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On the joint spectral radius of commuting matrices

by

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Abstract. For a commuting n-tuple of matrices we introduce the notion of a joint spectral radius with respect to the p-norm and prove a spectral radius formula.

1. Introduction. Let V_p, $1 \le p \le \infty$, be the d-dimensional complex vector space $\mathbb{C}^d$ equipped with the p-norm:

$$\|x\|_p = \Big(\sum_{i=1}^{d} |x_i|^p\Big)^{1/p}, \qquad x \in \mathbb{C}^d.$$

Let $\mathbf{T} = (T_1, \ldots, T_n)$ be an n-tuple of $d \times d$ matrices. The joint spectrum $\sigma_{\mathrm{pt}}(\mathbf{T})$ of the n-tuple $\mathbf{T}$ is the set of all points $\boldsymbol{\lambda}$ in $\mathbb{C}^n$ for which there exists a non-zero vector x (called a joint eigenvector) in V_p satisfying

$$T_j x = \lambda_j x, \qquad j = 1, \ldots, n. \tag{1}$$

If the T_j commute then there exists a unitary matrix U such that $U^* T_j U$ is upper-triangular for all $1 \le j \le n$, i.e.,

$$U^* T_j U = \begin{pmatrix} \lambda_1^{(j)} & & & * \\ & \lambda_2^{(j)} & & \\ & & \ddots & \\ 0 & & & \lambda_n^{(j)} \end{pmatrix}.$$

We then have

$$\sigma_{\mathrm{pt}}(\mathbf{T}) = \{(\lambda_i^{(1)}, \ldots, \lambda_i^{(n)}) : i = 1, \ldots, d\}. \tag{2}$$

Let $|\boldsymbol{\lambda}|_p$ denote the p-norm of a vector $\boldsymbol{\lambda}$ in $\mathbb{C}^n$. We define the *geometric spectral radius* of $\mathbf{T}$ as

$$r_p(\mathbf{T}) = \max\{|\boldsymbol{\lambda}|_p : \boldsymbol{\lambda} \in \sigma_{\mathrm{pt}}(\mathbf{T})\}. \tag{3}$$

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The n-tuple $\mathbf{T}$ can be identified with an operator from V_p to the space V_p^n, the direct sum of n copies of V_p equipped with the natural p-norm. The norm of this operator is given by

$$\|\mathbf{T}\|_p = \sup_{\|x\|_p=1} \Big(\sum_{j=1}^{n} \|T_j x\|_p^p\Big)^{1/p}. \tag{4}$$

If $\mathbf{S}$ is another m-tuple of commuting $d \times d$ matrices then define $\mathbf{TS}$ to be the nm-tuple whose entries are T_iS_j, $1 \le i \le n$, $1 \le j \le m$, arranged in lexicographic order. Using this multiplication rule, one can successively define the powers $\mathbf{T}^2, \mathbf{T}^3, \dots$ Then $\mathbf{T}^m$ is a vector with n^m entries; these are the products $T_{i_1} \dots T_{i_m}$ where the indices are chosen from $\{1, \dots, n\}$ with repetitions allowed, and are then arranged lexicographically. The *algebraic spectral radius* of the n-tuple $\mathbf{T}$ is defined as

$$\varrho_p(\mathbf{T}) = \inf \|\mathbf{T}^m\|_p^{1/m}, \qquad 1 \le p \le \infty. \tag{5}$$

One of the basic theorems in matrix theory is the *spectral radius formula* which asserts that for a (single) matrix T the inf above is actually a limit, is independent of the norm $\|\cdot\|_p$, and is equal to the geometric spectral radius $r(T)$. See, e.g., [HJ, page 299]. The main result of the present note is an analogue for the joint spectral radius:

THEOREM 1. *Let $\mathbf{T} = (T_1, \dots, T_n)$ be an n-tuple of commuting $d \times d$ matrices. Then*

$$r_p(\mathbf{T}) = \varrho_p(\mathbf{T}), \qquad 1 \le p \le \infty. \tag{6}$$

For $p = 2$ alone this has been proved in [CH]. In fact, in that case this formula can be extended to infinite dimensions as shown in [MS].

One of the basic ideas of our proof lies in the introduction of a new operator $\widehat{\mathbf{T}}$ corresponding to any n-tuple $\mathbf{T}$. This is an operator on V_p^∞, the Banach space of all sequences $x = (x_1, x_2, \dots)$ with $x_j \in V_p$ and $\sum_{j=1}^\infty \|x_j\|_p^p < \infty$, equipped with its natural norm $\|x\|_p = (\sum_{j=1}^\infty \|x_j\|_p^p)^{1/p}$. The operator $\widehat{\mathbf{T}}$ is defined as

$$\widehat{\mathbf{T}} = \begin{pmatrix} T_1 & 0 & \dots \\ \dots & \dots & \dots \\ T_n & 0 & \dots \\ 0 & T_1 & \dots \\ \dots & \dots & \dots \\ 0 & T_n & \dots \\ \dots & \dots & \dots \end{pmatrix}, \tag{7}$$

i.e., $\widehat{\mathbf{T}}$ is an infinite matrix each of whose columns contains one copy of $\mathbf{T}$ according to the rule

$$\widehat{\mathbf{T}}_{jk} = T_{j\,(\mathrm{mod}\,n)}, \quad (k-1)n+1 \le j \le kn, \qquad k = 1, 2, \dots$$

In Section 2 we investigate the properties of $\widehat{\mathbf{T}}$ and show that $\varrho_p(\mathbf{T})$ is the ordinary spectral radius of $\widehat{\mathbf{T}}$ acting on the space V_p^∞. This is used in proving the main theorem. Section 3 is devoted to infinite-dimensional Hilbert spaces. Here we observe how the operator $\widehat{\mathbf{T}}$ leads to a simpler arrangement of the proof in [MS]. We also derive an analogue of a theorem of Rota.

2. Proof of the main theorem

LEMMA 1. *Let $\mathbf{T}$ be an n-tuple of commuting matrices and let $\widehat{\mathbf{T}}$ be the operator defined in* (7). *Then*

(i) $\|\mathbf{T}\|_p = \|\widehat{\mathbf{T}}\|_p$,
(ii) $\widehat{\mathbf{T}^m} = (\widehat{\mathbf{T}})^m$,
(iii) $\|\mathbf{T}^m\|_p \le \|\mathbf{T}\|_p^m$,
(iv) *$\varrho_p(\mathbf{T})$ is the* (*ordinary*) *spectral radius of* $\widehat{\mathbf{T}}$.

Proof. Let $x = (x_1, x_2, \dots)$ be an element of V_p^∞. Then

$$\widehat{\mathbf{T}}x = (T_1x_1, \dots, T_nx_1, T_1x_2, \dots, T_nx_2, T_1x_3, \dots) = (\mathbf{T}x_1, \mathbf{T}x_2, \dots),$$

so that $\|\widehat{\mathbf{T}}x\|_p^p = \sum_{i=1}^\infty \|\mathbf{T}x_i\|_p^p \le \|\mathbf{T}\|_p^p \sum_{i=1}^\infty \|x_i\|_p^p = \|\mathbf{T}\|_p^p \|x\|_p^p$. Hence $\|\widehat{\mathbf{T}}\|_p \le \|\mathbf{T}\|_p$. On the other hand, for $x \in V_p$,

$$\|\mathbf{T}x\|_p^p = \sum_{j=1}^n \|T_jx\|_p^p = \|\widehat{\mathbf{T}}(x, 0, 0, \dots)\|_p^p \le \|\widehat{\mathbf{T}}\|_p^p \|x\|_p^p.$$

So $\|\mathbf{T}\|_p \le \|\widehat{\mathbf{T}}\|_p$. This proves (i).

The statement (ii) is an obvious consequence of the definition of $\mathbf{T}^m$. The statement (iii) follows from (i), (ii) and the fact that any operator norm is submultiplicative. To prove (iv) note that

$$\begin{aligned} \varrho_p(\mathbf{T}) &= \inf \|\mathbf{T}^m\|_p^{1/m} = \inf \|\widehat{\mathbf{T}^m}\|_p^{1/m} && \text{by (i)} \\ &= \inf \|(\widehat{\mathbf{T}})^m\|_p^{1/m} && \text{by (ii)}, \end{aligned}$$

and this is the (ordinary) spectral radius of $\widehat{\mathbf{T}}$. ∎

Let S be an invertible matrix. The tuple $S\mathbf{T}S^{-1}$ is defined as

$$S\mathbf{T}S^{-1} = (ST_1S^{-1}, \dots, ST_nS^{-1}). \tag{8}$$

LEMMA 2. *We have*

$$\|S\mathbf{T}S^{-1}\|_p \le \|S\|_p \|\mathbf{T}\|_p \|S^{-1}\|_p. \tag{9}$$

Proof. Let $\mathbf{R} = S\mathbf{T}S^{-1}$. Let $\mathrm{diag}(S, S, \dots)$ be the infinite block-diagonal matrix with the diagonal blocks all equal to S. The operator $\widehat{\mathbf{R}}$ on V_p^∞

is then the same as the operator $\operatorname{diag}(S, S, \ldots)\widehat{\mathbf{T}}\operatorname{diag}(S^{-1}, S^{-1}, \ldots)$. So

$$\|\widehat{\mathbf{R}}\|_p \le \|\operatorname{diag}(S, S, \ldots)\|_p \|\widehat{\mathbf{T}}\|_p \|\operatorname{diag}(S^{-1}, S^{-1}, \ldots)\|_p = \|S\|_p \|\widehat{\mathbf{T}}\|_p \|S^{-1}\|_p.$$

Now use Lemma 1(i). ∎

For $\boldsymbol{\lambda}$ in $\mathbb{C}^n$ define $\boldsymbol{\lambda}^m$ in the same way as $\mathbf{T}^m$.

Lemma 3. *We have*

$$\sigma_{\mathrm{pt}}(\mathbf{T}^m) = \{\boldsymbol{\lambda}^m : \boldsymbol{\lambda} \in \sigma_{\mathrm{pt}}(\mathbf{T})\}. \tag{10}$$

Proof. If $\boldsymbol{\lambda}$ is a joint eigenvalue of $\mathbf{T}$ with a joint eigenvector x then any product of the form $\lambda_{i_1} \ldots \lambda_{i_m}$ is an eigenvalue of $T_{i_1} \ldots T_{i_m}$ with the same eigenvector x. So $\boldsymbol{\lambda}^m$ is a joint eigenvalue of $\mathbf{T}^m$. ∎

Lemma 4. *For any commuting n-tuple $\mathbf{T}$ of matrices,*

$$r_p(\mathbf{T}) \le \|\mathbf{T}\|_p. \tag{11}$$

Proof. Let $\boldsymbol{\lambda} \in \sigma_{\mathrm{pt}}(\mathbf{T})$, and let $x \in V_p$ be such that $T_j x = \lambda_j x$ for all $j = 1, \ldots, n$. Then $|\lambda_j| \|x\|_p = \|T_j x\|_p$ for all $j = 1, \ldots, n$. Hence

$$\sum_{j=1}^{n} |\lambda_j|^p = \frac{1}{\|x\|_p^p} \sum_{j=1}^{n} \|T_j x\|_p^p \le \|\mathbf{T}\|_p^p.$$

This shows $r_p(\mathbf{T}) \le \|\mathbf{T}\|_p$. ∎

Lemma 5. *Let $\mathbf{T}$ be a commuting n-tuple of matrices. Then*

$$r_p(\mathbf{T}) \le \varrho_p(\mathbf{T}). \tag{12}$$

Proof. Applying Lemma 4 to the tuple $\mathbf{T}^m$ we get $r_p(\mathbf{T}^m) \le \|\mathbf{T}^m\|_p$. But $r_p(\mathbf{T}^m) = (r_p(\mathbf{T}))^m$ by Lemma 3. Hence

$$r_p(\mathbf{T}) \le \|\mathbf{T}^m\|_p^{1/m} \quad \text{for all } m = 1, 2, \ldots$$

So $r_p(\mathbf{T}) \le \varrho_p(\mathbf{T})$. ∎

We have noted before that for a commuting n-tuple $\mathbf{T}$ there exists a unitary matrix U such that $U^* T_j U$ is upper-triangular for all $j = 1, \ldots, n$. We denote the diagonal part of $U^* T_j U$ by D_j and the strictly upper-triangular part by N_j. Then $D_j = \operatorname{diag}(\lambda_1^{(j)}, \ldots, \lambda_d^{(j)})$. Let $\mathbf{D} = (D_1, \ldots, D_n)$ and $\mathbf{N} = (N_1, \ldots, N_n)$.

Lemma 6. *For the n-tuple $\mathbf{D}$ the geometric and algebraic spectral radii are equal, i.e.,*

$$\varrho_p(\mathbf{D}) = r_p(\mathbf{D}) = r_p(\mathbf{T}). \tag{13}$$

Proof. Let $x = (x_1, \ldots, x_d) \in V_p$. Then $D_j x = (\lambda_1^{(j)} x_1, \ldots, \lambda_d^{(j)} x_d)$. The norm of $\mathbf{D}$ as an operator from V_p to V_p^n is given by

$$\begin{aligned} \|\mathbf{D}\|_p &= \sup_{\|x\|_p = 1} \Big(\sum_{j=1}^{n} \|D_j x\|_p^p \Big)^{1/p} \\ &= \sup_{\|x\|_p=1} \Big(\sum_{j=1}^{n} \sum_{i=1}^{d} |\lambda_i^{(j)} x_i|^p \Big)^{1/p} = \sup_{\|x\|_p=1} \Big(\sum_{i=1}^{d} \sum_{j=1}^{n} |\lambda_i^{(j)}|^p |x_i|^p \Big)^{1/p} \\ &\le r_p(\mathbf{D}) \sup_{\|x\|_p=1} \Big(\sum_{i=1}^{d} |x_i|^p \Big)^{1/p} = r_p(\mathbf{D}). \end{aligned} \tag{14}$$

So $\varrho_p(\mathbf{D}) = \inf \|\mathbf{D}^m\|_p^{1/m} \le \|\mathbf{D}\|_p \le r_p(\mathbf{D})$. By Lemma 5, $\varrho_p(\mathbf{D}) = r_p(\mathbf{D})$. ∎

For any real t, let C_t be the $d \times d$ diagonal matrix with entries $t, t^2, \ldots, t^d$.

Lemma 7. *For any $\varepsilon > 0$, there exists t such that*

$$\|C_t U^* \mathbf{T} U C_t^{-1}\|_p < r_p(\mathbf{T}) + \varepsilon. \tag{15}$$

Proof. Let A be any $d \times d$ matrix. Then

$$C_t A C_t^{-1} = \begin{pmatrix} a_{11} & t^{-1} a_{12} & \ldots & t^{-d+1} a_{1d} \\ t a_{21} & a_{22} & \ldots & t^{-d+2} a_{2d} \\ \ldots & \ldots & \ldots & \ldots \\ t^{d-1} a_{d1} & t^{d-2} a_{d2} & \ldots & a_{dd} \end{pmatrix}. \tag{16}$$

If A is strictly upper-triangular then for large t we can make the p-norm of $C_t A C_t^{-1}$ as small as we want. We apply this fact to N_j (the strictly upper-triangular part of $U^* T_j U$) for all $j = 1, \ldots, n$. We choose t large enough so that $\|C_t N_j C_t^{-1}\|_p < \varepsilon/n$ for all $j = 1, \ldots, n$. Then

$$\begin{aligned} \|C_t U^* \mathbf{T} U C_t^{-1}\|_p &= \|C_t (\mathbf{D} + \mathbf{N}) C_t^{-1}\|_p \le \|C_t \mathbf{D} C_t^{-1}\|_p + \|C_t \mathbf{N} C_t^{-1}\|_p \\ &= \|\mathbf{D}\|_p + \|C_t \mathbf{N} C_t^{-1}\|_p < r_p(\mathbf{T}) + \varepsilon. \ ∎ \end{aligned}$$

The next lemma is the final step in the proof of the theorem.

Lemma 8. *For a commuting n-tuple $\mathbf{T}$, if $r_p(\mathbf{T}) < 1$ then $\|\mathbf{T}^m\|_p \to 0$ as $m \to \infty$.*

Proof. If $r_p(\mathbf{T}) < 1$ then by Lemma 7 there exists t such that $\|C_t U^* \mathbf{T} U C_t^{-1}\|_p < 1$. We have

$$\begin{aligned} \|\mathbf{T}^m\|_p &= \|(U C_t^{-1})(C_t U^* \mathbf{T}^m U C_t^{-1})(U C_t^{-1})^{-1})\|_p \\ &= \|(U C_t^{-1})(C_t U^* \mathbf{T} U C_t^{-1})^m (U C_t^{-1})^{-1})\|_p \\ &\le \|U C_t^{-1}\|_p \|(C_t U^* \mathbf{T} U C_t^{-1})^m\|_p \|(U C_t^{-1})^{-1}\|_p \quad \text{by Lemma 2} \\ &\le \|U C_t^{-1}\|_p \|(C_t U^* \mathbf{T} U C_t^{-1})\|_p^m \|(U C_t^{-1})^{-1}\|_p \quad \text{by Lemma 1(iii)}. \end{aligned}$$

Now as $m \to \infty$ the middle term tends to zero because $\|C_t U^* \mathbf{T} U C_t^{-1}\|_p < 1$. ■

To complete the proof of the theorem define for any $\varepsilon > 0$ a new n-tuple

$$\mathbf{S} = \frac{1}{r_p(\mathbf{T}) + \varepsilon}\mathbf{T}.$$

Then

$$\|\mathbf{S}^m\|_p = \left(\frac{1}{r_p(\mathbf{T}) + \varepsilon}\right)^m \|\mathbf{T}^m\|_p.$$

Since $r_p(\mathbf{S}) < 1$, Lemma 8 says that $\|\mathbf{S}^m\|_p \to 0$. So for sufficiently large m, $\|\mathbf{T}^m\|_p < (r_p(\mathbf{T}) + \varepsilon)^m$. Hence

$$\varrho_p(\mathbf{T}) \le r_p(\mathbf{T}).$$

In view of Lemma 5, this proves the theorem.

Remark. Let Σ be any bounded set of matrices and let Σ^m be the set consisting of products of matrices from Σ of length m. Let $\|\cdot\|$ be any operator norm on the space $\mathbb{C}^d$. The *Rota–Strang joint spectral radius* (see [RS]) of Σ is defined as $\nu(\Sigma) = \limsup_m \nu_m(\Sigma)$, where $\nu_m(\Sigma) = \sup\{\|A\| : A \in \Sigma_m\}$. In two recent papers, [BW] and [E], it has been shown that $\nu(\Sigma)$ is equal to the *generalized spectral radius* $\tau(\Sigma)$ (introduced in [DL]) defined by $\tau(\Sigma) = \limsup_m \tau_m(\Sigma)$, where $\tau_m(\Sigma) = \sup\{r(A) : A \in \Sigma_m\}$. If Σ is taken to be the set $\{T_1, \ldots, T_n\}$ and if the ∞-norm is used then it is easy to see that $\tau_m(\Sigma) = (r_\infty(\mathbf{T}))^m$ and $\nu_m(\Sigma) = \|\mathbf{T}^m\|_\infty$. Since $\nu(\Sigma)$ and $\tau(\Sigma)$ are equal, one gets $r_\infty(\mathbf{T}) = \varrho_\infty(\mathbf{T})$. Hence this gives another proof of Theorem 1 for the special case $p = \infty$.

3. The infinite-dimensional case. Now let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the space of all bounded operators on $\mathcal{H}$. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be an n-tuple of commuting elements of $\mathcal{L}(\mathcal{H})$. As before, we consider $\mathbf{T}$ to be a bounded operator from $\mathcal{H}$ to $\mathcal{H}^n$, the direct sum of n copies of $\mathcal{H}$. We will denote by $\sigma(\mathbf{T})$ the (*Taylor*) *joint spectrum* of $\mathbf{T}$ and by $\sigma_{\mathrm{app}}(\mathrm{T})$ the *joint approximate point spectrum* of $\mathbf{T}$. See [C2] for definitions. The *geometric spectral radius* is then defined to be

$$r(\mathbf{T}) = \sup\{|\boldsymbol{\lambda}|_2 : \boldsymbol{\lambda} \in \sigma(\mathbf{T})\}. \tag{17}$$

In [CZ], Chō and Żelazko have shown, in the more general context of a Banach space, that $r(T)$ does not change if, in (17), $\sigma(\mathbf{T})$ is replaced by any other joint spectrum having the polynomial spectral mapping property. In particular,

$$r(\mathbf{T}) = \sup\{|\boldsymbol{\lambda}|_2 : \boldsymbol{\lambda} \in \sigma_{\mathrm{app}}(\mathbf{T})\}. \tag{18}$$

The *algebraic spectral radius* is defined as

$$\varrho(\mathbf{T}) = \inf \|\mathbf{T}^m\|^{1/m}. \tag{19}$$

The operator $\widehat{\mathbf{T}}$ defined as in (7) is now an operator on $\mathcal{H}^\infty$ for which all the facts proved in Lemma 1 remain true. So the inf in (19) is actually a limit.

Let $\mathbf{M_T} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be the operator defined as

$$\mathbf{M_T}(A) = \sum_{j=1}^n T_j^* A T_j. \tag{20}$$

We then have

THEOREM 2 (Müller–Sołtysiak). *Let $\mathbf{T}$ be a commuting n-tuple of Hilbert space operators. Then*

$$r(\mathbf{T}) = r(\mathbf{M_T})^{1/2} = \varrho(\mathbf{T}) = \lim_{m\to\infty} \|\mathbf{T}^m\|^{1/m}. \tag{21}$$

Our next two remarks are directed towards a simplification of the proof in [MS] even while following their essential ideas.

By a theorem of Curto [C1], the (ordinary) spectrum of $\mathbf{M_T}$ and the joint spectrum of $\mathbf{T}$ are related by

$$\sigma(\mathbf{M_T}) = \Big\{\sum_{j=1}^n \overline{\lambda}_j \mu_j : \lambda, \mu \in \sigma(\mathbf{T})\Big\}. \tag{22}$$

Using this and the Cauchy–Schwarz inequality one sees that

$$r(\mathbf{M_T}) \le r(\mathbf{T})^2. \tag{23}$$

From the definition (17) it is also clear that $r(\mathbf{T})^2$ is a point in $\sigma(\mathbf{M_T})$. Hence

$$r(\mathbf{T})^2 \le r(\mathbf{M_T}). \tag{24}$$

This proves the first equality in (21). Next note that the operator $\mathbf{M_T}$ is a completely positive map on $\mathcal{L}(\mathcal{H})$ (see [P]). Such maps attain their norm at the identity operator I. Applying this to all powers of $\mathbf{M_T}$ we see that

$$\|\mathbf{M_T^m}\| = \|\mathbf{M_T^m}(I)\| = \|\mathbf{T}^m\|^2. \tag{25}$$

So the ordinary spectral radius formula for $\mathbf{M_T}$ gives the second equality in (21).

It is clear from Theorem 2 that

$$r(\mathbf{T}) \le \|\mathbf{T}\|. \tag{26}$$

Let $\mathcal{L}_{\mathrm{inv}}(\mathcal{H})$ and $\mathcal{L}_+(\mathcal{H})$ denote, respectively, the set of all invertible operators and the set of all positive-definite operators. For any $S \in \mathcal{L}_{\mathrm{inv}}(\mathcal{H})$

we have $\sigma(S\mathbf{T}S^{-1}) = \sigma(\mathbf{T})$, where $S\mathbf{T}S^{-1}$ is the tuple defined in (8). So it follows from (26) that

$$r(\mathbf{T}) \le \inf\{\|S\mathbf{T}S^{-1}\| : S \in \mathcal{L}_{\mathrm{inv}}(\mathcal{H})\}. \tag{27}$$

We can prove more: there is equality here; for a single operator this was proved by Rota [R].

THEOREM 3. *Let $\mathbf{T}$ be a commuting n-tuple of Hilbert space operators. Then*

$$r(\mathbf{T}) = \inf\{\|S\mathbf{T}S^{-1}\| : S \in \mathcal{L}_{\mathrm{inv}}(\mathcal{H})\} = \inf\{\|S\mathbf{T}S^{-1}\| : S \in \mathcal{L}_{+}(\mathcal{H})\}. \tag{28}$$

Proof. The proof for the case of a single operator T in [FN] can be modified for the present situation. To prove the assertion, we need to produce for each $\eta > r(\mathbf{T})$ an $S \in \mathcal{L}_{+}(\mathcal{H})$ such that $\|S\mathbf{T}S^{-1}\| < \eta$. By Theorem 2, given such an η we can find a positive integer m such that $\|\mathbf{T}^m\| < \eta^m$. This means that the operator

$$R = \sum_{m=0}^{\infty} \eta^{-2m}(\mathbf{T}^*)^m\mathbf{T}^m = I + \frac{1}{\eta^2}\sum_j T_j^*T_j + \frac{1}{\eta^4}\sum_{i,j} T_i^*T_j^*T_jT_i + \ldots$$

is well defined. (Here in all summations all subscript indices vary over $1, 2, \ldots, n$.) Note that $R \ge I$. Further,

$$\sum T_j^*RT_j = \sum T_j^*T_j + \frac{1}{\eta^2}\sum T_i^*T_j^*T_jT_i + \ldots = \eta^2(R-1) \le \eta^2 R.$$

Put $S = R^{1/2}$. Then

$$\begin{aligned}\|S\mathbf{T}S^{-1}\|^2 &= \|(S^{-1}\mathbf{T}^*S)(S\mathbf{T}S^{-1})\| = \Big\|\sum S^{-1}T_j^*S^2T_jS^{-1}\Big\| \\ &= \Big\|S^{-1}\Big(\sum T_j^*RT_j\Big)S^{-1}\Big\| \le \eta^2\|S^{-1}RS^{-1}\| = \eta^2. \ \blacksquare\end{aligned}$$

We remark that in the finite-dimensional case we have, from the discussion in Section 2, a version of Theorem 3 for all p-norms, $1 \le p \le \infty$. More precisely, we have, for any commuting tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of matrices,

$$r_p(\mathbf{T}) = \inf\|S\mathbf{T}S^{-1}\|_p, \tag{29}$$

where the infimum is taken over all invertible matrices S.

Can the result of Theorem 1 be extended to infinite dimensions? For $p = 2$ this question is answered in the affirmative by the theorem of Müller and Sołtysiak. For other values of p we formulate the problem as follows.

Cosider the Banach space l_p for $1 \le p \le \infty$. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be an n-tuple of commuting bounded operators on l_p. Equip l_p^n with the natural p-norm and define $\|\mathbf{T}\|_p$ as in (4). Let $r_p(\mathbf{T}) = \sup\{|\boldsymbol{\lambda}|_p : \boldsymbol{\lambda} \in \sigma(\mathbf{T})\}$, where $\sigma(\mathbf{T})$ is the Taylor joint spectrum of $\mathbf{T}$. Let $\varrho_p(\mathbf{T}) = \inf\|\mathbf{T}^m\|_p^{1/m}$. We propose the following

CONJECTURE. *For a commuting n-tuple $\mathbf{T}$ of bounded operators on l_p,*

$$r_p(\mathbf{T}) = \varrho_p(\mathbf{T}). \tag{30}$$

One of the referees has pointed out that we need not restrict our analysis to l_p spaces alone. Let X be any Banach space with norm $\|\cdot\|$. Let X_p^n be the direct sum of n copies of X, with the norm of an n-tuple $x = (x_1, \ldots, x_n)$ defined to be $\|x\| = (\sum_{j=1}^n \|x_j\|^p)^{1/p}$. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be an n-tuple of commuting bounded operators on X. As before, we can identify $\mathbf{T}$ with an operator from X to X_p^n. The norm of this operator is

$$\|\mathbf{T}\|_p = \sup_{\|x\|=1}\Big(\sum_{j=1}^n \|T_jx\|^p\Big)^{1/p}. \tag{31}$$

The number $\varrho_p(\mathbf{T})$ can be defined as before with respect to the above norm on operator-tuples. The definition of $r_p(\mathbf{T})$ is unchanged. The problem then is to show that these two quantities are equal.

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Martingale operators and Hardy spaces generated by them

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Abstract. Martingale Hardy spaces and BMO spaces generated by an operator T are investigated. An atomic decomposition of the space H_p^T is given if the operator T is predictable. We generalize the John–Nirenberg theorem, namely, we prove that the BMO_q spaces generated by an operator T are all equivalent. The sharp operator is also considered and it is verified that the L_p norm of the sharp operator is equivalent to the H_p^T norm. The interpolation spaces between the Hardy and BMO spaces are identified by the real method. Martingale inequalities between Hardy spaces generated by two different operators are considered. In particular, we obtain inequalities for the maximal function, for the q-variation and for the conditional q-variation. The duals of the Hardy spaces generated by these special operators are characterized.

1. Introduction. We consider martingale operators like Burkholder and Gundy did in their paper [10]. In the literature Hardy spaces generated by the maximal function or by the quadratic variations were dealt with. In this paper Hardy and BMO spaces generated by an operator T are investigated. Several new results are proved and many known results for the maximal function and quadratic variations are generalized to the case of an arbitrary operator T.

In Section 2 the basic definitions are given. In Section 3 the atoms are defined and the atomic decomposition of the H_p^T space generated by a predictable operator T is formulated. Two special cases of this result can be found in Herz [18] for the $\mathcal{P}_1$ space and in Weisz [35] for the conditional quadratic variation.

In the next section the sharp operator $T^\sharp$ of an operator T is introduced. The $\mathcal{BMO}_q$ spaces are defined and then generalized by considering the L_∞ norm of $T_q^\sharp$. The latter spaces are denoted by $\mathcal{BMO}_q^T$. We generalize the John–Nirenberg theorem [22] (see also Herz [18], Garsia [16]) and show that

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