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Martingale operators and Hardy spaces generated by them

by

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Abstract. Martingale Hardy spaces and BMO spaces generated by an operator T are investigated. An atomic decomposition of the space H_p^T is given if the operator T is predictable. We generalize the John–Nirenberg theorem, namely, we prove that the BMO_q spaces generated by an operator T are all equivalent. The sharp operator is also considered and it is verified that the L_p norm of the sharp operator is equivalent to the H_p^T norm. The interpolation spaces between the Hardy and BMO spaces are identified by the real method. Martingale inequalities between Hardy spaces generated by two different operators are considered. In particular, we obtain inequalities for the maximal function, for the q-variation and for the conditional q-variation. The duals of the Hardy spaces generated by these special operators are characterized.

1. Introduction. We consider martingale operators like Burkholder and Gundy did in their paper [10]. In the literature Hardy spaces generated by the maximal function or by the quadratic variations were dealt with. In this paper Hardy and BMO spaces generated by an operator T are investigated. Several new results are proved and many known results for the maximal function and quadratic variations are generalized to the case of an arbitrary operator T.

In Section 2 the basic definitions are given. In Section 3 the atoms are defined and the atomic decomposition of the H_p^T space generated by a predictable operator T is formulated. Two special cases of this result can be found in Herz [18] for the $\mathcal{P}_1$ space and in Weisz [35] for the conditional quadratic variation.

In the next section the sharp operator $T^\sharp$ of an operator T is introduced. The $\mathcal{BMO}_q$ spaces are defined and then generalized by considering the L_∞ norm of $T_q^\sharp$. The latter spaces are denoted by $\mathcal{BMO}_q^T$. We generalize the John–Nirenberg theorem [22] (see also Herz [18], Garsia [16]) and show that

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the $\mathcal{BMO}_q^T$ spaces are all equivalent for $0 < q < \infty$ (Theorem 2). In Theorem 3 it is proved that the L_p norm of $T_q^\sharp(f)$ is equivalent to the H_p^T norm of f $(0 < q < p < \infty)$. Some very special cases of this result can be found in Fefferman and Stein [15], Garsia [16] and Lepingle [24].

In Section 5 the interpolation spaces between the Hardy and BMO spaces are identified by the real method. We verify that

$$(H_{p_0}^T, X^T)_{\theta,q} = H_{p,q}^T, \qquad \frac{1}{p} = \frac{1-\theta}{p_0},$$

where $X^T = H_\infty^T$ or $X^T = \mathcal{BMO}^T$, $0 < \theta < 1$, $0 < q \le \infty$ and $0 < p_0 < \infty$ if T is predictable and $1 \le p_0 < \infty$ if T is adapted. Some special cases of this result can be found in Fefferman, Rivière and Sagher [14], Rivière and Sagher [29], Hanks [17] and Weisz [34].

In Section 6 martingale inequalities are verified. We show that if an inequality holds for a number p, then, by the atomic decomposition and interpolation, it also holds for all parameters less than p (Theorem 12 and Corollary 5). As special operators the maximal operator M, the q-variation S_q and the conditional q-variation s_q are considered. The well-known Burkholder–Davis–Gundy inequality is obtained from the general results. An inequality relative to the strong q-variation due to Lepingle [23] and Pisier and Xu [27] is proved.

In Section 7 the duals of the Hardy spaces $H_p^{S_q}$ and $H_p^{s_q}$ generated respectively by the q-variation and conditional q-variation are characterized. More exactly, the dual of $H_p^{S_q}$ is $H_{p'}^{S_{q'}}$ and the dual of $H_p^{s_q}$ is $H_{p'}^{s_{q'}}$ $(1 < p, q < \infty,\ 1/p + 1/q = 1/p' + 1/q' = 1)$. The duals of $H_1^{s_q}$ and $H_1^{S_q}$ are $\mathcal{BMO}_{q'}$ and $\mathcal{BMO}_{q'}^-$, respectively, where $1 \le q < \infty$ and $1/q + 1/q' = 1$. These duality results are known for $q = 2$ (see Garsia [16], Herz [18], [19], Pratelli [28], Weisz [35]). The third duality result is due to Lepingle [24]. As a consequence we get a generalization of an inequality due to Rosenthal [30] and Burkholder [8]. Furthermore, we show that the duals of $\mathcal{VMO}_{q'}$ and $\mathcal{VMO}_{q'}^-$, which are subspaces of $\mathcal{BMO}_{q'}$ and $\mathcal{BMO}_{q'}^-$, are $H_1^{s_q}$ and $H_1^{S_q}$ $(1 < q' < \infty, 1/q + 1/q' = 1)$, respectively.

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2. Preliminaries and notations. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathcal{F} = (\mathcal{F}_n, n \in \mathbb{N})$ be a non-decreasing sequence of σ-algebras. The σ-algebra generated by an arbitrary set system $\mathcal{H}$ will be denoted by $\sigma(\mathcal{H})$. We suppose that $\mathcal{A} = \sigma(\bigcup_{n\in\mathbb{N}}\mathcal{F}_n)$.

The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ $(n \in \mathbb{N})$ are denoted by E and E_n, respectively. We briefly write

L_p for the real or complex $L_p(\Omega, \mathcal{A}, P)$ space; the norm (or quasinorm) of this space is defined by $\|f\|_p := (E|f|^p)^{1/p}$ $(0 < p \le \infty)$. For simplicity, we assume that for every function $f \in L_1$ and every martingale $f = (f_n, n \in \mathbb{N})$ we have $E_0 f = 0$ and $f_0 = 0$, respectively.

The stochastic basis $\mathcal{F}$ is said to be *regular* if there exists a number $R > 0$ such that for every non-negative and integrable function f

$$E_n f \le R E_{n-1} f \quad (n \in \mathbb{N}).$$

We define $E_{-1} := E_0$. The simplest example of a regular stochastic basis is the sequence of dyadic σ-algebras, where $\Omega = [0, 1)$, $\mathcal{A}$ is the σ-algebra of Borel measurable sets, P is the Lebesgue measure and

$$\mathcal{F}_n = \sigma\{[k2^{-n}, (k+1)2^{-n}) : 0 \le k < 2^n\}.$$

In this paper the constants C_p depend only on p and may be different in different places.

We define the *martingale differences* as follows:

$$d_0 f := 0, \quad d_n f := f_n - f_{n-1} \quad (n \ge 1).$$

The concept of a *stopped martingale* is well known in martingale theory: if ν is a stopping time (briefly $\nu \in \mathcal{T}$) and f is a martingale then the stopped martingale $f^\nu = (f_n^\nu, n \in \mathbb{N})$ is defined by

$$f_n^\nu := \sum_{k=0}^{n} \chi(\nu \ge k) d_k f,$$

where $\chi(A)$ is the characteristic function of a set A. f_n^ν has the property that $f_n^\nu = f_m$ on the set $\{\nu = m\}$ whenever $n \ge m$. In particular, in case $\nu = n$ for any $n \in \mathbb{N}$ one has $f^n = (f_0, f_1, \ldots, f_n, f_n, \ldots)$. Moreover, define

$$f_n^{\nu^-} := \sum_{k=0}^{n} \chi(\nu > k) d_k f.$$

For these functions one has $f_n^{\nu^-} = f_{m-1}$ on the set $\{\nu = m\}$ whenever $n \ge m - 1$.

We shall consider the following special martingale operators. The *maximal function* of a martingale $f = (f_n, n \in \mathbb{N})$ is denoted by

$$f_n^* := \sup_{k \le n} |f_k|, \qquad f^* := \sup_{k\in\mathbb{N}} |f_k|.$$

The *q-variation* $S_q(f)$ and the *conditional q-variation* $s_q(f)$ $(1 \le q < \infty)$ of a martingale f are defined by

$$S_{q,n}(f) := \Big(\sum_{k=0}^{n} |d_k f|^q\Big)^{1/q}, \qquad S_q(f) := \Big(\sum_{k=0}^{\infty} |d_k f|^q\Big)^{1/q},$$

and

$$s_{q,n}(f) := \Big(\sum_{k=0}^{n} E_{k-1}|d_kf|^q\Big)^{1/q}, \quad s_q(f) := \Big(\sum_{k=0}^{\infty} E_{k-1}|d_kf|^q\Big)^{1/q},$$

while for $q = \infty$ let

$$S_{\infty,n}(f) := s_{\infty,n}(f) := \sup_{k\le n} |d_kf|, \quad S_\infty(f) := s_\infty(f) := \sup_{k\in\mathbb{N}} |d_kf|.$$

Usually the 2-variations are dealt with; however, in the papers of Lepingle [23], [24] and Pisier and Xu [27] the q-variations are also considered.

Following Burkholder and Gundy [10] we investigate more general martingale operators T that map the set of martingales stopped by n for any $n \in \mathbb{N}$ into the set of non-negative $\mathcal{A}$-measurable functions. Throughout the paper we will assume the following conditions:

(B1) T is subadditive, i.e. if $f = \sum_{k=0}^{\infty} f_k$ in the sense of $f_m = \sum_{k=0}^{\infty} f_{k,m}$ a.e. for all $m \in \mathbb{N}$, then $T(f^n) \le \sum_{k=0}^{\infty} T(f_k^n)$ $(n \in \mathbb{N})$ where f_k $(k \in \mathbb{N})$ are martingales.

(B2) T is homogeneous, i.e. $T(cf) = |c|T(f)$.

(B3) T is local, i.e. $T(f) = 0$ on the set $\{s_2(f) = 0\}$.

(B4) T is symmetric, i.e. $T(f) = T(-f)$.

Note that our condition (B1) is slightly stronger than the one in Burkholder and Gundy [10]. These operators were also investigated in Hitczenko [20].

For every martingale f we define $T_n(f) := T(f^n)$ $(n \in \mathbb{N})$, $T^*(f) := \sup_{n\in\mathbb{N}} T_n(f)$ and suppose that $T_0(f) = 0$. Under these conditions the operator T has some natural properties. For example, $T(f-g) \le T(f) + T(g)$ and $T(f^\mu - f^\nu) = 0$ on the set $\{\mu = \nu\}$. Moreover, if we set $T_\nu(f) = T_n(f)$ on $\{\nu = n\}$ where $\nu \in \mathcal{T}$ is a finite stopping time, then we have $T_\nu(f) = T(f^\nu)$. It is easy to see that the operator T^* also satisfies all the above conditions. For more details and examples we refer to Burkholder and Gundy [10].

An operator T is said to be *adapted* (resp. *predictable*) if $T_n(f)$ is $\mathcal{F}_n$- (resp. $\mathcal{F}_{n-1}$-) measurable for all martingales f and for all $n \in \mathbb{N}$. If $M(f^n) := |f_n|$ then $M_n^*(f) = f_n^*$ and $M^*(f) = f^*$ $(n \in \mathbb{N})$. One can easily check that the operators M, S_q and s_q $(1 \le q < \infty)$ satisfy the condition (B); moreover, M and S_q are adapted, and s_q is predictable.

The *predictable operator* of an operator T satisfying (B) is now introduced. We consider all the non-decreasing, non-negative and predictable sequences $\lambda = (\lambda_n, n \in \mathbb{N})$ of functions for which

$$T_n(f) \le \lambda_n \quad (n \in \mathbb{N}).$$

Set

$$T_n^-(f) := \inf_\lambda \lambda_n \quad (n \in \mathbb{N}), \quad T^-(f) := \sup_{n\in\mathbb{N}} T_n^-(f).$$

One can easily prove that T^- satisfies (B) and is predictable, and moreover, that $T_n^-(f)$ is non-decreasing in n. We remark that $T^-(f)$ is not necessarily finite a.e. whenever $T^*(f)$ is. T^- was introduced for the maximal operator by Garsia [16] and for S_2 by Weisz [35].

The *martingale Hardy space* H_p^T $(0 < p \le \infty)$ generated by T is the space of martingales for which

$$\|f\|_{H_p^T} := \|T^*(f)\|_p < \infty.$$

3. Atomic decomposition. The atomic decomposition is a useful characterization of Hardy spaces used in proving some duality theorems, martingale inequalities and interpolation results.

Let us introduce first the concept of an atom:

DEFINITION 1. A martingale a is a *p-atom relative to an operator* T if there exists a stopping time ν such that

(i) $a_n = 0$ if $\nu \ge n$,

(ii) $\|T^*(a)\|_\infty \le P(\nu \ne \infty)^{-1/p}$.

Note that atomic decompositions have already been investigated for special operators: for M^- see Herz [18], Bernard and Maisonneuve [5] and Chevalier [11], for S_2^- and s_2 see Weisz [35].

THEOREM 1. *Assume that T is a predictable operator. If the martingale $f = (f_n, n \in \mathbb{N})$ is in H_p^T $(0 < p < \infty)$ then there exist a sequence $(a_k, k \in \mathbb{Z})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{Z})$ of real numbers such that for all $n \in \mathbb{N}$,*

$$\sum_{k=-\infty}^{\infty} \mu_k a_{k,n} = f_n \tag{1}$$

and

$$\Big(\sum_{k=-\infty}^{\infty} |\mu_k|^p\Big)^{1/p} \le C_p \|f\|_{H_p^T}. \tag{2}$$

Conversely, if $0 < p \le 1$ and the martingale f has a decomposition of type (1) *then $f \in H_p^T$ and*

$$\|f\|_{H_p^T} \sim \inf \Big(\sum_{k=-\infty}^{\infty} |\mu_k|^p\Big)^{1/p}, \tag{3}$$

where the infimum is taken over all decompositions of f of the form (1) *and $\sim$ denotes equivalence of norms.*

Proof. Assume that $f \in H_p^T$. Consider the following stopping times for all $k \in \mathbb{Z}$:

$$\nu_k := \inf\{n \in \mathbb{N} : T_{n+1}^*(f) > 2^k\}.$$

It is easy to see that

$$f_n = \sum_{k=-\infty}^{\infty} (f_n^{\nu_{k+1}} - f_n^{\nu_k}). \tag{4}$$

Let

$$\mu_k := 2^k 3 P(\nu_k \neq \infty)^{1/p} \quad \text{and} \quad a_{k,n} := \frac{f_n^{\nu_{k+1}} - f_n^{\nu_k}}{\mu_k}.$$

It is clear that, for a fixed k, $a_k := (a_{k,n}, n \in \mathbb{N})$ is a martingale. Since T is local, we have

$$T^*(a_k) \le P(\nu_k \neq \infty)^{-1/p}.$$

If $n \le \nu_k$ then $a_{k,n} = 0$, thus we see that a_k is really a p-atom. As usual, by Abel rearrangement we get

$$\sum_{k=-\infty}^{\infty} |\mu_k|^p = 3^p \sum_{k=-\infty}^{\infty} 2^{kp} P[T^*(f) > 2^k] \le C_p E[T^*(f)^p], \tag{5}$$

which proves (2).

Assume that $0 < p \le 1$ and f has a decomposition of the form (1). By (B1),

$$E[T^*(f)^p] \le \sum_{k=-\infty}^{\infty} |\mu_k|^p E[T^*(a_k)^p]. \tag{6}$$

It follows from the definitions that

$$T^*(a_k) = T^*(a_k - a_k^{\nu_k}) = 0 \quad \text{on } \{\nu_k = \infty\}, \tag{7}$$

hence

$$E[T^*(a_k)^p] \le P(\nu_k \neq \infty)^{-1} P(\nu_k \neq \infty) = 1,$$

which proves the theorem. ∎

Note that this result yields that the Hardy spaces generated by s_q $(1 \le q < \infty)$ and by the predictable operator of an arbitrary operator have atomic decompositions.

4. BMO spaces and sharp operators. It is well known that the duals of the Hardy spaces H_p^M and $H_p^{s_2}$ are the BMO spaces that are usually defined with the norms

$$\|f\|_{\mathrm{BMO}_q(\alpha)} := \sup_{\nu \in \mathcal{T}} P(\nu \neq \infty)^{-1/q-\alpha} \|f - f^\nu\|_q,$$

$$\|f\|_{\mathrm{BMO}_q^-(\alpha)} := \sup_{\nu \in \mathcal{T}} P(\nu \neq \infty)^{-1/q-\alpha} \|f - f^{\nu^-}\|_q,$$

where $1 \le q < \infty$ and $\alpha \ge 0$ (cf. Garsia [16], Herz [18], [19], Weisz [35]). To characterize the duals of the spaces $H_p^{s_q}$ and $H_p^{S_q}$ we have to change slightly these formulas. For $1 \le q \le \infty$ and $\alpha \ge 0$ let

$$\|f\|_{\mathcal{BMO}_q(\alpha)} := \sup_{\nu \in \mathcal{T}} P(\nu \neq \infty)^{-1/q-\alpha} \Big[E\Big(\sum_{k=1}^{\infty} |d_k f|^q \chi(\nu < k)\Big)\Big]^{1/q}$$

and

$$\|f\|_{\mathcal{BMO}_q^-(\alpha)} := \sup_{\nu \in \mathcal{T}} P(\nu \neq \infty)^{-1/q-\alpha} \Big[E\Big(\sum_{k=1}^{\infty} |d_k f|^q \chi(\nu \le k)\Big)\Big]^{1/q}.$$

We say that $f \in \mathcal{BMO}_q(\alpha)$ (resp. $f \in \mathcal{BMO}_q^-(\alpha)$) if $\|f\|_{\mathcal{BMO}_q(\alpha)} < \infty$ (resp. $\|f\|_{\mathcal{BMO}_q^-(\alpha)} < \infty$). Set $\mathcal{BMO}_q := \mathcal{BMO}_q(0)$ and $\mathcal{BMO}_q^- := \mathcal{BMO}_q^-(0)$. Note that

$$\|f\|_{\mathcal{BMO}_\infty(\alpha)} = \sup_{\nu \in \mathcal{T}} P(\nu \neq \infty)^{-\alpha} \sup_{k \in \mathbb{N}} \|d_k f \chi(\nu < k)\|_\infty,$$

$$\|f\|_{\mathcal{BMO}_\infty^-(\alpha)} = \sup_{\nu \in \mathcal{T}} P(\nu \neq \infty)^{-\alpha} \sup_{k \in \mathbb{N}} \|d_k f \chi(\nu \le k)\|_\infty$$

and $\mathcal{BMO}_\infty = \mathcal{BMO}_\infty^- = H_\infty^{S_\infty}$. It is easy to see that

$$\|f\|_{\mathrm{BMO}_2(\alpha)} = \|f\|_{\mathcal{BMO}_2(\alpha)}, \qquad \|f\|_{\mathrm{BMO}_2^-(\alpha)} = \|f\|_{\mathcal{BMO}_2^-(\alpha)}.$$

Finally, one can verify that

$$\|f\|_{\mathcal{BMO}_q} = \Big\| \sup_{n \in \mathbb{N}} \Big(E_n \sum_{k=n+1}^{\infty} |d_k f|^q\Big)^{1/q} \Big\|_\infty \tag{8}$$

and

$$\|f\|_{\mathcal{BMO}_q^-} = \Big\| \sup_{n \in \mathbb{N}} \Big(E_n \sum_{k=n}^{\infty} |d_k f|^q\Big)^{1/q} \Big\|_\infty. \tag{9}$$

The *sharp operator* of an operator T satisfying (B) is defined by

$$T_q^\sharp(f) := \sup_{k \in \mathbb{N}} \sup_{k-1 \le \nu < \infty} [E_k(T_{\nu+1-i}(f - f^{k-i})^q)]^{1/q}, \tag{10}$$

where $\nu \in \mathcal{T}$ and, in the sequel of this section, $i = 0$ if T is predictable and $i = 1$ if T is adapted.

Note that one can write T^* instead of $T_{\nu+1-i}$ in (10) if T_n $(n \in \mathbb{N})$ is non-decreasing.

The sharp function generated by M was first investigated by Fefferman and Stein [15] in the classical case and by Garsia [16] and later by Lepingle [24] in the martingale case. Garsia ([16], pp. 31, 115) has proved that the L_p norm of $M_1^\sharp(f)$ (resp. $M_2^\sharp(f)$) is equivalent to the L_p norm of f whenever

$1 < p < \infty$ (resp. $2 < p < \infty$). This result will be generalized to every operator satisfying (B) and every $0 < p < \infty$.

Obviously, we have $\|M_q^\sharp(f)\|_\infty = \|f\|_{\mathrm{BMO}_q^-}$ $(1 < q < \infty)$, $\|(S_q)_q^\sharp(f)\|_\infty = \|f\|_{\mathcal{BMO}_q^-}$ and $\|(s_q)_q^\sharp(f)\|_\infty = \|f\|_{\mathcal{BMO}_q}$ $(1 \le q < \infty)$.

We introduce the $\mathcal{BMO}_q^T$ space generated by an operator T satisfying (B) with the norm

$$\|f\|_{\mathcal{BMO}_q^T} := \|T_q^\sharp(f)\|_\infty \quad (0 < q < \infty).$$

It was proved by John and Nirenberg [22], Garsia [16] and Herz [18] that the BMO_q^- spaces are all equivalent for $1 \le q < \infty$ and, in the classical case, by Strömberg [33] and Hanks [17] for $0 < q < \infty$. This result is generalized to every operator and every $0 < q < \infty$ in the next theorem.

THEOREM 2. *If T satisfies* (B) *then the $\mathcal{BMO}_q^T$ spaces are all equivalent for $0 < q < \infty$.*

Proof. An adapted ($i = 1$) or predictable ($i = 0$) sequence (A_n) is a $\mathrm{BMO}(B)$ *sequence* if

$$E_k|A_{n+1-i} - A_{k-i}| \le B \quad (n \ge k).$$

It is proved in Garsia [16] (p. 66) that if (A_n) is $\mathrm{BMO}(B)$ and non-decreasing then for all $n \in \mathbb{N}$,

$$E_i(e^{tA_{n+1-i}}) \le \frac{1}{1 - tB} \quad (tB < 1,\ t > 0). \tag{11}$$

Let $0 < q \le 1$ be fixed and $f \in \mathcal{BMO}_q^T$. Since $a^q - b^q \le (a-b)^q$ $(a > b \ge 0)$ and, by (B1),

$$T_{n+1-i}(f) - T_{k-i}(f) \le T_{n+1-i}(f - f^{k-i}), \tag{12}$$

we see that $T_n(f)^q$ is a $\mathrm{BMO}(B)$ sequence with $B := \|f\|_{\mathcal{BMO}_q^T}$. Consequently, $T_n^*(f)^q$ is $\mathrm{BMO}(8B)$ (see Garsia [16], p. 75) and, by (11),

$$E_i(e^{tT_{n+1-i}^*(f)^q}) \le \frac{1}{1 - 8tB} \quad (8tB < 1).$$

Applying this to the martingale $f - f^{k-i}$ we can see that

$$E_k[e^{t(T_{\nu+1-i}(f-f^{k-i}))^q}] \le E_k(e^{tT^*(f-f^{k-i})^q}) \le \frac{1}{1 - 8tB} \quad (8tB < 1, \nu \in \mathcal{T}).$$

The proof can be finished as in Garsia [16] (p. 65). ∎

Since

$$E_k(s_{r,n+1}^r(f) - s_{r,k}^r(f)) \le \|f\|_{\mathcal{BMO}_r}^r$$

and

$$E_k(S_{r,n}^r(f) - S_{r,k-1}^r(f)) \le \|f\|_{\mathcal{BMO}_r^-}^r,$$

the sequence $(s_{r,n}^r(f))$ (resp. $(S_{r,n}^r(f))$) $(1 \le r < \infty)$ is $\mathrm{BMO}(B)$ with $B = \|f\|_{\mathcal{BMO}_r}^r$ (resp. $B = \|f\|_{\mathcal{BMO}_r^-}^r$). Consequently, the inequalities

$$E(e^{ts_r^r(f)}) \le \frac{1}{1 - t\|f\|_{\mathcal{BMO}_r}^r}, \quad E(e^{tS_r^r(f)}) \le \frac{1}{1 - t\|f\|_{\mathcal{BMO}_r^-}^r}$$

follow from (11). Note that, for $r = 2$, the second inequality was proved by Burkholder [8] and by Garsia [16] (p. 69) and the first one by Weisz [34].

Garsia [16] proved in Theorem IV.4.4 that every function $f \in H_p^{s_2}$ $(p > 1)$ can be derived as a martingale transform of an $h \in \mathcal{BMO}_2 = \mathcal{BMO}_s^{s_2}$ and conjectured that this does not hold for $p = 1$. However, he proved this result for $h \in \mathcal{BMO}_1^{s_2}$. In other words, the conjecture says that $\mathcal{BMO}_1^{s_2}$ is not equivalent to $\mathcal{BMO}_2$, though this is true by Theorem 2.

Finally, we are going to prove the theorem relative to the sharp operators mentioned at the beginning of this section.

The following lemma is a slight modification of Theorem 1 of Bassily and Mogyoródi [2].

LEMMA 1. *Let $0 < q < \infty$ be fixed and T be a predictable ($i = 0$) or adapted ($i = 1$) operator satisfying* (B). *If γ is a function such that*

$$\sup_{\nu \in \mathcal{T},\, k-1 \le \nu < \infty} E_k(T_{\nu+1-i}(f - f^{k-i})^q) \le E_k\gamma^q$$

for all $k \in \mathbb{N}$ then for all $\beta > \alpha > 0$,

$$(\beta - \alpha)^q E_i[\chi(T_{m+1-i}^*(f) > \beta)] \le E_i[\chi(T_{m+1-i}^*(f) > \alpha)\gamma^q] \quad (m \in \mathbb{N}).$$

Proof. Let us introduce the following stopping times:

$$\nu_\lambda := \begin{cases} \inf\{n \le m : T_{n+1-i}(f) > \lambda\} & \text{if } T_{m+1-i}^*(f) > \lambda, \\ m+1 & \text{if } T_{m+1-i}^*(f) \le \lambda, \end{cases}$$

where $\lambda > 0$ is arbitrary. Obviously,

$$E_i[\chi(T_{m+1-i}^*(f) > \beta)] = E_i[\chi(\nu_\beta \le m)] = E_i\Big[\sum_{n=1}^m \sum_{k=1}^n \chi(\nu_\beta = n,\ \nu_\alpha = k)\Big].$$

On the set $\{\nu_\beta = n,\ \nu_\alpha = k\}$ we have $T_{n+1-i}(f) > \beta$ and $T_{k-i}(f) \le \alpha$, so $T_{n+1-i}(f) - T_{k-i}(f) > \beta - \alpha$. Using (12) we obtain

$$\begin{aligned} E_i[\chi(T_{m+1-i}^*(f) > \beta)] &\le E_i\Big[\sum_{n=1}^m \sum_{k=1}^n \chi(\nu_\beta = n,\ \nu_\alpha = k) \frac{(T_{n+1-i}(f) - T_{k-i}(f))^q}{(\beta -)^q}\Big] \\ &\le E_i\Big[\sum_{k=1}^m \chi(\nu_\alpha = k) \frac{T_{(\nu_\beta+1-i)\vee(k-i)}(f - f^{k-i})^q}{(\beta -)^q}\Big]. \end{aligned}$$

From this it follows that

$$(\beta-\alpha)^q E_i[\chi(T^*_{m+1-i}(f)>\beta)] \le E_i\Big[\sum_{k=1}^{m}\chi(\nu_\alpha=k)\gamma^q\Big],$$

which proves the lemma. ∎

THEOREM 3. *If T satisfies* (B) *then*

$$c_p\|f\|_{H_p^T} \le \|T_q^\sharp(f)\|_p \le C_{p,q}\|f\|_{H_p^T} \quad (0<q<p<\infty).$$

Proof. Write $\beta=2\alpha$ in Lemma 1. Multiplying by $p\alpha^{p-q-1}$ the inequality

$$\alpha^q E[\chi(T^*(f)/2>\alpha)] \le E[\chi(T^*(f)>\alpha)\gamma^q],$$

integrating it in α from 0 to ∞ and using Fubini's theorem we obtain

$$E(|T^*(f)/2|^p) \le E\Big(p\gamma^q \int_0^\infty \alpha^{p-q-1}\chi(T^*(f)>\alpha)\,d\alpha\Big) = c_p E(\gamma^q T^*(f)^{p-q}).$$

By Hölder's inequality

$$\|T^*(f)\|_p \le c_p\|\gamma\|_p \quad (p>q).$$

If we choose $\gamma=T_q^\sharp(f)$ then the left hand inequality of the assertion is verified.

To prove the other inequality, let us estimate $T_q^\sharp(f)$ by the quantity $2\sup_{n\in\mathbb{N}}[E_n(T^*(f)^q)]^{1/q}$. Using Doob's inequality we get

$$\|T_q^\sharp(f)\|_p \le 2(E(\sup_{n\in\mathbb{N}} E_n[T^*(f)^q])^{p/q})^{1/p} \le C_{p,q}\|T^*(f)\|_p \quad (p>q)$$

and this completes the proof. ∎

5. Interpolation of martingale spaces. In this section the interpolation spaces between the martingale Hardy and BMO spaces generated by an operator T satisfying (B) are identified with the real method. For this we shall need some additional definitions.

For a measurable function f the *non-increasing rearrangement* is introduced by

$$\widetilde{f}(t) := \inf\{y : P(\{x : |f(x)|>y\}) \le t\}.$$

The Lorentz space $L_{p,q}$ is defined as follows: for $0<p<\infty$ and $0<q<\infty$,

$$\|f\|_{p,q} := \Big(\int_0^\infty \widetilde{f}(t)^q t^{q/p}\,\frac{dt}{t}\Big)^{1/q},$$

while for $0<p\le\infty$,

$$\|f\|_{p,\infty} := \sup_{t>0} t^{1/p}\widetilde{f}(t).$$

Set

$$L_{p,q} := L_{p,q}(\Omega,\mathcal{A},P) := \{f : \|f\|_{p,q}<\infty\}.$$

One can show that $L_{p,p}=L_p$ and $L_{p,\infty}$ is the weak L_p space $(0<p\le\infty)$.

For $0<p,q\le\infty$ the *martingale Hardy–Lorentz space* $H_{p,q}^T$ is the set of martingales f for which

$$\|f\|_{H_{p,q}^T} := \|T^*(f)\|_{p,q}<\infty.$$

In case $p=q$ we recover the original definition of the Hardy space $H_{p,p}^T = H_p^T$.

The basic definitions concerning the real method of interpolation are now briefly recalled. For the details see Bennett and Sharpley [3] or Bergh and Löfström [4]. Suppose that A_0 and A_1 are quasi-normed spaces continuously embedded in a topological vector space A. The *interpolation spaces* between A_0 and A_1 are defined by means of an interpolating function $K(t,f,A_0,A_1)$. If $f\in A_0+A_1$, define

$$K(t,f,A_0,A_1) := \inf_{f=f_0+f_1}\{\|f_0\|_{A_0}+t\|f_1\|_{A_1}\},$$

where the infimum is taken over all choices of f_0 and f_1 such that $f_0\in A_0$, $f_1\in A_1$ and $f=f_0+f_1$. The interpolation space $(A_0,A_1)_{\theta,q}$ is defined as the space of all functions f in A_0+A_1 such that

$$\|f\|_{(A_0,A_1)_{\theta,q}} := \Big(\int_0^\infty [t^{-\theta}K(t,f,A_0,A_1)]^q\,\frac{dt}{t}\Big)^{1/q}<\infty,$$

where $0<\theta<1$ and $0<q\le\infty$. We use the conventions $(A_0,A_1)_{0,q}=A_0$ and $(A_0,A_1)_{1,q}=A_1$ for each $0<q\le\infty$.

It is well known that

$$K(t,f,L_{p_0},L_\infty) \sim \Big(\int_0^{t^{p_0}} \widetilde{f}(x)^{p_0}\,dx\Big)^{1/p_0} \quad (0<p_0<\infty) \tag{13}$$

and the interpolation spaces of Lorentz spaces are Lorentz spaces again, more precisely, if $p_0\ne p_1$ then

$$(L_{p_0,q_0},L_{p_1,q_1})_{\eta,q} = L_{p,q}, \quad \frac{1}{p}=\frac{1-\eta}{p_0}+\frac{\eta}{p_1}, \tag{14}$$

provided that $0<\eta<1$ and $0<p_0,p_1,q_0,q_1,q\le\infty$ (see e.g. Bergh and Löfström [4]).

Suppose that B_0 and B_1 are also quasi-normed spaces continuously embedded in a topological vector space B. A map

$$U : A_0+A_1 \to B_0+B_1$$

is said to be *quasilinear* from (A_0, A_1) to (B_0, B_1) if for given $a \in A_0 + A_1$ and $a_i \in A_i$ with $a_0 + a_1 = a$ there exist $b_i \in B_i$ satisfying

$$U(a) = b_0 + b_1$$

and

$$\|b_i\|_{B_i} \le K_i \|a_i\|_{A_i} \quad (K_i > 0,\ i = 0, 1).$$

The following theorem, which is used several times in this paper, shows that the boundedness of a quasilinear operator is hereditary for interpolation spaces.

THEOREM 4 (Rivière and Sagher [29]). *If $0 < q \le \infty$, $0 \le \theta \le 1$ and U is a quasilinear map from (A_0, A_1) to (B_0, B_1) then*

$$U : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$$

and

$$\|U(a)\|_{(B_0,B_1)_{\theta,q}} \le K_0^{1-\theta} K_1^{\theta} \|a\|_{(A_0,A_1)_{\theta,q}}.$$

We prove the following consequence of this theorem.

COROLLARY 1. *Suppose that B_i has the lattice property, i.e. $|g| \le |f|$ a.e. implies that $\|g\|_{B_i} \le \|f\|_{B_i}$ whenever $f \in B_i$ $(i = 0, 1)$. If $0 < q \le \infty$, $0 \le \theta \le 1$ and U is a subadditive operator which is bounded from A_i to B_i with norms K_i $(i = 0, 1)$, then*

$$\|U(a)\|_{(B_0,B_1)_{\theta,q}} \le K_0^{1-\theta} K_1^{\theta} \|a\|_{(A_0,A_1)_{\theta,q}}.$$

Proof. Let $a \in A_0 + A_1$ and $a_i \in A_i$ $(i = 0, 1)$ with $a_0 + a_1 = a$. By subadditivity, $|U(a)| \le |U(a_0)| + |U(a_1)|$. Define

$$b_0 := \operatorname{sign}(U(a)) \min\{|U(a_0)|, |U(a)|\}, \qquad b_1 := U(a) - b_0.$$

Since $|b_0| \le |U(a_0)|$ and $|b_1| \le |U(a_1)|$, we have

$$\|b_i\|_{B_i} \le \|U(a_i)\|_{B_i} \le K_i \|a_i\|_{A_i} \quad (i = 0, 1),$$

which shows that U is quasilinear from (A_0, A_1) to (B_0, B_1). The corollary follows from Theorem 4. ∎

Note that the Lorentz spaces $L_{p,q}$ $(0 < p, q \le \infty)$ have the lattice property (see Bennett and Sharpley [3], p. 41).

With the help of the atomic decomposition a new decomposition theorem for martingales is now given.

THEOREM 5. *Let T be a predictable operator satisfying* (B), $f \in H_p^T$, $y > 0$ *and* $0 < p \le 1$. *Then f can be decomposed into the sum of two martingales g and h such that*

$$\|g\|_{H_\infty^T} \le 4y$$

and

$$\|h\|_{H_p^T} \le C_p \Big(\int_{\{T^*(f) > y\}} T^*(f)^p \, dP \Big)^{1/p}$$

where the positive constant C_p depends only on p.

Proof. Choose $N \in \mathbb{Z}$ such that $2^{N-1} < y \le 2^N$. Take the same stopping times ν_k, atoms a_k and real numbers μ_k $(k \in \mathbb{Z})$ as in Theorem 1. Set

$$g_n := \sum_{k=-\infty}^{N} \mu_k a_{k,n} \quad \text{and} \quad h_n := \sum_{k=N+1}^{\infty} \mu_k a_{k,n}.$$

It was proved in Theorem 1 that $f_n = g_n + h_n$ for all $n \in \mathbb{N}$ and $g = f^{\nu_{N+1}}$, and moreover, that

$$T^*(g) = T^*(f^{\nu_{N+1}}) \le 2^{N+1} \le 4y,$$

which proves the first inequality of the theorem.

On the other hand, the inequality

$$\|h\|_{H_p^T}^p \le \sum_{k=N+1}^{\infty} |\mu_k|^p = C_p \sum_{k=N+1}^{\infty} (2^k)^p P(T^*(f) > 2^k)$$

follows from Theorem 1. Similarly to (5), by Abel rearrangement, we obtain

$$\|h\|_{H_p^T}^p \le C_p \int_{\{T^*(f) > 2^N\}} T^*(f)^p \, dP \le C_p \int_{\{T^*(f) > y\}} T^*(f)^p \, dP.$$

The proof of the theorem is complete. ∎

The interpolation spaces between the martingale Hardy spaces generated by predictable operators can be identified.

THEOREM 6. *Assume that T is predictable. If $0 < \theta < 1$, $0 < p_0 \le 1$ and $0 < q \le \infty$ then*

$$(H_{p_0}^T, H_\infty^T)_{\theta,q} = H_{p,q}^T, \qquad \frac{1}{p} = \frac{1-\theta}{p_0}.$$

The main step in the proof is the following result.

LEMMA 2. *If T is predictable and $0 < p_0 \le 1$ then*

$$K(t, f, H_{p_0}^T, H_\infty^T) \le C \Big(\int_0^{t^{p_0}} \widetilde{T^*(f)}(x)^{p_0} \, dx \Big)^{1/p_0}.$$

Proof. Choose y in Theorem 5 such that, for a fixed $t \in [0, 1]$, $y = \widetilde{T^*(f)}(t^{p_0})$. For this y denote the two martingales in Theorem 5 by g_t and h_t. By the definition of the functional K,

$$K(t, f, H_{p_0}^T, H_\infty^T) \le \|h_t\|_{H_{p_0}^T} + t \|g_t\|_{H_\infty^T}.$$

By Theorem 5 we get

$$\|h_t\|_{H^T_{p_0}} \le C\Big(\int_{\{T^*(f)>\widetilde{T^*(f)}(t^{p_0})\}} T^*(f)^{p_0}\,dP\Big)^{1/p_0} = C\Big(\int_0^{t^{p_0}} \widetilde{T^*(f)}(x)^{p_0}\,dx\Big)^{1/p_0}.$$

On the other hand,

$$t\|g_t\|_{H^T_\infty} \le Ct\widetilde{T^*(f)}(t^{p_0}) \le C\Big(\int_0^{t^{p_0}} \widetilde{T^*(f)}(x)^{p_0}\,dx\Big)^{1/p_0},$$

which shows the lemma. ■

Proof of Theorem 6. By (13) the right hand side of the inequality in Lemma 2 is equivalent to $K(t, T^*(f), L_{p_0}, L_\infty)$. Applying (14) we conclude that

$$\|f\|^q_{(H^T_{p_0},H^T_\infty)_{\theta,q}} \le C\int_0^1 t^{-\theta q}K(t,T^*(f),L_{p_0},L_\infty)^q\,\frac{dt}{t} \le C\|T^*(f)\|^q_{p,q}.$$

To prove the converse observe that $T^*: H^T_\infty \to L_\infty$ and $T^*: H^T_{p_0} \to L_{p_0}$ are bounded. Therefore, by Corollary 1 and (14),

$$\|f\|_{H^T_{p,q}} = \|T^*(f)\|_{p,q} \le C\|f\|_{(H^T_{p_0},H^T_\infty)_{\theta,q}}.$$

The proof of Theorem 6 is complete if $0 < q < \infty$. With a fine modification of the previous proof the theorem can also be shown in case $q = \infty$. ■

Applying the reiteration theorem (see e.g. Bergh and Löfström [4]) we get the following result.

COROLLARY 2. *Suppose that T is predictable, $0 < \eta < 1$ and $0 < p_0, p_1, q_0, q_1, q \le \infty$. If $p_0 \ne p_1$ then*

$$(H^T_{p_0,q_0}, H^T_{p_1,q_1})_{\eta,q} = H^T_{p,q}, \qquad \frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}.$$

This result was proved by Fefferman, Rivière and Sagher [14] in the classical case and by Weisz [34] for the operator s_2. In the classical case Janson and Jones [21] verified the analogous result with the complex method.

The interpolation spaces between H^T_p and $\mathcal{BMO}^T$ will now be identified, where $\mathcal{BMO}^T$ denotes one of the spaces $\mathcal{BMO}^T_q$. Recall that these spaces are all equivalent.

THEOREM 7. *Assume that T is predictable. If $0 < \theta < 1$, $0 < q \le \infty$ and $0 < r < \infty$ then*

$$(H^T_r, \mathcal{BMO}^T)_{\theta,q} = H^T_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{r}.$$

Proof. It is obvious that

$$\|f\|_{\mathcal{BMO}^T} \le C\|f\|_{H^T_\infty}.$$

Thus

$$\|f\|_{(H^T_r,\mathcal{BMO}^T)_{\theta,q}} \le C\|f\|_{(H^T_r,H^T_\infty)_{\theta,q}} = C\|f\|_{H^T_{p,q}}.$$

To see the converse consider the operator $T^\sharp_u$ for a fixed $0 < u < r$. In Theorem 3 it was proved that $T^\sharp_u: H^T_r \to L_r$ is bounded. Furthermore, so is $T^\sharp_u: \mathcal{BMO}^T \to L_\infty$. Using Corollary 1 with $q = p$ and Theorem 3, one can see that $f \in (H^T_r, \mathcal{BMO}^T)_{\theta,p}$ implies

$$\|f\|_{H^T_p} \le C_p\|T^\sharp_u(f)\|_p \le C_p\|f\|_{(H^T_r,\mathcal{BMO}^T)_{\theta,p}},$$

which proves the theorem for $p = q$, namely,

$$(H^T_r, \mathcal{BMO}^T)_{\theta,p} = H^T_p, \qquad \frac{1}{p} = \frac{1-\theta}{r}.$$

Applying the reiteration theorem we can prove the theorem with a usual argument (cf. Hanks [17]). ■

As a further application of the reiteration theorem we get the following

COROLLARY 3. *Assume that T is predictable. If $0 < \theta < 1$, $0 < p_0 < \infty$ and $0 < q_0, q \le \infty$ then*

$$(H^T_{p_0,q_0}, \mathcal{BMO}^T)_{\theta,q} = H^T_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0}.$$

This result is due to Hanks [17] in the classical case and to Weisz [34] in the martingale case for s_2.

Let us turn to the adapted operators and prove similar interpolation theorems for them. First of all notice that the following two equivalences can be verified with the same method as (14):

$$(L_1(l_1), L_{p_1}(l_1))_{\theta,q} = L_{p,q}(l_1), \qquad \frac{1}{p} = 1-\theta+\frac{\theta}{p_1}, \tag{15}$$

where $0 < \theta < 1$, $0 < q \le \infty$, $1 < p_1 \le \infty$, and

$$(H^{S_1}_{p_0}, H^{S_1}_{p_1})_{\theta,q} = H^{S_1}_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{16}$$

where $0 < \theta < 1$, $0 < q \le \infty$ and $1 = p_0 < p_1 < \infty$. Moreover, with that method one can show (16) for S_r $(1 < r < \infty)$ instead of S_1, but for $1 < p_0 < p_1 < \infty$ only. We are going to extend this result. The idea of the following proof is due to Milman [25].

THEOREM 8. *If* $0<\theta<1$, $0<q\le\infty$ *and* $1\le r<\infty$ *then*

$$(H_1^{S_r},H_\infty^{S_r})_{\theta,q}=H_{p,q}^{S_r},\qquad \frac{1}{p}=1-\theta.$$

As we have seen in Theorem 6, this theorem follows from Corollary 1 and from the next lemma.

LEMMA 3. *If* $1\le r<\infty$ *then*

$$K(t,f,H_1^{S_r},H_\infty^{S_r})\le C\int_0^t \widetilde{S_r(f)}(x)\,dx.$$

Proof. For a fixed t consider the following two stopping times:

$$\nu:=\inf\{n\in\mathbb{N}:S_{r,n}(f)>\widetilde{S_r(f)}(t)\},$$
$$\tau:=\inf\Big\{n\in\mathbb{N}:\Big(\sum_{k=1}^{n+1}|E_{k-1}[d_kf\chi(\nu=k)]|^r\Big)^{1/r}>\widetilde{S_r(f)}(t)\Big\}.$$

Set

$$g_n:=f_n^{\nu\wedge\tau}-\sum_{k=1}^{n}[d_kf\chi(\nu=k)-E_{k-1}(d_kf\chi(\nu=k))]\chi(\tau\ge k),$$
$$h_n:=f_n-g_n.$$

By the definitions of ν and τ,

$$\begin{aligned}S_r(g)&=\Big(\sum_{k=1}^{\infty}|d_kf\chi(\nu\ge k)-d_kf\chi(\nu=k)+E_{k-1}(d_kf\chi(\nu=k))|^r\chi(\tau\ge k)\Big)^{1/r}\\&\le\Big(\sum_{k=1}^{\infty}|d_kf|^r\chi(\nu>k)\chi(\tau\ge k)\Big)^{1/r}\\&\quad+\Big(\sum_{k=1}^{\infty}|E_{k-1}(d_kf\chi(\nu=k))|^r\chi(\tau\ge k)\Big)^{1/r}\le 2\widetilde{S_r(f)}(t).\end{aligned}$$

Hence

$$t\|S_r(g)\|_\infty\le Ct\widetilde{S_r(f)}(t)\le C\int_0^t\widetilde{S_r(f)}(x)\,dx.$$

Since $S_r(\cdot)\le S_1(\cdot)$, we have

$$\begin{aligned}\|S_r(h)\|_1&\le\|S_r(f-f^{\nu\wedge\tau})\|_1\\&\quad+\Big\|S_1\Big(\sum_{k=1}^{\infty}[d_kf\chi(\nu=k)-E_{k-1}(d_kf\chi(\nu=k))]\chi(\tau\ge k)\Big)\Big\|_1.\end{aligned}$$

On the one hand, the second term of the right hand side can be estimated by

$$2\sum_{k=1}^{\infty}E(|d_kf|\chi(\nu=k))\le 2\int_{\{S_r(f)>\widetilde{S_r(f)}(t)\}}S_r(f)\,dP\le 2\int_0^t\widetilde{S_r(f)}(x)\,dx.\tag{17}$$

On the other hand, by the definitions of ν and τ,

$$\begin{aligned}\|S_r(f-f^{\nu\wedge\tau})\|_1&\le\int_{\{\nu<\infty\}}S_r(f)\,dP+\int_{\{\nu=\infty\}\cap\{\tau<\infty\}}S_r(f)\,dP\\&\le\int_{\{S_r(f)>\widetilde{S_r(f)}(t)\}}S_r(f)\,dP+P(\tau<\infty)\widetilde{S_r(f)}(t).\end{aligned}$$

By the Markov inequality

$$P(\tau<\infty)\widetilde{S_r(f)}(t)\le\Big\|\Big(\sum_{k=1}^{\infty}|E_{k-1}[d_kf\chi(\nu=k)]|^r\Big)^{1/r}\Big\|_1,$$

and this is estimated in (17) by $\int_0^t\widetilde{S_r(f)}(x)\,dx$. The proof of the lemma is complete. ∎

The convexity theorem states that

$$\Big\|\sum_{k=0}^{\infty}E_k|f_k|\Big\|_{p,q}\le C_{p,q}\Big\|\sum_{k=0}^{\infty}|f_k|\Big\|_{p,q}\qquad(1<p<\infty,\ 0<q\le\infty)\tag{18}$$

for $1<p=q<\infty$, where f_k $(k\in\mathbb{N})$ are arbitrary functions (see Garsia [16], p. 113). Therefore (15) and Corollary 1 yield that (18) is true for all $1<p<\infty$ and $0<q\le\infty$.

The following result is a generalization of the Davis decomposition for the operator M (see Garsia [16], p. 91).

LEMMA 4. *Suppose that T is adapted and*

$$c|d_nf|\le T_n(d_nf)\le C|d_nf|\tag{19}$$

for all martingales f and all $n\in\mathbb{N}$. If $f\in H_{p,q}^T$ with either $1<p<\infty$ and $0<q\le\infty$, or $p=q=1$, then there exist $h\in H_{p,q}^{S_1}$ and $e\in H_{p,q}^{T^-}$ such that $f=h+e$ and

$$\|h\|_{H_{p,q}^{S_1}}\le C_{p,q}\|f\|_{H_{p,q}^T},\qquad\|e\|_{H_{p,q}^{T^-}}\le C_{p,q}\|f\|_{H_{p,q}^T}.$$

Proof. The martingales h and e are given by

$$h := \sum_{k=1}^{\infty} [d_k f \chi(T_k^*(f) > 2T_{k-1}^*(f)) - E_{k-1}(d_k f \chi(T_k^*(f) > 2T_{k-1}^*(f)))]$$

and

$$e := \sum_{k=1}^{\infty} [d_k f \chi(T_k^*(f) \le 2T_{k-1}^*(f)) - E_{k-1}(d_k f \chi(T_k^*(f) \le 2T_{k-1}^*(f)))],$$

respectively. One can prove that

$$\sum_{k=1}^{n} |d_k h| \le 4T_n^*(f) + 4 \sum_{k=1}^{n} E_{k-1}(T_k^*(f) - T_{k-1}^*(f))$$

and

$$T_n^*(g) \le 13T_{n-1}^*(f) + 4 \sum_{k=1}^{n-1} E_{k-1}(T_k^*(f) - T_{k-1}^*(f))$$

(see Garsia [16], p. 91). Now the lemma follows from (18). ∎

Observe that

$$\|f\|_{H_p^T} \le C \|f\|_{H_p^{S_1}} \quad (0 < p \le \infty) \tag{20}$$

follows from (19).

The condition (19) looks like the equivalence condition $\|\sup_n U_n(d_n f)\|_p \sim \|\sup_n T_n(d_n f)\|_p$ which was used by Burkholder and Gundy [10] to derive martingale inequalities (cf. Section 6).

THEOREM 9. *Suppose that T is adapted and satisfies* (19), *and moreover, that $0 < \theta < 1$ and either $1 < p_0 < p_1 \le \infty$ and $0 < q_0, q_1, q \le \infty$, or $p_0 = q_0 = 1$. Then*

$$(H_{p_0,q_0}^T, H_{p_1,q_1}^T)_{\theta,q} = H_{p,q}^T, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Proof. By the reiteration theorem we only have to verify that

$$(H_1^T, H_\infty^T)_{\theta,q} = H_{p,q}^T, \quad \frac{1}{p} = 1 - \theta.$$

Applying (20), Lemma 4, and Theorems 6 and 8 we can conclude that

$$\begin{aligned}\|f\|_{(H_1^T, H_\infty^T)_{\theta,q}} &\le \|e\|_{(H_1^T, H_\infty^T)_{\theta,q}} + \|h\|_{(H_1^T, H_\infty^T)_{\theta,q}} \\ &\le \|e\|_{(H_1^{T-}, H_\infty^{T-})_{\theta,q}} + C\|h\|_{(H_1^{S_1}, H_\infty^{S_1})_{\theta,q}} \\ &\le C_{p,q}\|e\|_{H_{p,q}^{T-}} + C_{p,q}\|h\|_{H_{p,q}^{S_1}} \le C_{p,q}\|f\|_{H_{p,q}^T}.\end{aligned}$$

The converse inequality follows from Corollary 1. ∎

Observe that

$$(H_{p_0,q_0}^T, H_{p_1,q_1}^T)_{\theta,q} \subset H_{p,q}^T, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

follows for all $0 < p_0 < p_1 \le \infty$ in this case, too. The analogous result for Hardy spaces consisting of vector-valued harmonic functions can be found in Blasco [6] and Blasco and Xu [7].

Analogously to Theorem 7 and Corollary 3 the following result can be formulated.

THEOREM 10. *Suppose that T is adapted and satisfies* (19), *and moreover, that $0 < \theta < 1$ and either $1 < p_0 < \infty$ and $0 < q_0, q \le \infty$, or $p_0 = q_0 = 1$. Then*

$$(H_{p_0,q_0}^T, \mathcal{BMO}^T)_{\theta,q} = H_{p,q}^T, \quad \frac{1}{p} = \frac{1-\theta}{p_0}.$$

An analogous result was proved for $T = M$ by Janson and Jones [21] with the complex method.

Observe again that

$$(H_{p_0,q_0}^T, \mathcal{BMO}^T)_{\theta,q} \subset H_{p,q}^T, \quad \frac{1}{p} = \frac{1-\theta}{p_0},$$

holds for every $0 < p_0 < \infty$. Analogous results in the vector-valued classical case can be found in Blasco [6] and Blasco and Xu [7].

Note that Theorems 9 and 10 do not extend to $p_0 < 1$ (cf. Janson and Jones [21]).

In the proof of the last result in this section Wolff's reiteration theorem is used:

THEOREM 11 (Wolff [37]). *Let A_1, A_2, A_3 and A_4 be quasi-Banach spaces satisfying $A_1 \cap A_4 \subset A_2 \cap A_3$. Suppose that*

$$A_2 = (A_1, A_3)_{\phi,q}, \quad A_3 = (A_2, A_4)_{\psi,r}$$

for any $0 < \phi, \psi < 1$ and $0 < q, r \le \infty$. Then

$$A_2 = (A_1, A_4)_{\varrho,q}, \quad A_3 = (A_1, A_4)_{\theta,r},$$

where

$$\varrho = \frac{\phi\psi}{1 - \phi + \phi\psi}, \quad \theta = \frac{\psi}{1 - \phi + \phi\psi}.$$

Applying Theorem 10 to $T = M$ and Wolff's theorem we get

COROLLARY 4. *If $0 < \theta < 1$, $0 < p_0 < \infty$ and $0 < q_0, q \le \infty$ then*

$$(L_{p_0,q_0}, \mathcal{BMO}^M)_{\theta,q} = L_{p,q}, \quad \frac{1}{p} = \frac{1-\theta}{p_0}. \tag{21}$$

Proof. By Theorem 10 we have (21) for $1 < p_0 < \infty$ and $p_0 = q_0$ and, by the reiteration theorem, for all $0 < q_0 \le \infty$. Set $A_1 = L_{p_0,q_0}$ for any $0 < p_0 \le 1$, $A_2 = L_{p_1,q_1}$ for any $1 < p_1 < \infty$, $A_3 = L_{p,q}$ for any $p_1 < p < \infty$ and $A_4 = \mathcal{BMO}^M$. By (14) we can apply Theorem 11 to get (21) for $1 < p < \infty$. Let us apply Wolff's theorem again. Now set $A_1 = L_{p_0,q_0}$ for any $0 < p_0 < 1$, $A_2 = L_{p,q}$ for any $p_0 < p \le 1$, $A_3 = L_{p_1,q_1}$ for any $1 < p_1 < \infty$ and $A_4 = \mathcal{BMO}^M$. Applying (21) to $1 < p < \infty$ together with (14) and Theorem 11 we obtain (21) for all $0 < p < \infty$. The proof is complete. ∎

6. Martingale inequalities. In this section the connections between martingale Hardy spaces are investigated. The idea of the method is the following. If an inequality holds for a number p, then by the atomic decomposition we can verify it for all parameters not greater than 1 and by interpolation for all parameters less than p. For special operators one can also derive the inequality for parameters greater than p; however, for general operators this is not the case. We single out the results for some special operators. As a consequence the well-known Burkholder–Davis–Gundy inequality is obtained.

THEOREM 12. *Assume that T is predictable and U is adapted, and moreover, that there exists $0 < p_1 \le \infty$ such that for all martingales f,*

$$\|U^*(f)\|_{p_1} \le C\|T^*(f)\|_{p_1}. \tag{22}$$

Then

$$\|f\|_{H_p^U} \le C_p\|f\|_{H_p^T} \quad (0 < p \le p_1).$$

Proof. Suppose that $f \in H_p^T$ and $0 < p \le 1 \wedge p_1$. Let $f = \sum_{k=-\infty}^{\infty} \mu_k a_k$ be an atomic decomposition of f satisfying (2). As (6) also holds for the operator U, we only have to show that, for every p-atom a,

$$E[U^*(a)^p] \le C.$$

Indeed, applying (7), (22) and Hölder's inequality we can see that

$$\begin{aligned} E[U^*(a)^p] &\le E^{p/p_1}[U^*(a)^{p_1}]P(\nu \ne \infty)^{1-p/p_1} \\ &\le CE^{p/p_1}[T^*(a)^{p_1}]P(\nu \ne \infty)^{1-p/p_1} \\ &\le C[P(\nu \ne \infty)^{-p_1/p}P(\nu \ne \infty)]^{p/p_1}P(\nu \ne \infty)^{1-p/p_1} = C. \end{aligned}$$

If $1 < p_1$ then we see that $U^* : H_{p_1}^T \to L_{p_1}$ and $U^* : H_1^T \to L_1$ are bounded. It follows from Corollaries 1 and 2 that $U^* : H_p^T \to L_p$ is also bounded whenever $1 \le p \le p_1$. The proof of the theorem is complete. ∎

The equivalence $H_\infty^T \sim H_\infty^{T^-}$ and the inequality

$$\|f\|_{H_p^T} \le \|f\|_{H_p^{T^-}} \quad (0 < p \le \infty) \tag{23}$$

are clear from the definition. The next result is a consequence of Theorem 12 and (23).

COROLLARY 5. *Assume that U and T are adapted operators, and moreover, that there exists $0 < p_1 \le \infty$ such that (22) holds. Then*

$$\|f\|_{H_p^U} \le C_p\|f\|_{H_p^{T^-}} \quad (0 < p \le p_1).$$

For a regular stochastic basis the converse of (23) is also valid.

PROPOSITION 1. *If $\mathcal{F}$ is regular then $H_p^T \sim H_p^{T^-}$ for all $0 < p \le \infty$.*

We omit the proof because it is the same as the one of Proposition 2 in Weisz [35].

COROLLARY 6. *Let $\mathcal{F}$ be regular. Assume that U and T are adapted and for all martingales f,*

$$\|U^*(f)\|_{p_1} \sim \|T^*(f)\|_{p_1}, \tag{24}$$

where $0 < p_1 \le \infty$. Then

$$\|f\|_{H_p^U} \sim \|f\|_{H_p^T} \quad (0 < p \le p_1).$$

Proof. By Corollary 5 and Proposition 1,

$$\|U^*(f)\|_p \le C_p\|T^-(f)\|_p \le C_p\|T^*(f)\|_p.$$

The reverse inequality can be proved in the same way. ∎

THEOREM 13. *Assume that U and T are adapted and satisfy* (19) *and* (22). *Then*

$$\|f\|_{H_p^U} \le C_p\|f\|_{H_p^T} \quad (1 \le p \le p_1).$$

Proof. Let $f \in H_p^T$. Using Lemma 4 with $p = q$ and Corollary 5 we can conclude that

$$\|f\|_{H_p^U} \le \|h\|_{H_p^U} + \|e\|_{H_p^U} \le \|h\|_{H_p^{s_1}} + C_p\|e\|_{H_p^{T^-}} \le C_p\|f\|_{H_p^T},$$

which yields the assumption. ∎

COROLLARY 7. *Assume that U and T are adapted and satisfy* (19) *and* (24). *Then*

$$\|f\|_{H_p^U} \sim \|f\|_{H_p^T} \quad (1 \le p \le p_1).$$

In the sequel of this section some results for special operators will be given. The following basic inequalities are used:

$$\|S_q(f)\|_q = \|s_q(f)\|_q \quad (1 \le q < \infty), \tag{25}$$

$$\|f^*\|_q \le C_q\|s_q(f)\|_q \quad (1 \le q \le 2), \tag{26}$$

$$\|f^*\|_2 \sim \|S_2(f)\|_2. \tag{27}$$

(26) can be found in Lepingle [24], while (27) comes from Doob's inequality.

PROPOSITION 2. *If* $1 \le q < \infty$ *then*

$$\|f\|_{H_p^{S_q}} \le C_p \|f\|_{H_p^{s_q}} \quad (0 < p \le q),$$
$$\|f\|_{H_p^{s_q}} \le C_p \|f\|_{H_p^{S_q}} \quad (q \le p < \infty),$$
$$\|f\|_{H_p^M} \le C_p \|f\|_{H_p^{s_q}} \quad (0 < p \le q \le 2).$$

Proof. The first and third inequalities follow from (25), (26) and Theorem 12, and the second one from (18) with $p = q$. ■

From Corollary 6 and (25), (27) we obtain

COROLLARY 8. *If* $\mathcal{F}$ *is regular and* $1 \le q < \infty$ *then* $H_p^{S_q} \sim H_p^{s_q}$ $(0 < p < \infty)$ *and* $H_p^M \sim H_p^{S_2}$ $(0 < p \le 2)$.

Note that $H_p^{S_q} \sim H_p^{s_q}$ for $q \le p < \infty$ comes easily from the regularity. Furthermore, in the regular case $\|f\|_{H_1^{s_{q_1}}} \le C_{q_1,q_2} \|f\|_{H_1^{s_{q_2}}}$ $(q_1 > q_2)$, which is not true for non-regular $\mathcal{F}$.

COROLLARY 9 (Burkholder–Davis–Gundy [10], [13]). $H_p^{S_2}$ *is equivalent to* H_p^M *for* $1 \le p < \infty$.

Proof. This can be derived from Corollary 7 and from (27) for $1 \le p \le 2$. With the duality method (see e.g. Garsia [16], pp. 32–33) one can verify the equivalence for all $1 \le p < \infty$. ■

The strong q-variation

$$W_q(f) := \sup \Big\{ \Big(\sum_{k=1}^{\infty} |f_{n_k} - f_{n_{k-1}}|^q \Big)^{1/q} \Big\},$$

where the supremum runs over all increasing sequences of integers $0 = n_0 \le n_1 \le \ldots$, was investigated by Lepingle [23] and Pisier and Xu [27]. They have shown that

$$\|W_q(f)\|_q \le C_q \|S_q(f)\|_q (= C_q \|s_q(f)\|_q) \quad (1 \le q < 2)$$

and

$$\|W_q(f)\|_q \le C_q \|f^*\|_q \quad (2 < q < \infty).$$

By our method we obtain

$$\|W_q(f)\|_p \le C_p \|S_q(f)\|_p \quad (1 \le p \le q < 2), \tag{28}$$
$$\|W_q(f)\|_p \le C_p \|s_q(f)\|_p \quad (0 < p \le q < 2)$$

and

$$\|W_q(f)\|_p \le C_p \|f^*\|_p \quad (1 \le p \le q > 2). \tag{29}$$

Since $\|f^*\|_p \le C_p \|s_2(f)\|_p$ $(0 < p \le 2)$, from the preceding inequality we get

$$\|W_q(f)\|_p \le C_p \|s_2(f)\|_p \quad (0 < p \le 2 < q).$$

Note that Pisier and Xu [27] have also shown (28) and (29) for $q < p < \infty$.

7. Duality results. The dual spaces of the Hardy spaces generated by the operators S_q and s_q are now characterized.

THEOREM 14. *The dual of* $H_p^{S_q}$ *is* $H_{p'}^{S_{q'}}$, *where* $1 < p, q < \infty$ *or* $1 = q \le p < \infty$, *and* $1/p + 1/p' = 1/q + 1/q' = 1$.

Proof. Define the linear functional l_g by

$$l_g(f) = E\Big(\sum_{k=1}^{\infty} d_k f d_k g \Big), \tag{30}$$

where $g \in H_{p'}^{S_{q'}}$ is fixed and $f \in H_p^{S_q}$. Then it is obvious that

$$|l_g(f)| \le \|f\|_{H_p^{S_q}} \|g\|_{H_{p'}^{S_{q'}}}.$$

On the other hand, it follows from the well-known duality between $L_p(l_q)$ and $L_{p'}(l_{q'})$ that if l is in the dual of $H_p^{S_q}$, then there exist functions h_k $(k \in \mathbb{N})$ such that

$$l(f) = E\Big(\sum_{k=1}^{\infty} d_k f h_k \Big) = E\Big(\sum_{k=1}^{\infty} d_k f d_k h_k \Big)$$

and

$$\|(h_k)\|_{L_{p'}(l_{q'})} := \Big\| \Big(\sum_{k=1}^{\infty} |h_k|^{q'} \Big)^{1/q'} \Big\|_{p'} \le \|l\|.$$

Set $H := \sum_{k=1}^{\infty} d_k h_k$. By a generalization of Stein's inequality [32] (see Asmar and Montgomery-Smith [1], Theorem 3.1) we conclude that

$$\|H\|_{H_{p'}^{S_{q'}}} = \Big\| \Big(\sum_{k=1}^{\infty} |E_k h_k - E_{k-1} h_k|^{q'} \Big)^{1/q'} \Big\|_{p'} \le 2 \|(h_k)\|_{L_{p'}(l_{q'})} \le 2\|l\|,$$

which proves the theorem. ■

Note that the dual of $H_1^{S_1}$ was described by Herz [18]. The next theorem was proved by Pratelli for continuous time and for the operator s_2. Since some new ideas are needed, we will outline the major steps. First we need a lemma.

LEMMA 5. *Let* $0 < p < \infty$ *and* $p \vee 1 \le q < \infty$. *Then* $H_q^{S_q}$ *is dense in* $H_p^{s_q}$ *and in* $H_p^{S_q}$.

Proof. Since $\|f\|_{H_p^{s_q}}$ is not greater than the right hand side of (25), it is clear that $H_q^{S_q} \subset H_p^{s_q}$. It is also clear that every p-atom relative to s_q is in $H_q^{S_q}$. Suppose that $T = s_q$ in Theorem 1 and take the same stopping times ν_k, atoms a_k and real numbers μ_k $(k \in \mathbb{Z})$ as in Theorem 1. We now show that the sum $\sum_{k=l}^{m} \mu_k a_k$ converges to f in $H_p^{s_q}$ norm as $m \to \infty$ and $l \to -\infty$. Obviously,

$$f - \sum_{k=l}^{m} \mu_k a_k = (f - f^{\nu_{m+1}}) + f^{\nu_l}.$$

Notice that $f - f^{\nu_{m+1}} \to 0$ in $H_p^{s_q}$ norm as $m \to \infty$ because the a.e. limit of

$$s_q^p(f - f^{\nu_{m+1}}) = [s_q^q(f) - s_q^q(f^{\nu_{m+1}})]^{p/q}$$

is equal to zero and it can be majorized by the integrable function $s_q^p(f)$. Since $s_q(f^{\nu_l}) \le 2^l$, we obtain our statement. So the lemma is proved for $H_p^{s_q}$. Considering the atomic decomposition for the operator S_q^- one can verify the lemma for $H_p^{S_q}$ with the same method. ∎

THEOREM 15. *The dual of $H_p^{s_q}$ is $H_{p'}^{s_{q'}}$, where $1 < p \le q < \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1$.*

Proof. For the linear functional l_g ($g \in H_{p'}^{s_{q'}}$ is fixed) defined in (30) we again have

$$|l_g(f)| \le \|f\|_{H_p^{s_q}} \|g\|_{H_{p'}^{s_{q'}}} \qquad (f \in H_p^{s_q}).$$

Conversely, if l is in the dual of $H_p^{s_q}$ then it is also in the dual of $H_q^{S_q}$. Hence, there exists $g \in H_{q'}^{S_{q'}}$ such that l has the form (30) for all $f \in H_q^{S_q}$. By Lemma 5, $H_q^{S_q}$ is dense in $H_p^{s_q}$, hence the linear functional l is uniquely determined by (30).

First assume that $g \in H_\infty^{s_{q'}}$. Set

$$v_k := \frac{s_{q',k}^{p'-q'}(g)}{\|g\|_{H_{p'}^{s_{q'}}}^{p'-1}}.$$

We define the martingale h in the following way:

$$d_k h := v_k[|d_k g|^{q'-1} \operatorname{sign}(d_k g) - E_{k-1}(|d_k g|^{q'-1} \operatorname{sign}(d_k g))] \qquad (k \in \mathbb{N}).$$

Since v_k is $\mathcal{F}_{k-1}$-measurable, h is really a martingale. Thus

$$s_q^q(h) \le C_q \sum_{k=0}^{\infty} \frac{s_{q',k}^{qp'-qq'}(g)}{\|g\|_{H_{p'}^{s_{q'}}}^{qp'-q}} E_{k-1}|d_k g|^{q'} \le C_q \frac{s_{q'}^{qp'-(q-1)q'}(g)}{\|g\|_{H_{p'}^{s_{q'}}}^{qp'-q}}.$$

As $g \in H_\infty^{s_{q'}}$, this implies that $h \in H_q^{S_q}$. On the other hand,

$$E[s_q^p(h)] \le C_{p,q} \frac{E[s_{q'}^{p'}(g)]}{\|g\|_{H_{p'}^{s_{q'}}}^{p'}} = C_{p,q}.$$

Hence

$$C_{p,q}\|l\| \ge |l(h)| = \frac{1}{\|g\|_{H_{p'}^{s_{q'}}}^{p'-1}} E\Big[\sum_{k=0}^{\infty} s_{q',k}^{p'-q'}(g) E_{k-1}|d_k g|^{q'}\Big]$$

$$= \frac{1}{\|g\|_{H_{p'}^{s_{q'}}}^{p'-1}} E\Big[\sum_{k=0}^{\infty} s_{q',k}^{p'-q'}(g)(s_{q',k}^{q'}(g) - s_{q',k-1}^{q'}(g))\Big].$$

Applying the classical inequality

$$x^\alpha - 1 \le \alpha(x-1)x^{\alpha-1} \qquad (1 \le \alpha, x)$$

to $x = s_{q',k}^{q'}(g)/s_{q',k-1}^{q'}(g)$ and to $\alpha = p'/q' \ge 1$ we get

$$\frac{q'}{p'}[s_{q',k}^{p'}(g) - s_{q',k-1}^{p'}(g)] \le [s_{q',k}^{q'}(g) - s_{q',k-1}^{q'}(g)] s_{q',k}^{p'-q'}(g).$$

From this we can conclude that

$$C_{p,q}\|l\| \ge \frac{1}{\|g\|_{H_{p'}^{s_{q'}}}^{p'-1}} E(s_{q'}^{p'}(g)) = \|g\|_{H_{p'}^{s_{q'}}}.$$

This inequality for an arbitrary $g \in H_{q'}^{S_{q'}}$ satisfying (30) can be proved with a stopping time argument (see Pratelli [28]). ∎

Note that applying the inequality

$$|x+y|^q + |x-y|^q \le 2^{q-1}(|x|^q + |y|^q) \qquad (x, y \in \mathbb{R},\ q \ge 2)$$

we can prove that $H_p^{s_q}$ $(p \ge q \ge 2)$ is uniformly convex (cf. Pratelli [28]), hence it is reflexive (see Yosida [38]). This yields that Theorem 15 also holds for $p \ge q \ge 2$.

With the help of the atomic decomposition we generalize this result to $0 < p \le 1$.

THEOREM 16. *The dual of $H_p^{s_q}$ is $\mathcal{BMO}_{q'}(\alpha)$ where $0 < p \le 1$, $1 \le q < \infty$, $\alpha = 1/p - 1$ and $1/q + 1/q' = 1$.*

Proof. As we have seen in Lemma 5, $H_q^{S_q}$ is dense in $H_p^{s_q}$. Let $g \in \mathcal{BMO}_{q'}(\alpha)$; then, of course, $g \in H_{q'}^{S_{q'}}$. For $f \in H_q^{S_q}$ define l_g as in (30). Since

$$d_k f = \sum_{l=-\infty}^{\infty} \mu_l d_k a_l \qquad (k \in \mathbb{N}),$$

with convergence also in $H_q^{s_q}$ norm (see Lemma 5), where μ_l and a_l are the same as in Theorem 1, we can conclude that

$$l_g(f) = \sum_{k=1}^{\infty} \sum_{l=-\infty}^{\infty} E(\mu_l d_k a_l d_k g).$$

Applying the identity $a_{l,n} = a_{l,n}\chi(\nu_l < n)$ and Hölder's inequality we get

$$|l_g(f)| \le \sum_{l=-\infty}^{\infty} |\mu_l| E\Big(\sum_{k=1}^{\infty} |d_k a_l| \chi(\nu_l < k) |d_k g|\Big)$$
$$\le \sum_{l=-\infty}^{\infty} |\mu_l| \Big(E \sum_{k=1}^{\infty} |d_k a_l|^q\Big)^{1/q} \Big(E \sum_{k=1}^{\infty} |d_k g|^{q'} \chi(\nu_l < k)\Big)^{1/q'}.$$

By the definition of a p-atom,

$$\Big(E \sum_{k=1}^{\infty} |d_k a_l|^q\Big)^{1/q} = \|a_l\|_{H_q^{S_q}} = \|a_l\|_{H_q^{s_q}} \le P(\nu_l \ne \infty)^{-1/p+1/q}.$$

Therefore

$$|l_g(f)| \le \sum_{l=-\infty}^{\infty} |\mu_l| \|g\|_{\mathcal{BMO}_{q'}(\alpha)}$$
$$\le \Big(\sum_{l=-\infty}^{\infty} |\mu_l|^p\Big)^{1/p} \|g\|_{\mathcal{BMO}_{q'}(\alpha)} \le C_p \|f\|_{H_p^{s_q}} \|g\|_{\mathcal{BMO}_{q'}(\alpha)}.$$

We verify the converse similarly to Theorem 15 by using the test martingale

$$h := \frac{\sum_{k=1}^{\infty} [|d_k g|^{q'-1} \mathrm{sign}(d_k g) - E_{k-1}(|d_k g|^{q'-1} \mathrm{sign}(d_k g))] \chi(\nu < k)}{P(\nu \ne \infty)^{1/p-1/q} (E \sum_{k=1}^{\infty} |d_k g|^{q'} \chi(\nu < k))^{1/q}}.$$

In this case

$$\|h\|_{H_p^{s_q}} \le C_{p,q} \frac{[E(\sum_{k=1}^{\infty} E_{k-1} |d_k g|^{q'} \chi(\nu < k))^{p/q}]^{1/p}}{P(\nu \ne \infty)^{1/p-1/q} (E \sum_{k=1}^{\infty} |d_k g|^{q'} \chi(\nu < k))^{1/q}}$$
$$\le C_{p,q} \frac{(E \sum_{k=1}^{\infty} |d_k g|^{q'} \chi(\nu < k))^{1/q} P(\nu \ne \infty)^{1/p-1/q}}{P(\nu \ne \infty)^{1/p-1/q} (E \sum_{k=1}^{\infty} |d_k g|^{q'} \chi(\nu < k))^{1/q}} = C_{p,q}.$$

Hence

$$C_{p,q} \|l\| \ge |l(h)| = \frac{E(\sum_{k=1}^{\infty} |d_k g|^{q'} \chi(\nu < k))}{P(\nu \ne \infty)^{1/p-1/q} (E \sum_{k=1}^{\infty} |d_k g|^{q'} \chi(\nu < k))^{1/q}}.$$

Taking the supremum over all stopping times we obtain

$$C_{p,q} \|l\| \ge \|g\|_{\mathcal{BMO}_{q'}(\alpha)}.$$

With a slight modification the theorem can also be shown in the $q = 1$ case. ∎

Note that this theorem was proved by Lepingle [23] for $p = 1$ and by Herz [19] and Weisz [35] for $q = 2$.

Now Theorem 14 is extended to $p = 1$.

THEOREM 17. *The dual space of $H_p^{S_q}$ $(1 \le p \le q < \infty)$ can be given the norm*

$$\|g\| := \|g\|_{H_{p'}^{s_{q'}}} + \|g\|_{H_{p'}^{S_\infty}} \quad (1 < q' \le p' \le \infty),$$

where $1/p + 1/p' = 1/q + 1/q' = 1$ and $H_\infty^{s_{q'}} := \mathcal{BMO}_{q'}$.

Proof. If $g \in H_{p'}^{s_{q'}} \cap H_{p'}^{S_\infty}$ then clearly $g \in H_{q'}^{S_{q'}}$. Let l_g be defined on $H_q^{S_q}$ by (30). Recall that $H_q^{S_q}$ is dense in $H_p^{S_q}$ by Lemma 5, so l_g is also well-defined on $H_p^{S_q}$. Since $f_n \to f$ in $H_q^{S_q}$ norm $(f \in H_q^{S_q})$ as $n \to \infty$, we have

$$l_g(f) = \lim_{n\to\infty} E\Big(\sum_{k=1}^{n} d_k f d_k g\Big).$$

Let $f \in H_q^{S_q}$, $T = S_q$ and consider the martingales h and e defined in Lemma 4. It is easy to see that the stopped martingales h^n and e^n are in $H_q^{S_q}$ for all $n \in \mathbb{N}$. By Theorems 14–16 we obtain

$$\Big|E\Big(\sum_{k=1}^{n} d_k f d_k g\Big)\Big| \le \Big|E\Big(\sum_{k=1}^{n} d_k e d_k g\Big)\Big| + \Big|E\Big(\sum_{k=1}^{n} d_k h d_k g\Big)\Big|$$
$$\le C_p \|e\|_{H_p^{s_q}} \|g\|_{H_{p'}^{s_{q'}}} + \|h\|_{H_p^{S_1}} \|g\|_{H_{p'}^{S_\infty}}.$$

It follows from (25) and Corollary 5 that

$$\|e\|_{H_p^{s_q}} \le C_p \|e\|_{H_p^{S_q^-}} \quad (0 < p \le q).$$

Applying Lemma 4 for $p = q$ we get

$$|l_g(f)| \le C_p \|f\|_{H_p^{S_q}} (\|g\|_{H_{p'}^{s_{q'}}} + \|g\|_{H_{p'}^{S_\infty}}),$$

which yields that l_g is really a bounded linear functional on $H_p^{S_q}$.

Conversely, if l is an arbitrary bounded linear functional on $H_p^{S_q}$, then there exists $g \in H_{q'}^{S_{q'}}$ such that (30) holds for all $f \in H_q^{S_q}$. Obviously, l is bounded on $H_p^{S_1}$ and, by Proposition 2, it is also bounded on $H_p^{s_q}$. Consequently, from Theorems 14–16 we get

$$\|g\|_{H_{p'}^{s_{q'}}} \le C_q \|l\|, \quad \|g\|_{H_{p'}^{S_\infty}} \le C_q \|l\|,$$

which completes the proof of the theorem. ∎

Let us single out this result for $p = 1$. The formulas (8) and (9) imply that
$$\|\cdot\|_{\mathcal{BMO}^-_{q'}} \sim \|\cdot\|_{\mathcal{BMO}_{q'}} + \|\cdot\|_{H^{S_\infty}_\infty}.$$
So the following corollary follows.

COROLLARY 10. *The dual of* $H_1^{S_q}$ *is* $\mathcal{BMO}^-_{q'}$, *where* $1 \le q < \infty$ *and* $1/q + 1/q' = 1$.

This corollary is well known for $q = 2$ (see Garsia [16] and Herz [18]).

Note that using this result, Theorems 16 and 2 and Corollary 8 we conclude that $\mathcal{BMO}^{s_q}_p$ is equivalent to $\mathcal{BMO}^{S_q}_r$ for every $0 < p, r < \infty$ provided that the stochastic basis is regular.

Taking into account Theorem 14 we find that the $H^{S_{q'}}_{p'}$ norm and the norm given in Theorem 17 are equivalent.

COROLLARY 11. *For a martingale* f *we have*
$$\|S_{q'}(f)\|_{p'} \le C_{p,q}(\|s_{q'}(f)\|_{p'} + \|\sup_{n\in\mathbb{N}}|d_n f|\|_{p'}) \quad (1 < q' < \infty, 0 < p' < \infty).$$

Proof. The inequality follows from Theorems 14 and 17 for $q' \le p'$ and from Proposition 2 for $p' \le q'$. ■

Note that the converse of this inequality for $q' \le p'$ also follows from Proposition 2. Corollary 11 was proved by Rosenthal [30] and Burkholder [8] for $q' = 2$.

Of course, the duals of $\mathcal{BMO}_{q'}$ and $\mathcal{BMO}^-_{q'}$ are not $H_1^{s_q}$ and $H_1^{S_q}$, respectively. However, a kind of special subspaces of $\mathcal{BMO}_{q'}$ and $\mathcal{BMO}^-_{q'}$ can be defined, having duals $H_1^{s_q}$ and $H_1^{S_q}$, respectively. From now on until the end of the paper we suppose that every σ-algebra $\mathcal{F}_n$ is generated by *finitely many (set) atoms*. Denote by L the set of functions with mean zero which are $\mathcal{F}_n$-measurable for any $n \in \mathbb{N}$. Let $\mathcal{VMO}_q$ and $\mathcal{VMO}^-_q$ be the closures of L in $\mathcal{BMO}_q$ and in $\mathcal{BMO}^-_q$ norm $(1 < q' < \infty)$, respectively. It is simple to verify that a function $f \in \mathcal{BMO}_q$ is in $\mathcal{VMO}_q$ if and only if
$$\lim_{n\to\infty} \Big\| \Big(E_n \sum_{k=n+1}^{\infty} |d_k f|^q\Big)^{1/q} \Big\|_\infty = 0.$$
The analogous result holds for $\mathcal{VMO}^-_q$.

Now we can identify the duals of $\mathcal{VMO}_{q'}$ and $\mathcal{VMO}^-_{q'}$.

THEOREM 18. *If every* σ*-algebra* $\mathcal{F}_n$ *is generated by finitely many atoms then the dual of* $\mathcal{VMO}_{q'}$ *is* $H_1^{s_q}$ *and the dual of* $\mathcal{VMO}^-_{q'}$ *is* $H_1^{S_q}$, *where* $1 < q' < \infty$ *and* $1/q + 1/q' = 1$.

Proof. We only prove the first statement, the second one can be proved similarly. By Theorem 16 we know that
$$l_f(g) := \sum_{k=1}^{\infty} E(d_k f d_k g) \quad (g \in L)$$
is a bounded linear functional on $\mathcal{VMO}_{q'}$.

To verify the converse, we embed the normed vector space $(L, \|\cdot\|_{\mathcal{VMO}_{q'}})$ isometrically in a space whose dual can easily be found. Let X_n and Y_n $(n \in \mathbb{N})$ denote the spaces of function sequences $\xi = (\xi_k, k \ge n+1)$ for which
$$\|\xi\|_{X_n} := \Big\| \Big(E_n \sum_{k=n+1}^{\infty} |\xi_k|^{q'}\Big)^{1/q'} \Big\|_\infty < \infty$$
and
$$\|\xi\|_{Y_n} := \Big\| \Big(E_n \sum_{k=n+1}^{\infty} |\xi_k|^{q}\Big)^{1/q} \Big\|_1 < \infty,$$
respectively. Since every σ-algebra is generated by finitely many atoms, using the duality result concerning the $L_{q'}(l_{q'})$ spaces we can easily show that the dual of X_n is Y_n $(n \in \mathbb{N})$, more exactly, if $\xi = (\xi_k, k \ge n+1) \in X_n$ and $(f_k, k \ge n+1) \in Y_n$ then
$$\Big|\sum_{k=n+1}^{\infty} E(\xi_k f_k)\Big| \le \Big\| \Big(E_n \sum_{k=n+1}^{\infty} |\xi_k|^{q'}\Big)^{1/q'} \Big\|_\infty \Big\| \Big(E_n \sum_{k=n+1}^{\infty} |f_k|^{q}\Big)^{1/q} \Big\|_1;$$
on the other hand, if Ψ is a bounded linear functional on X_n then there exists a unique sequence $(f_k, k \ge n+1) \in Y_n$ such that
$$\Psi(\xi) = \sum_{k=n+1}^{\infty} E(\xi_k f_k) \quad (\xi \in X_n)$$
and
$$\|\Psi\| = \Big\| \Big(E_n \sum_{k=n+1}^{\infty} |f_k|^{q}\Big)^{1/q} \Big\|_1.$$
Let $X := \bigtimes_{n\in\mathbb{N}} X_n$ with the norm
$$\|\xi\|_X := \sup_{n\in\mathbb{N}} \|\xi^n\|_{X_n} \quad (\xi = (\xi^n, n \in \mathbb{N}) \in X).$$
Denote by X_0 those elements $\xi \in X$ for which $\xi^n = 0$ except for finitely many $n \in \mathbb{N}$. It is easy to see that if Λ is a bounded linear functional on X_0 then there exist $f^n = (f_k^n, k \ge n+1) \in Y_n$ $(n \in \mathbb{N})$ such that
$$\Lambda(\xi) = \sum_{n=0}^{\infty} \Lambda(\xi^n) = \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} E(\xi_k^n f_k^n) \quad (\xi \in X_0)$$

and

$$\|\Lambda\| = \sum_{n=0}^{\infty} \Big\| \Big(E_n \sum_{k=n+1}^{\infty} |f_k^n|^q \Big)^{1/q} \Big\|_1 < \infty.$$

Now we embed $(L, \|\cdot\|_{\mathcal{VMO}_{q'}})$ in X_0 in the following way:

$$R : L \to X_0, \qquad Rg := ((d_k g, k \ge n+1); n \in \mathbb{N}).$$

If l is in the dual of $\mathcal{VMO}_{q'}$ then $l \circ R^{-1}$ is a bounded linear functional on the range of R, thus, by Banach–Hahn's theorem, $l \circ R^{-1}$ can be extended onto X_0 preserving its norm. Consequently, there exist $f^n = (f_k^n, k \ge n+1) \in Y_n$ $(n \in \mathbb{N})$ such that

$$\|l\| = \|l \circ R^{-1}\| = \sum_{n=0}^{\infty} \Big\| \Big(E_n \sum_{k=n+1}^{\infty} |f_k^n|^q \Big)^{1/q} \Big\|_1 \tag{31}$$

and

$$l(g) = \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} E(d_k g f_k^n) = \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} E(d_k g d_k f_k^n) \qquad (g \in L).$$

The martingale $\sum_{k=n+1}^{\infty} d_k f_k^n$ is denoted by $g^{(n)}$. Set $f := \sum_{n=0}^{\infty} g^{(n)}$, which exists in the sense described in (B1). Thus f is a martingale and $d_k f = \sum_{n=0}^{k-1} d_k f_k^n$. Hence

$$l(g) = \sum_{k=1}^{\infty} E(d_k g d_k f) \qquad (g \in L).$$

Moreover, $g^{(n)} \in H_1^{s_q}$, $f \in H_1^{s_q}$ and, by (31),

$$\begin{aligned} \|f\|_{H_1^{s_q}} &\le \sum_{n=0}^{\infty} \|g^{(n)}\|_{H_1^{s_q}} \le \sum_{n=0}^{\infty} E\Big(\sum_{k=n+1}^{\infty} E_{k-1} |d_k f_k^n|^q \Big)^{1/q} \\ &= \sum_{n=0}^{\infty} E\Big[E_n \Big(\sum_{k=n+1}^{\infty} E_{k-1} |d_k f_k^n|^q \Big)^{1/q} \Big] \\ &\le \sum_{n=0}^{\infty} E\Big(E_n \sum_{k=n+1}^{\infty} |d_k f_k^n|^q \Big)^{1/q} \le 2\|l\|, \end{aligned}$$

which completes the proof of the theorem. ■

Note that this result was known for $q' = 2$ (for $\mathcal{VMO}_2^-$ see Schipp [31], for $\mathcal{VMO}_2$ see Weisz [35]).

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Relatively perfect σ-algebras for flows

by

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Abstract. We show that for every ergodic flow, given any factor σ-algebra $\mathcal{F}$, there exists a σ-algebra which is relatively perfect with respect to $\mathcal{F}$. Using this result and Ornstein's isomorphism theorem for flows, we give a functorial definition of the entropy of flows.

Introduction. Perfect σ-algebras play an important role in ergodic theory and statistical mechanics, especially in the spectral theory of dynamical systems with discrete time (measure preserving $\mathbb{Z}^d$-actions). The existence of these σ-algebras in the case $d \geq 2$ has been proved by the use of their relative versions (for $\mathbb{Z}^{d-1}$-actions), the so-called relatively perfect σ-algebras ([K1]). In [K2] the relatively perfect σ-algebras have been used to give a functorial definition of entropy of a $\mathbb{Z}^d$-action.

Blanchard in [B1] and Gurevič in [G2] have shown that for every ergodic flow there exists a perfect σ-algebra. The main purpose of the present paper is to prove a relative version of this result (Theorem B). The motivations of this theorem are, on the one hand, expected applications of relatively perfect σ-algebras to the investigation of the spectral structure of multidimensional flows and, on the other hand, an application to an axiomatic, i.e. functorial definition of entropy of one-dimensional flows. Such definitions have been given for $\mathbb{Z}^d$-action by Rokhlin ([Ro]) in the case $d = 1$ and by Kamiński in [K2] for $d \geq 2$, but it was not known whether such a characterization exists for flows. Section 1 contains definitions and auxiliary results needed in the sequel. In Section 2 we prove a relative version of the Abramov formula for the entropy of a special flow. Section 3 is devoted to relatively excellent σ-algebras. Results of these sections together with a relative version of the Ambrose–Kakutani–Rudolph theorem allow us to prove in Section 4 the existence of relatively perfect σ-algebras.

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