2]).

THEOREM B. *For every ergodic flow T and a Lebesgue space $(X, \mathcal{B}, \mu)$ and every factor σ-algebra $\mathcal{H}$ of T there exists a relatively perfect σ-algebra with respect to $\mathcal{H}$.*

Proof. We may assume that $\mathcal{H}$ is nonatomic. Indeed, in the opposite case, due to the ergodicity of T, $\mathcal{H}$ is finite, therefore $\mathcal{H} \subset \pi(T)$. Then it is easy to show, using formulas (1) and (2), that any perfect σ-algebra $\mathcal{A}$ for T (such σ-algebras exist by [B1], [G2]) is also relatively perfect with respect to $\mathcal{H}$.

Suppose now that $\mathcal{H}$ is nonatomic. Due to Lemma 8 we may assume that $X = Y_f$, $\mathcal{B} = \mathcal{C}_f$, $\mu = \nu_f$, $T^t = \tau_f^t$ and $\mathcal{H} = \mathcal{F}_f$, where $\mathcal{F}$ is a factor σ-algebra of the automorphism τ of $(Y, \mathcal{C}, \nu)$ and f is a $\mathcal{F}$-measurable function with two values. We put

$$\alpha = \min\{f(y) : y \in Y\}.$$

In view of the corollary to Lemma 5 there exists a relatively excellent σ-algebra $\mathcal{D} \subset \mathcal{C}$ for τ with respect to $\mathcal{F}$ such that f is $\mathcal{D}$-measurable.

We put $\mathcal{A} = \mathcal{D}_f$. The $\mathcal{D}$-measurability and the conditions (5) and (6) of relatively excellent σ-algebras imply that

$$\mathcal{A} \supset \mathcal{H}, \quad T^t\mathcal{A} \supset \mathcal{A}, \quad t > 0, \quad \bigvee_{t\in\mathbb{R}} T^t\mathcal{A} = \mathcal{B}.$$

Applying Lemma 3.1 of [G1] and the equality (9) we get

$$H(T^t\mathcal{A} \mid \mathcal{A}) = t \cdot (E(f))^{-1} \cdot H(\tau\mathcal{D} \mid \mathcal{D}) = t \cdot (E(f))^{-1} \cdot h(\tau \mid \mathcal{F}), \quad t > 0.$$

On the other hand, Lemma 3 gives

$$h(T^t \mid \mathcal{H}) = h(\tau_f^t \mid \mathcal{F}_f) = t \cdot (E(f))^{-1} \cdot h(\tau \mid \mathcal{F}), \quad t > 0.$$

Therefore we have

$$h(T^t \mid \mathcal{H}) = H(T^t\mathcal{A} \mid \mathcal{A}), \quad t > 0.$$

It follows from Lemma 7 that $\mathcal{A}$ is relatively excellent for $T^\alpha = \tau_f^\alpha$ with respect to $\mathcal{H}$. Applying the equality (8) to T^α we get

$$\bigcap_{t\in\mathbb{R}} T^t\mathcal{A} = \bigcap_{n=-\infty}^{+\infty} T^{n\alpha}\mathcal{A} = \pi(T^\alpha \mid \mathcal{H}) = \pi(T \mid \mathcal{H}),$$

which completes the proof.

5. Principal factors and an axiomatic definition of entropy.

Let $T = (T^t)$ be a measurable flow on a Lebesgue space $(X, \mathcal{B}, \mu)$.

DEFINITION 3. A factor σ-algebra $\mathcal{H}$ of T is said to be *principal* if every increasing σ-algebra $\mathcal{A} \supset \mathcal{H}$ is a factor σ-algebra.

DEFINITION 4. A factor flow $S = (S^t)$ of T is said to be *principal* if every factor σ-algebra $\mathcal{H}$ of T such that the flows $T_\mathcal{H}$ and S are isomorphic is principal.

LEMMA 9. *If a flow S is a principal factor of T then $h(T) = h(S)$. Conversely, if $h(T) < \infty$ then the reverse implication is also true.*

Proof. Let $\mathcal{H}$ be a principal σ-algebra such that S and $T_\mathcal{H}$ are isomorphic. It follows from Theorem B that there exists an increasing σ-algebra $\mathcal{A} \supset \mathcal{H}$ with

$$h(T^t \mid \mathcal{H}) = H(T^t\mathcal{A} \mid \mathcal{A}), \quad t > 0.$$

It follows from the assumption that $h(T^t \mid \mathcal{H}) = 0$, $t > 0$. Therefore the formula (1) implies

$$h(T) = h(T_\mathcal{H}) = h(S).$$

Now suppose $h(T) < \infty$ and $h(T) = h(S)$. Let $\mathcal{H}$ be a factor σ-algebra such that S and $T_\mathcal{H}$ are isomorphic. Therefore we have $h(T) = h(T_\mathcal{H})$, i.e. $h(T \mid \mathcal{H}) = 0$. Let $\mathcal{A} \supset \mathcal{H}$ be increasing. Since

$$H(T^t\mathcal{A} \mid \mathcal{A}) = H(\mathcal{A} \mid T^{-t}\mathcal{A}) \le h(T^t \mid \mathcal{H})$$

we have $H(T^t\mathcal{A} \mid \mathcal{A}) = 0$, $t > 0$, and so $\mathcal{A}$ is a factor σ-algebra.

Let now τ be an automorphism of a Lebesgue space $(Y, \mathcal{C}, \nu)$ and $f : Y \to \mathbb{R}^+$ a measurable function with $\inf\{f(y) : y \in Y\} > 0$. From Lemma 9 and Abramov's formula ([A]) for the entropy of a special flow one obtains at once the following

COROLLARY. *If an automorphism σ of $(Y, \mathcal{C}, \nu)$ is a principal factor of τ then the special flow σ_f is a principal factor of τ_f.*

Let $\operatorname{Act}\mathbb{R}$ denote the set of all ergodic flows on Lebesgue probability spaces. We denote by $T_{\mathcal{O}}$ the flow defined as follows (cf. [O1]). Let τ be a Bernoulli 2-shift which acts on a Lebesgue space $(Y, \mathcal{C}, \nu)$. Let $P = \{A, B\}$ be an independent generating partition of Y for τ and let

$$f = p\chi_A + q\chi_B$$

where p and q are positive reals such that $p + q = 2$ and pq^{-1} is irrational. We define $T_{\mathcal{O}} = (T_{\mathcal{O}}^t)$ as the flow built under f with base automorphism τ. It follows from [O1] that $T_{\mathcal{O}}$ is a Bernoulli flow. The Abramov formula implies

$$h(T_{\mathcal{O}}) = (E(f))^{-1} \cdot h(\tau) = \log 2.$$

Applying the Ornstein isomorphism theorem for Bernoulli flows ([O2]) and Lemma 9 one may prove, using Rokhlin's idea (cf. [Ro]), the following

PROPOSITION. *Let $H : \operatorname{Act}\mathbb{R} \to [0, +\infty]$ be a function such that $H(T_{\mathcal{O}}) = \log 2$ and for all $T, S \in \operatorname{Act}\mathbb{R}$ the following conditions are satisfied:*

(i) *if S is a factor of T then $H(T) \geq H(S)$,*
(ii) *if S is a principal factor of T then $H(T) = H(S)$,*
(iii) $H(T \times S) = H(T) + H(S)$.

Then $H(T) = h(T)$ for all $T \in \operatorname{Act}\mathbb{R}$.

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