Relatively perfect $\sigma$-algebras for flows

by

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Abstract. We show that for every ergodic flow, given any factor $\sigma$-algebra $\mathcal{F}$, there exists a $\sigma$-algebra which is relatively perfect with respect to $\mathcal{F}$. Using this result and Ornstein's isomorphism theorem for flows, we give a functorial definition of the entropy of flows.

Introduction. Perfect $\sigma$-algebras play an important role in ergodic theory and statistical mechanics, especially in the spectral theory of dynamical systems with discrete time (measure preserving $\mathbb{Z}^d$-actions). The existence of these $\sigma$-algebras in the case $d \geq 2$ has been proved by the use of their relative versions (for $\mathbb{Z}^{d-1}$-actions), the so-called relatively perfect $\sigma$-algebras ([K1]). In [K2] the relatively perfect $\sigma$-algebras have been used to give a functorial definition of entropy of a $\mathbb{Z}^d$-action.

Blanchard in [B1] and Gurevič in [G2] have shown that for every ergodic flow there exists a perfect $\sigma$-algebra. The main purpose of the present paper is to prove a relative version of this result (Theorem B). The motivations of this theorem are, on the one hand, expected applications of relatively perfect $\sigma$-algebras to the investigation of the spectral structure of multidimensional flows and, on the other hand, an application to an axiomatic, i.e. functorial definition of entropy of one-dimensional flows. Such definitions have been given for $\mathbb{Z}^d$-action by Rohlin ([R0]) in the case $d = 1$ and by Kamiński in [K2] for $d \geq 2$, but it was not known whether such a characterisation exists for flows. Section 1 contains definitions and auxiliary results needed in the sequel. In Section 2 we prove a relative version of the Abramov formula for the entropy of a special flow. Section 3 is devoted to relatively excellent $\sigma$-algebras. Results of these sections together with a relative version of the Ambrose Kakutani–Rudolph theorem allow us to prove in Section 4 the existence of relatively perfect $\sigma$-algebras.

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In Section 5 we first introduce a concept of a principal factor for a flow in terms of increasing \( \sigma \)-algebras. Next, using Theorem B we describe principal factors in terms of entropy. It follows from the Ornstein isomorphism theorem for Bernoulli flows and the result above that, as in the case of \( \mathbb{Z}^d \)-actions, the behaviour of the entropy of flows under factor and principal factor maps and direct products determines it uniquely up to normalization.

1. Preliminaries. Let \((X, B, \mu)\) be a Lebesgue probability space and let \(\mathcal{A}_X\) be the trivial sub-\(\sigma\)-algebra of \(B\).

With every measurable partition \(P\) of \(X\) we associate the \(\sigma\)-algebra \(\mathcal{P} \hat{\mathcal{P}}\) of \(P\)-sets, i.e. \(\hat{\mathcal{P}}\) consists of all measurable sums of elements of \(P\). It is well known that for every sub-\(\sigma\)-algebra \(A \subset B\) there is a unique (mod \(\mu\)) measurable partition \(\mathcal{A}\) such that \(\mathcal{A}\) is the \(\sigma\)-algebra of \(\hat{\mathcal{A}}\)-sets.

The symbol \(\mathcal{Z}\) stands for the set of all countable measurable partitions with finite entropy.

Let \(P = \{P_i\} \in \mathcal{Z}\) and let \(\mathcal{A}\) be a sub-\(\sigma\)-algebra of \(B\). The information of \(P\) given \(\mathcal{A}\) is

\[
I(P \mid \mathcal{A}) = - \sum_i \chi_{P_i} \cdot \log \mu(P_i \mid \mathcal{A}).
\]

For a given function \(f \in L^1(X, \mu)\) we put

\[
E(f) = \int_X f \, d\mu.
\]

Let now \(\tau\) be an automorphism of \((X, B, \mu)\). With every partition \(P \in \mathcal{Z}\) we associate the two \(\sigma\)-algebras

\[
P^\tau = \bigvee_{i=1}^{+\infty} \tau^{-i} P, \quad \tau P = \bigvee_{i=-\infty}^{+\infty} \tau^i P.
\]

Let \(T = (T^t)\) be a measurable flow on \((X, B, \mu)\).

A sub-\(\sigma\)-algebra \(A \subset B\) is said to be increasing if \(T^t A \supset A\) for all \(t > 0\).

It is called a factor \(\sigma\)-algebra if \(T^t A = A\) for all \(t \in \mathbb{R}\).

We denote by \(h(T^t)\) and \(\pi(T^t)\) the entropy and the Pinsker \(\sigma\)-algebra of the automorphism \(T^t\) respectively, \(t \in \mathbb{R}\). It is well known ([A2], [G1]) that

\[
h(T^t) = |t| \cdot h(T^1), \quad \pi(T^t) = \pi(T^1), \quad t \in \mathbb{R}.
\]

Recall that the entropy \(h(T)\) and the Pinsker \(\sigma\)-algebra \(\pi(T)\) of the flow \(T\) are defined as follows:

\[
h(T) = h(T^1), \quad \pi(T) = \pi(T^1).
\]

Let now \(\mathcal{H}\) be a fixed factor sub-\(\sigma\)-algebra of \(T\) and let \(T_{\mathcal{H}}\) be the factor flow induced by \(\mathcal{H}\). We denote by \(h(T^t \mid \mathcal{H})\) and \(\pi(T^t \mid \mathcal{H})\) the relative entropy and the relative Pinsker \(\sigma\)-algebra of \(T^t\) with respect to \(\mathcal{H}\) respectively,

\[
t \in \mathbb{R}. \text{ Proceeding similarly to [A1] and [G1] one easily obtains}
\]

\[
h(T^t \mid \mathcal{H}) = |t| \cdot h(T^1 \mid \mathcal{H}), \quad \pi(T^t \mid \mathcal{H}) = \pi(T^1 \mid \mathcal{H}), \quad t \in \mathbb{R}.
\]

We define the relative entropy and the relative Pinsker \(\sigma\)-algebra of \(T\) with respect to \(\mathcal{H}\) as

\[
h(T \mid \mathcal{H}) = h(T^1 \mid \mathcal{H}), \quad \pi(T \mid \mathcal{H}) = \pi(T^1 \mid \mathcal{H})
\]

respectively. Clearly

\[
h(T) = h(T \mid \mathcal{N}) \text{ and } \pi(T) = \pi(T \mid \mathcal{N}).
\]

We shall use in the sequel the following equalities:

\[
(1) \quad h(T) = h(T_\mathcal{H}) + h(T \mid \mathcal{H}),
\]

\[
(2) \quad \pi(T \mid \pi(T)) = \pi(T).
\]

They have been shown for \(\mathbb{Z}\)-actions in [K2] and [K1] respectively.

Now we recall the concept of a special flow built under a function.

Let \((Y, \mathcal{C}, \nu)\) be a Lebesgue probability space, \(\sigma\) be an automorphism of \(Y\) and \(f: Y \to \mathbb{R}^\geq 0\) a measurable function such that \(\inf\{f(y) : y \in Y\} > 0\) and \(f \in L^1(Y, \nu)\). Let \(Y_f = \{(y, u) : y \in Y \times \mathbb{R}^\geq 0 : u < f(y)\}\) and let \(\mathcal{C}_f\) be the restriction of the product \(\sigma\)-algebra \(\mathcal{C} \otimes \mathcal{C}\) to \(Y_f\), where \(\mathcal{C}\) denotes the \(\sigma\)-algebra of Lebesgue sets of \(\mathbb{R}^\geq 0\). We denote by \(\nu_f\) the measure on \(\mathcal{C}_f\) defined by

\[
\nu_f = (E(f))^{-1} \cdot (\nu \times \lambda),
\]

where \(\lambda\) stands for Lebesgue measure. Let \(\tau_f\) be the measurable flow on \((Y_f, \mathcal{C}_f, \nu_f)\) defined as follows. For \(0 \leq t < \inf\{f(y) : y \in Y\}\) we put

\[
\tau_f(y, u) = \begin{cases} (y, u + t) & \text{if } u + t < f(y), \\ (\tau^t y, u + t - f(y)) & \text{if } u + t \geq f(y). \end{cases}
\]

For other values of \(t\) the automorphism \(\tau_f\) is uniquely determined by the condition that \(\tau_f\) is a one-parameter group of automorphisms.

The flow \(\tau_f\) is called the special flow built under the function \(f\), the automorphism \(\tau\) is the base automorphism and \(f\) is the ceiling function of \(\tau_f\).

Since Bernoulli flows will play an essential role in Section 5 we recall their definition and the Ornstein result which we will need.

A flow \(T\) is said to be a Bernoulli flow if for every \(t \neq 0\) the automorphism \(T^t\) is a Bernoulli shift.

The existence of Bernoulli flows has been proved by Ornstein in [O1]. The following “Sinai type” theorem may be easily deduced from the corollary to the Main Lemma from [O2].

**Theorem A.** For every ergodic flow \(T\) with \(h(T) > 0\) and every \(a \in (0, h(T))\) there exists a Bernoulli flow \(S\) which is isomorphic to a factor of \(T\), with \(h(S) = a\).
2. Relative version of the Abramov formula for flows. Let \((Y, \mathcal{C}, \nu)\) be a Lebesgue probability space, \(\tau\) an automorphism of \(Y\) and \(\mathcal{F} \subset \mathcal{C}\) a factor \(\sigma\)-algebra of \(\tau\). Let \(A \in \mathcal{C}\) be a set of positive measure such that \(\bigcup_{n=0}^{\infty} \tau^n A = Y\). It is well known that if \(\tau\) is ergodic then every set \(A\) of positive measure satisfies this condition. The automorphism of \(\mathcal{F}\) induced by \(\tau\) is denoted by \(\tau_{\mathcal{F}}\).

Let \(\mathcal{F}^A\) be the sub-\(\sigma\)-algebra of \(A\) consisting of the sets of the form \(F \cap A\) where \(F \in \mathcal{F}\).

Remark 1. If \(A \in \mathcal{F}\) then \(\mathcal{F}^A\) is a factor \(\sigma\)-algebra of \(\tau_{\mathcal{F}}\). Indeed, the Poincaré recurrence theorem implies that \(A = \bigcup_{m=1}^{\infty} A_m\), where \(A_m\) denotes the set of the \(m\)th return time. The desired property follows at once from the equality

\[
\tau_{\mathcal{F}}(F \cap A) = \bigcup_{m=1}^{\infty} \tau^m(F \cap A \cap \tau^{-1} A \cap \ldots \cap \tau^{-(m-1)} A) \cap A
\]

where \(F \in \mathcal{F}\).

Lemma 1. For every \(A \in \mathcal{F}\) with \(\nu(A) > 0\) we have

\[
h(\tau_A \mid \mathcal{F}^A) = (\nu(A))^{-1} \cdot h(\tau \mid \mathcal{F})\]

We omit the proof because it may be easily obtained from the proof in the absolute case (see [A1]).

Let now \(\sigma\) be an automorphism of \((Y, \mathcal{C}, \nu)\) and let \(\mathcal{F} \subset \mathcal{C}\) be a factor \(\sigma\)-algebra of \(\sigma\). Let \(\mathcal{L}\) denote the \(\sigma\)-algebra of Lebesgue sets of the interval \([0,1]\), \(\lambda\) the Lebesgue measure on \([0,1]\) and \(\varphi : Y \to [0,1]\) a \(\mathcal{C}\)-measurable function.

We consider the product measure space

\[(X, B, \mu) = (Y, \mathcal{C}, \nu) \times ([0,1], \mathcal{L}, \lambda)\]

and the automorphism \(\tau = \tau_{\varphi}\) of \((X, B, \mu)\) defined by

\[
\tau(y, u) = (\sigma y, u + \varphi(y)),
\]

where + means addition mod 1.

We put \(\mathcal{H} = \mathcal{F} \otimes \mathcal{L}\). One easily checks the following

Remark 2. If \(\varphi\) is \(\mathcal{F}\)-measurable then \(\mathcal{H}\) is a factor \(\sigma\)-algebra of \(\tau_{\varphi}\).

Lemma 2. If \(\varphi\) is \(\mathcal{F}\)-measurable, then

\[
h(\tau \mid \mathcal{H}) = h(\sigma \mid \mathcal{F}).
\]

We omit the proof for the same reason as in the case of Lemma 1 (see the Lemma in [A2]).

Let now \((Y_f, \mathcal{C}_f, \nu_f, \tau_f)\) be the special flow over \((Y, \mathcal{C}, \nu, \tau)\) under a function \(f : Y \to \mathbb{R}^+\).

For a \(\sigma\)-algebra \(\mathcal{F} \subset \mathcal{C}\) we denote by \(\mathcal{F}_f\) the restriction of the product \(\sigma\)-algebra \(\mathcal{F} \otimes \mathcal{C}\) to \(Y_f\).

Remark 3 ([B2]). If \(\mathcal{F}\) is a factor \(\sigma\)-algebra for \(\tau\) and \(f\) is \(\mathcal{F}\)-measurable then \(\mathcal{F}_f\) is a factor \(\sigma\)-algebra for \(\tau_f\).

Lemma 3. If \(f\) is \(\mathcal{F}\)-measurable then

\[
h(\tau_f \mid \mathcal{F}_f) = |t| \cdot (B(f))^{-1} \cdot h(\tau \mid \mathcal{F}), \quad t \in \mathbb{R}.
\]

Proof. It is enough to show the equality for \(0 < t < \inf\{f(y) : y \in Y\}\).

We consider, as in the absolute case ([A2]), the product space \(\tilde{Y}_f = Y \times [0, t]\) equipped with the product \(\sigma\)-algebra \(\tilde{\mathcal{F}}_f = \mathcal{F} \otimes \mathcal{L}^t\), where \(\mathcal{L}^t\) denotes the \(\sigma\)-algebra of Lebesgue sets of \([0, t]\), and the natural product measure.

Let \(\tilde{\tau}_f\) be the automorphism of \(\tilde{Y}_f\) defined by

\[
\tilde{\tau}_f(y, u) = (\tau_f y, u + \varphi_f(y)),
\]

where + means addition mod \(t\) and

\[
\varphi_f(y) = t - f(y) + \left[\frac{1}{t} f(y)\right].
\]

As Abramov observed, \(\tilde{\tau}_f\) is the automorphism induced by \(\tau_f\) on \(\tilde{Y}_f\). Therefore, by Remark 1, \(\mathcal{F}_f = \tilde{\mathcal{F}}_f\). It follows from Lemma 1 that

\[
h(\tilde{\tau}_f \mid \tilde{\mathcal{F}}_f) = (\nu(\varphi_f(Y)))^{-1} \cdot h(\tau_f \mid \mathcal{F}_f) = t^{-1} \cdot B(f) \cdot h(\tau_f \mid \mathcal{F}_f).
\]

Applying Lemma 2 we have

\[
h(\tilde{\tau}_f \mid \tilde{\mathcal{F}}_f) = h(\tau \mid \mathcal{F}).
\]

Combining (3) and (4) finishes the proof.

3. Relatively excellent \(\sigma\)-algebras. Let \(\tau\) be an automorphism of a Lebesgue probability space \((Y, \mathcal{C}, \nu)\) and let \(\mathcal{F}\) be a factor \(\sigma\)-algebra of \(\tau\).

Definition 1. A sub-\(\sigma\)-algebra \(A \subset \mathcal{C}\) is said to be relatively excellent for \(\tau\) with respect to \(\mathcal{F}\) if

\[
\begin{align*}
(\text{i}) & \quad \mathcal{F} \subset A, \quad \tau A \supset A, \\
(\text{ii}) & \quad \mathcal{F} \subset A, \quad \tau^n A = C, \quad \text{for some } n \geq 0.
\end{align*}
\]

There exists a sequence \(\{P_n\} \subset \mathcal{C}\) with \(\mathcal{F} \narrow \mathcal{A}\) such that

\[
\lim_{n \to \infty} h(P_n, \tau \mid \mathcal{F}) - H(P_n \mid \mathcal{A}) = 0.
\]

The proof of Theorem 1 of [K2] yields

Lemma 4. For every factor \(\sigma\)-algebra \(\mathcal{F}\) of \(\tau\) there exists a relatively excellent \(\sigma\)-algebra \(A\) with respect to \(\mathcal{F}\). Every such \(\sigma\)-algebra is relatively
perfect with respect to \( \mathcal{F} \), i.e., it also satisfies the following two equalities:

\[
\begin{align*}
\tau^n \mathcal{A} &= \pi(\tau | \mathcal{F}), \\
\mathcal{H}(\tau | \mathcal{F}) &= \mathcal{H}(\tau \mathcal{A} | \mathcal{A}).
\end{align*}
\]

(8) \hspace{2cm} (9)

If \( \mathcal{F} = \mathcal{N} \), then \( \mathcal{A} \) is simply an excellent \( \sigma \)-algebra as defined in [B1].

In the sequel we shall use the relative Pinsker formula ([K2]):

\[
\mathcal{H}(\tau | \mathcal{F}) = \mathcal{H}(\tau \mathcal{A} | \mathcal{A}).
\]

(10)

for \( P, Q \in \mathcal{Z} \), \( \mathcal{H}(P \vee Q, \tau | \mathcal{F}) = \mathcal{H}(P, \tau | \mathcal{F}) + \mathcal{H}(Q, \tau | P \vee \mathcal{F}). \)

**Lemma 5.** If \( \mathcal{A} \subseteq \mathcal{C} \) is relatively excellent with respect to \( \mathcal{F} \) and \( Q \in \mathcal{Z} \) then the \( \mathcal{A} \)-\( \sigma \)-algebra \( \mathcal{A} \vee \tau Q^{-} \) is also relatively excellent with respect to \( \mathcal{F} \).

**Proof.** It is clear that the \( \mathcal{A} \)-\( \sigma \)-algebra \( \mathcal{A} \vee \tau Q^{-} \) satisfies (5) and (6). Let a sequence \( (P_n) \subseteq \mathcal{Z} \) satisfy (7) and let \( Q_n = \bigvee_{m=0}^{\infty} \tau^{-m} Q, n \geq 0 \). We claim that the sequence \( (P_n \vee Q_n) \) also satisfies (7) (for the \( \mathcal{A} \)-\( \sigma \)-algebra \( \mathcal{A} \vee \tau Q^{-} \)).

In the sequel we shall use some ideas from the proof of Proposition 1.3 of [B1].

For any natural numbers \( n, m \), formula (10) gives

\[
\begin{align*}
\mathcal{H}(P_n \vee Q_m, \tau | \mathcal{F}) &= \mathcal{H}(P_n, \tau | \mathcal{F}) + \mathcal{H}(Q_m, \tau | P_n \vee \mathcal{F}) \\
&= \mathcal{H}(P_n, \tau | \mathcal{F}) + \mathcal{H}(Q_m, \tau | (P_n) \vee \mathcal{F}).
\end{align*}
\]

(11)

Assume \( n < N \). Since \( \mathcal{H}(P_n | P_n) < \infty \) there exists a partition \( P_N^n \) with finite entropy such that \( P_n \vee P_N^n = P_n \). Simple properties of the conditional entropy give

\[
\begin{align*}
\mathcal{H}(P_n \vee Q_n | P_N^n \vee Q^{-} \vee \mathcal{F}) &= \mathcal{H}(P_n \vee Q_n | P_N^n \vee Q^{-} \vee \mathcal{F}) \\
&= \mathcal{H}(P_n \vee Q_n | P_N^n \vee Q^{-} \vee \mathcal{F}) - \mathcal{H}(P_N^n | P_n \vee P_N^n \vee Q_n \vee Q^{-} \vee \mathcal{F}) \\
&= \mathcal{H}(P_n | P_N^n \vee \mathcal{F}) + \mathcal{H}(P_N^n | P_N^n \vee P_n \vee \mathcal{F}) \\
&+ \mathcal{H}(Q \vee Q^{-} \vee (P_n) \vee \mathcal{F}) - \mathcal{H}(P_N^n | P_n \vee P_N^n \vee Q_n \vee Q^{-} \vee \mathcal{F}) \\
&\geq \mathcal{H}(P_n | P_N^n \vee \mathcal{F}) + \mathcal{H}(Q \vee Q^{-} \vee (P_n) \vee \mathcal{F}).
\end{align*}
\]

Hence, in view of (11) for \( m = n \), we get

\[
\begin{align*}
0 &\leq \mathcal{H}(P_n \vee Q_n | P_N^n \vee Q^{-} \vee \mathcal{F}) - \mathcal{H}(P_n \vee Q_n | P_N^n \vee Q^{-} \vee \mathcal{F}) \\
&\leq \mathcal{H}(P_n | P_N^n \vee \mathcal{F}) - \mathcal{H}(P_n | P_N^n \vee \mathcal{F}) \\
&+ \mathcal{H}(Q \vee Q^{-} \vee (P_n) \vee \mathcal{F}) - \mathcal{H}(Q \vee Q^{-} \vee (P_n) \vee \mathcal{F}), \quad n < N.
\end{align*}
\]

Since \( \mathcal{A} \) is generating and \( (P_n) \) satisfies (7), it follows that taking the limit, first as \( N \to \infty \), and then as \( n \to \infty \), we obtain the desired result.

**Corollary.** If \( f : Y \to \mathbb{R} \) is measurable with an a.e. finite set of values then for every factor \( \sigma \)-algebra \( \mathcal{F} \) there exists a \( \sigma \)-algebra \( \mathcal{A} \) relatively excellent with respect to \( \mathcal{F} \) such that \( f \) is \( \mathcal{A} \)-measurable.

**Proof.** Let \( \mathcal{D} \) be an arbitrary relatively excellent \( \sigma \)-algebra with respect to \( \mathcal{F} \). Lemma 4 assures that such a \( \sigma \)-algebra exists. Consider the partition \( Q = \{ Q_1, \ldots, Q_n \} \) of \( Y \) into sets where \( f \) is constant. It follows from Lemma 5 that the \( \mathcal{A} \)-\( \sigma \)-algebra \( \mathcal{A} = \mathcal{D} \vee \tau Q^{-} \) is also relatively excellent with respect to \( \mathcal{F} \). It is clear that \( f \) is \( \mathcal{A} \)-measurable.

Now suppose \( g \) is an integrable function on \( Y \) with values in \( \mathbb{N} \). Let \( (Y^0, \mathcal{C}^0, \nu^0, \tau^0) \) be the integral dynamical system over \( (Y, \mathcal{C}, \nu, \tau) \) under the function \( g \) (cf. [CPS]). We denote by \( Q^g \) the partition of \( Y \) generated by \( g \), i.e.,

\[
Q^g = \{ g^{-1}(\{ k \}) : k \in \mathbb{N} \}.
\]

It follows from Lemma 1.1 of [B3] that \( Q^g \in \mathcal{Z} \).

For a given sub-\( \sigma \)-algebra \( \mathcal{F} \subseteq \mathcal{C} \) we denote by \( \mathcal{F}^0 \) the sub-\( \sigma \)-algebra of \( \mathcal{C}^0 \) defined in the same way as \( \mathcal{C}^0 \), i.e.,

\[
A \in \mathcal{F}^0 \iff A = \bigcup_{i=1}^{\infty} A_i \times \{ i \} \quad \text{iff} \quad A_i \in \mathcal{F}, \quad i \in \mathbb{N}.
\]

One easily checks the following

**Remark 4.** If the function \( g \) is \( \mathcal{F} \)-measurable and \( \mathcal{F} \) is a factor \( \sigma \)-algebra of \( \tau \) then \( \mathcal{F}^0 \) is a factor \( \sigma \)-algebra of \( \tau^0 \).

Let \( P = (P_k) \) be a countable measurable partition of \( Y \). We associate with it the partition \( \overline{P} \) of \( Y^0 \) as follows. The atoms of \( \overline{P} \) are all the sets \( P_k \times \{ k \} \), \( k \in \mathbb{N} \), and the set \( Y^0 \setminus \{ Y \times \{ 1 \} \} \).

**Lemma 6.** If \( \mathcal{F} \) is a factor \( \sigma \)-algebra of \( \tau \) such that \( g \) is \( \mathcal{F} \)-measurable then for every \( P \in \mathcal{Z} \) we have

\[
\mathcal{H}(P \vee \tau Q^g, \tau^0 | \mathcal{F}^0) = (E(g))^{-1} \cdot \mathcal{H}(P \vee \tau Q^g, \tau | \mathcal{F}).
\]

**Proof.** Let \( Y_0 = Y \times \{ 1 \} \), \( R = P \vee \tau Q^g \) and \( T = \tau^0 \). It follows easily from the definition of \( T \) that

\[
\begin{align*}
Y_0 \in R \mathcal{T}^0, \\
(R \mathcal{T}^0 \vee \mathcal{F}^0) \cap Y_0 = [(P \vee \tau Q^g) \vee \mathcal{F}] \times \{ 1 \}.
\end{align*}
\]

Hence

\[
\mathcal{H}(Y_0 | R \mathcal{T}^0 \vee \mathcal{F}^0) = \chi_{Y_0}
\]

(13)

(14)

(15)
and

\[ \nu^\varphi(A \times \{1\} \mid R_T^{-} \vee \mathcal{F}^g)(y, 1) = \nu(A \mid (P \vee \varphi \mathcal{Q}^g)^- \vee \mathcal{F})(y), \quad A \in \mathcal{C}, \ y \in Y. \]

From (16) it follows that

\[ \int_{Y^g} I(R \mid R_T^{-} \vee \mathcal{F}^g) \, d\nu^\varphi \]

\[ = - \int_{Y^g} \sum_{A \in P \vee \varphi \mathcal{Q}^g} \chi_A \chi(1)(y, 1) \cdot \log \nu^\varphi(A \times \{1\} \mid R_T^{-} \vee \mathcal{F}^g)(y, 1) \, d\nu^\varphi \]

\[ = (E(g))^{-1} \cdot \int_{Y} I(P \vee \varphi \mathcal{Q}^g \mid (P \vee \varphi \mathcal{Q}^g)^- \vee \mathcal{F})(y) \, d\nu \]

\[ = (E(g))^{-1} \cdot h(P \vee \varphi \mathcal{Q}^g, \tau \mid \mathcal{F}). \]

The equality (15) implies

\[ \int_{Y^g} I(R \mid R_T^{-} \vee \mathcal{F}^g) \, d\nu^\varphi \]

\[ = - \int_{Y^g} \chi_Y \chi(y, 1) \cdot \log \nu^\varphi(Y^g \mid R_T^{-} \vee \mathcal{F}^g)(y, 1) \, d\nu^\varphi = 0. \]

Comparing (17) and (18) one gets

\[ h(R, T \mid \mathcal{F}^g) = \int_{Y^g} I(R \mid R_T^{-} \vee \mathcal{F}^g) \, d\nu^\varphi = (E(g))^{-1} \cdot h(P \vee \varphi \mathcal{Q}^g, \tau \mid \mathcal{F}), \]

which completes the proof.

Now suppose \( f : Y \to \mathbb{R}^+ \) is an integrable function such that

\[ \inf \{ f(y) : y \in Y \} = \alpha > 0. \]

Let \( \tau_f \) be the special flow on the space \((Y_f, C_f, \nu_f)\), built under \( f \) and over \( \tau \).

For a given sub-\( \sigma \)-algebra \( A \subseteq C \) we denote by \( A_f \) the sub-\( \sigma \)-algebra of \( C_f \) defined by

\[ A_f = \{ A \cap Y_f : A \in A \otimes \mathcal{C} \}. \]

Remark 5 \((B2))\). If \( A \subseteq C \) is an increasing sub-\( \sigma \)-algebra for \( \tau \) and \( f \) is \( A \)-measurable then \( A_f \) is increasing for the flow \( \tau_f \).

Let \( Y_1 = Y \times [0, 1] \). With any measurable partition \( P \) of \( Y \) we associate the partition \( P^1 = P \times [0, 1] \) of \( Y_1 \). For a sub-\( \sigma \)-algebra \( A \subseteq C \) put \( A^1 = A \otimes \mathcal{L}^1 \).

Let \( R_k = \{ R_{k,i} : 0 \leq i < 2^{k-1} \} \) be the partition of \( Y_1 \) defined by

\[ R_{k,i} = \{ (y, u) \in Y_1 : i \cdot 2^{k-1} \leq u < (i+1) \cdot 2^{k-1} \}, \quad 0 \leq i < 2^{k-1} - 1, \ k \geq 1. \]

It is clear that the smallest \( \sigma \)-algebra \( \mathcal{R} \) containing all \( R_k \), \( k \geq 1 \), coincides with the \( \sigma \)-algebra \( \mathcal{N} \otimes \mathcal{L}^1 \).

Let \( \tau_1 \) be the automorphism of \( Y_1 \) induced by \( T^1 = \tau_f^1 \). It follows from Abramov’s remark (see the proof of Lemma 3) that

\[ \tau_1(y, u) = (\tau u, u + \tau(y)), \]

where \( \varphi(y) = 1 - \{ \tau(y) \} \). The \( \sigma \)-algebra \( C \otimes \mathcal{N} \otimes \mathcal{L}^1 \) is a principal factor \( \sigma \)-algebra of \( \tau_1 \) and the corresponding factor automorphism is isomorphic to \( \tau \).

Since \( \tau_1 \) is induced by \( T^1 \) on the set \( Y_1 \), \( T^1 \) is an integral automorphism over \( \tau_1 \). Let \( g \) be the corresponding ceiling function on \( Y_1 \), i.e., the Poincaré cocycle for \( \tau_1 \). If the flow \( \tau_f \) is ergodic then the well-known Katz theorem implies \( \int_{Y_1} g \, d\nu_f = 1 \), i.e.

\[ E(g) = \int_{Y_1} g(y, u) \, d\nu \, du = E(f). \]

For a given measurable partition \( Q \) of \( Y \) we denote by \( \overline{Q} \) the partition of \( Y_f \) which consists of all atoms of \( Q \) and the set \( Y_f \). If \( \mathcal{D} \) is a sub-\( \sigma \)-algebra of \( Y_f \), \( \overline{\mathcal{D}} \) stands for the \( \sigma \)-algebra \( \overline{\mathcal{D}} = \{ A \cup Y_f : A \in \mathcal{D} \} \).

Let \( \mathcal{F} \) be a factor \( \sigma \)-algebra of \( \tau \).

Lemma 7. If \( A \subseteq C \) is a relatively excellent for \( \tau \) with respect to \( \mathcal{F} \) and \( f \) is \( A \)-measurable then \( A_f \) is

(19) \( f \) increasing for the flow \( \tau_f \),

(20) \( f \) relatively excellent for \( \tau_f^1 \) with respect to \( \mathcal{F} \).

Proof. Since \( A \subseteq \mathcal{F} \) and \( A \) is generating, \( A_f \) is of course generating and \( A_f \supseteq \mathcal{F} \).

We may assume \( \alpha = 1 \). Put \( T^1 = \tau_f^1 \) as above. It is easy to check that \( T^1 A_f = (\overline{\tau A})^1 \vee \mathcal{R} \). Since, by Remark 6, \( A_f \) is increasing with respect to \( T^1 \), the above equality implies

\[ A_f = (\overline{\tau A})^1 \vee \mathcal{R}. \]

Let \( (P_n) \) be a sequence of partitions of \( Y \) with finite entropy such that

\[ \lim_{n \to \infty} (h(p_n, \tau \mid \mathcal{F}) - H(P_n \mid \mathcal{A}^1 \vee \mathcal{F})) = 0. \]

Let \( Q_n,k = (\tau p_n)^1 \vee R_k \), \( n, k \geq 1 \). We shall show that there exists an increasing sequence \( (u_k) \) of natural numbers such that

\[ h(\overline{Q}_{n,k}, T_f^1 \mid \mathcal{F}_f) - H(\overline{Q}_{n,k} \mid A_f \vee \mathcal{F}) \to 0 \]

as \( k \to \infty \). Since \( \overline{Q}_{n,k} \not\subseteq (\overline{\tau A})^1 \vee \mathcal{R} \) the equality (21) implies that \( A_f \) is relatively excellent for \( T^1 \) with respect to \( \mathcal{F} \). One easily checks the following.
equations:
\[ \nu_f(\tau A \times [0,1] \mid A_f \vee F_f)(y, u) = \nu(\tau A \mid A \vee F)(y) \]
and
\[ \nu_f(Y_1 \mid A_f \vee F_f) = \chi_{Y_1}, \quad (y, u) \in Y_1, \ A \in \mathcal{P}, \ n \geq 1. \]
From these equalities a straightforward computation yields
\[ H((\tau P_n)^1 \mid A_f \vee F_f) = (E(f))^{-1} \cdot H(\tau P_n \mid A \vee F). \tag{22} \]
For every set \( A \in \mathcal{C} \) we have
\[ \tau_f^{-1}(A \times [0,1]) = \tau^{-1}(A) \times [0,1], \]
\[ (\nu \times \lambda)(A \times [0,1] \mid D \otimes \mathcal{L}^1) = \nu(A \mid D), \]
where \( D \) is an arbitrary sub-\( \sigma \)-algebra of \( C \). Therefore we get
\[ h((\tau P_n)^1, \tau_f \mid F \otimes \mathcal{L}^1) = h(P_n, \tau \mid F) \tag{23} \]
and
\[ H((\tau P_n)^1 \mid [\tau(A)^1]_{\tau_f} \vee F \otimes \mathcal{L}^1) = H(P_n, A^{-} \vee F). \tag{24} \]
We have
\[ h(\overline{Q}_{n,k}, T^1 \mid F_f) - H(\overline{Q}_{n,k} \mid A_f \vee F_f) = a_{n,k} + b_{n,k} + c_{n,k}, \]
where
\[ a_{n,k} = h(\overline{Q}_{n,k}, T^1 \mid F_f) - (E(g))^{-1} \cdot h(Q_{n,k} \vee \tau Q^q, \tau_1 \mid F \otimes \mathcal{L}^1), \]
\[ b_{n,k} = (E(g))^{-1} \cdot h(Q_{n,k} \vee \tau Q^q, \tau_1 \mid F \otimes \mathcal{L}^1) \]
\[ - (E(f))^{-1} \cdot h((\tau P_n)^1, \tau_1 \mid F \otimes \mathcal{L}^1), \]
\[ c_{n,k} = (E(f))^{-1} \cdot h((\tau P_n)^1, \tau_1 \mid F \otimes \mathcal{L}^1) - H(\overline{Q}_{n,k} \mid A_f \vee F_f), \]
for \( n, k \geq 1 \). By Lemma 6 we have
\[ a_{n,k} = h(\overline{Q}_{n,k}, T^1 \mid F_f) - h(\overline{Q}_{n,k} \vee \tau Q^q, T^1 \mid F_f) \leq 0. \]
It follows from the relative Pinsker formula (10) and the equality \( E f = E g \) that
\[ b_{n,k} = (E(f))^{-1} \cdot h(R_k \vee \tau Q^q, \tau_1 \mid (\tau P_n)^1) \tau_1. \]
Since \( R_k \subset \mathcal{A} \) we have, by (22)-(24),
\[ c_{n,k} = (E(f))^{-1} \cdot h(\tau P_n \mid F \vee F_f) = (E(f))^{-1} \cdot h(P_n, \tau \mid F) \]
\[ = (E(f))^{-1} \cdot h(P_n, \tau \mid F) - H(P_n, A^{-} \vee F), \quad n, k \geq 1. \]
Since \( \overline{P}_n \not\subset \mathcal{A} \) and \( \mathcal{A} \) is generating we have \([\tau P_n]_{\tau_f} = [P_n]_{\tau_f} \not\subset \mathcal{C} \otimes \mathcal{L}^1 \). But \( \mathcal{C}^1 \) is a principal factor \( \sigma \)-algebra for \( \tau_1 \) so \( \lim_{n \to \infty} b_{n,k} = 0 \) for every \( k \geq 1 \). Therefore there exists an increasing sequence \((n_k)\) of natural numbers such that \( \lim_{n \to \infty} b_{n_k,k} = 0 \). Thus
\[ \lim_{k \to \infty} (h(\overline{Q}_{n_k,k}, T^1 \mid F_f) - H(\overline{Q}_{n_k,k} \mid A_f \vee F_f)) = 0, \]
i.e. \( A_f \) is relatively excellent for \( T^1 \) with respect to \( F_f \), which completes the proof.

4. Relatively perfect \( \sigma \)-algebras. Our proof of Theorem B below requires a relative version of the well-known Ambrose–Kakutani–Rudolph (AKR) theorem ([AK], [Rud]).

Lemma 8. For every ergodic flow \( T \) on a Lebesgue probability space \((X, \mathcal{B}, \mu)\), given a nonatomic factor \( \sigma \)-algebra \( \mathcal{H} \) of \( T \) and two positive real numbers \( p \) and \( q \) with \( p/q \) irrational, there exists a special flow \((Y_f, \mathcal{C}_f, \nu_f, \tau_f)\), where \( f \) is a measurable function with values \( p \) and \( q \), a factor \( \sigma \)-algebra \( \mathcal{F} \) for \( \tau \) such that \( f \) is \( \mathcal{F} \)-measurable and an isomorphism \( \varphi : X \to Y_f \) of the flows \( T \) and \( \tau_f \) such that \( \varphi(\mathcal{H}) = \mathcal{F} \).

Proof. Let \( \xi \) be a measurable partition of \( X \) associated with \( \mathcal{H} \), i.e. \( \xi = \overline{H} \). It is clear that \( T^1 \xi = \xi, \ t \in \mathbb{R} \). We consider the quotient Lebesgue space \((X/\xi, \mathcal{B}_\xi, \mu_\xi)\) equipped with the quotient flow \( T_\xi \). We denote by \( H_\xi : X \to X/\xi \) the natural homomorphism. It follows from the proof of the AKR theorem ([cf. [CFS]]) that there exists a measurable partition \( \zeta_\xi \) of \( X/\xi \) which is regular for \( T_\xi \), i.e.
\[ \zeta_\xi \text{ is a measurable partition of } X/\xi \text{ into intervals of trajectories with lengths } p \text{ and } q, \ i.e. \text{ sets of the form } \{T_\xi^t C : 0 \leq t \leq \tilde{f}(C)\}, \text{ where } \tilde{f}(C) = p \text{ or } \tilde{f}(C) = q, \]
\[ \text{the functions } F, C \text{ defined by } F(D) = \tilde{f}(C), C(D) = t, \text{ where } D = T_\xi^t (C), \text{ are } \mathcal{B}_\xi \text{-measurable}. \]

Now we define a measurable partition \( \zeta \) of \( X \) which is regular for \( T \). Let \( E \) be an element of \( \zeta_\xi \) and let \( \mathcal{C}_E (E) \subset X/\xi \), \( \mathcal{C}_E (E) \subset E \) denoting the beginning of the trajectory of \( T_\xi \) included in \( E \). We denote by \( \zeta \) the partition of \( X \) consisting of the following intervals of trajectories of \( T \):
\[ \{T^t x : x \in \mathcal{C}_E^{-1} (E)\} \]
where \( E \subset \zeta_\xi \).

It is easy to check that \( \zeta \) is regular for \( T \) with the same lengths \( p \) and \( q \) of trajectories.

Now we construct the desired probability space \((Y, \mathcal{C}, \nu)\) and the automorphism \( \tau \) of \( Y \) in the same way as in the proof of the AKR theorem. Recall that \( Y \) is the set of left ends of elements of \( \zeta_\xi \), i.e. the points belonging to \( H_\xi^{-1} \mathcal{C}_E (E) \), \( E \in \zeta_\xi \).
Let $\eta$ be the measurable partition of $Y$ whose elements are the sets $H_{\xi}^{-1}C_{0}(E), \ E \in \xi$, and let $\mathcal{F}$ be the $\sigma$-algebra of $\eta$-sets.

It is clear that $\mathcal{F}$ is a factor $\sigma$-algebra of $\tau$. For every $y \in Y$, $y \in H_{\xi}^{-1}(C), \ C = C_{0}(E), \ E \in \xi$, the length $f(y)$ of the trajectory of $y$ is equal to $\widetilde{f}(C)$. Hence $f$ is $\mathcal{F}$-measurable.

Denoting by $\varphi$ the isomorphism between $X$ and $Y$ defined in [CFS] we obtain the equality $\varphi(\eta) = \mathcal{F}$.

Let $T$ be a measurable flow on a Lebesgue space $(X, B, \mu)$ and let $\mathcal{H}$ be a factor $\sigma$-algebra of $T$.

**Definition 2.** A sub-$\sigma$-algebra $A \subset B$ is said to be relatively perfect with respect to $\mathcal{H}$ if

(i) $A \supseteq \mathcal{H}$, $T^{t}A \supseteq \mathcal{H}$, $t > 0$,
(ii) $\bigvee_{t \in \mathbb{R}} T^{t}A = B$,
(iii) $\bigcap_{t \in \mathbb{R}} T^{t}A = \pi(T \mid \mathcal{H})$,
(iv) $h(T^{t} \mid \mathcal{H}) = H(T^{t}A \mid A)$, $t > 0$.

In the case $\mathcal{H} = \mathcal{N}$ the concept of a relatively perfect $\sigma$-algebra reduces to the concept of a perfect $\sigma$-algebra ([B1], [G2]).

**Theorem B.** For every ergodic flow $T$ and a Lebesgue space $(X, B, \mu)$ and every factor $\sigma$-algebra $\mathcal{H}$ of $T$ there exists a relatively perfect $\sigma$-algebra with respect to $\mathcal{H}$.

**Proof.** We may assume that $\mathcal{H}$ is nonatomic. Indeed, in the opposite case, due to the ergodicity of $T$, $\mathcal{H}$ is finite, therefore $\mathcal{H} \subset \pi(T)$. Then it is easy to show, using formulas (1) and (2), that any perfect $\sigma$-algebra $A$ for $T$ (such $\sigma$-algebras exist by [B1], [G2]) is also relatively perfect with respect to $\mathcal{H}$.

Suppose now that $\mathcal{H}$ is nonatomic. Due to Lemma 8 we may assume that $X = Y_{f}, \mathcal{S} = \mathcal{C}_{f}, \mu = \nu_{f}, T^{t} = \tau_{f}$ and $\mathcal{H} = \mathcal{F}_{f}$, where $\mathcal{F}$ is a factor $\sigma$-algebra of the automorphism $\tau$ of $(Y, \mathcal{C}, \nu)$ and $f$ is a $\mathcal{F}$-measurable function with two values. We put $\alpha = \min\{f(y) : y \in Y\}$.

In view of the corollary to Lemma 5 there exists a relatively acceptable $\alpha$-algebra $D \subset C$ for $\tau$ with respect to $\mathcal{F}$ such that $f$ is $D$-measurable.

We put $A = D_{f}$. The $D$-measurability and the conditions (5) and (6) of relatively acceptable $\sigma$-algebras imply that

$$A \supseteq \mathcal{H}, \quad T^{t}A \supseteq A, \quad t > 0, \quad \bigvee_{t \in \mathbb{R}} T^{t}A = B.$$ 

Applying Lemma 3.1 of [G1] and the equality (9) we get

$$h(T^{t} \mid \mathcal{H}) = t \cdot (E(f))^{-1} \cdot h(\tau \mid D) = t \cdot (E(f))^{-1} \cdot h(\tau \mid \mathcal{F}), \quad t > 0.$$ 

On the other hand, Lemma 3 gives

$$h(T^{t} \mid \mathcal{H}) = h(\tau_{f} \mid \mathcal{F}_{f}) = t \cdot (E(f))^{-1} \cdot h(\tau \mid \mathcal{F}), \quad t > 0.$$ 

Therefore we have

$$h(T^{t} \mid \mathcal{H}) = H(T^{t}A \mid A), \quad t > 0.$$ 

It follows from Lemma 7 that $\mathcal{A}$ is relatively excellent for $T^{\alpha} = \tau_{f}$ with respect to $\mathcal{H}$. Applying the equality (8) to $T^{\alpha}$ we get

$$\bigcap_{t \in \mathbb{R}} T^{t}A = \bigcap_{t = -\infty}^{\infty} T^{t} \mathcal{A} = \pi(T^{\alpha} \mid \mathcal{H}) = \pi(T \mid \mathcal{H}),$$

which completes the proof.

5. **Principal factors and an axiomatic definition of entropy.** Let $T = (T^{t})$ be a measurable flow on a Lebesgue space $(X, B, \mu)$.

**Definition 3.** A factor $\sigma$-algebra $\mathcal{H}$ of $T$ is said to be principal if every increasing $\sigma$-algebra $A \supseteq \mathcal{H}$ is a factor $\sigma$-algebra.

**Definition 4.** A factor flow $S = (S^{t})$ of $T$ is said to be principal if every factor $\sigma$-algebra $\mathcal{H}$ of $T$ such that the flows $T_{\mathcal{H}}$ and $S$ are isomorphic is principal.

**Lemma 9.** If a flow $S$ is a principal factor of $T$ then $h(T) = h(S)$. Conversely, if $h(T) < \infty$ then the reverse implication is also true.

**Proof.** Let $\mathcal{H}$ be a principal $\sigma$-algebra such that $S$ and $T_{\mathcal{H}}$ are isomorphic. It follows from Theorem B that there exists an increasing $\sigma$-algebra $A \supseteq \mathcal{H}$ with

$$h(T^{t} \mid \mathcal{H}) = H(T^{t}A \mid A), \quad t > 0.$$ 

It follows from the assumption that $h(T^{t} \mid \mathcal{H}) = 0, \ t > 0$. Therefore the formula (1) implies

$$h(T) = h(T_{\mathcal{H}}) = h(S).$$

Now suppose $h(T) < \infty$ and $h(T) = h(S)$. Let $\mathcal{H}$ be a factor $\sigma$-algebra such that $S$ and $T_{\mathcal{H}}$ are isomorphic. Therefore we have $h(T) = h(T_{\mathcal{H}})$, i.e.

$$h(T \mid \mathcal{H}) = 0, \text{ let } A \supseteq \mathcal{H} \text{ be increasing. Since}$$

$$H(T^{t}A \mid A) = H(A \mid T^{-t}A) \leq h(T^{t} \mid \mathcal{H})$$

we have $H(T^{t}A \mid A) = 0, \ t > 0$, and so $A$ is a factor $\sigma$-algebra.

Let now $\tau$ be an automorphism of a Lebesgue space $(Y, C, \nu)$ and $f : Y \to \mathbb{R}^{+}$ a measurable function with $\inf(f(y) : y \in Y) > 0$. From Lemma 9 and Abramov's formula ([A]) for the entropy of a special flow one obtains at once the following
Corollary. If an automorphism $\sigma$ of $(Y, C, \nu)$ is a principal factor of $\tau$ then the special flow $\sigma_f$ is a principal factor of $\tau_f$.

Let $X$ denote the set of all ergodic flows on Lebesgue probability spaces. We denote by $T_0$ the flow defined as follows (cf. [O2]). Let $\tau$ be a Bernoulli 2-shift which acts on a Lebesgue space $(Y, C, \nu)$. Let $P = \{A, B\}$ be an independent generating partition of $Y$ for $\tau$ and let

$$f = pA + qB,$$

where $p$ and $q$ are positive reals such that $p + q = 2$ and $pq^{-1}$ is irrational.

We define $T_0 = (T_0)^{\#$} as the flow built under $f$ with base automorphism $\tau$. It follows from [O1] that $T_0$ is a Bernoulli flow. The Abramov formula implies

$$h(T_0) = (E(f))^{-1} \cdot h(\tau) = \log 2.$$

Applying the Ornstein isomorphism theorem for Bernoulli flows ([O2]) and Lemma 9 one may prove, using Rokhlin’s idea (cf. [Ro]), the following

**Proposition.** Let $H : \text{Act} \to [0, +\infty]$ be a function such that $H(T_0)$ = $\log 2$ and for all $T, S \in \text{Act}$ the following conditions are satisfied:

(i) if $S$ is a factor of $T$ then $H(T) \geq H(S)$,

(ii) if $S$ is a principal factor of $T$ then $H(T) = H(S)$,

(iii) $H(T \times S) = H(T) + H(S)$.

Then $H(T) = h(T)$ for all $T \in \text{Act}$.

References


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