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Operational quantities characterizing semi-Fredholm operators

by

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Abstract. Several operational quantities have appeared in the literature characterizing upper semi-Fredholm operators. Here we show that these quantities can be divided into three classes, in such a way that two of them are equivalent if they belong to the same class, and are comparable and not equivalent if they belong to different classes. Moreover, we give a similar classification for operational quantities characterizing lower semi-Fredholm operators.

1. Introduction. Several authors [2], [3], [5], [8], [9], [12]–[14], [16]–[21] have considered operational quantities in order to obtain characterizations and perturbation results for various classes of operators of Fredholm theory. For example, Schechter introduced in [13] operational quantities derived from the norm in the following way:

Let X, Y be infinite-dimensional Banach spaces, $L(X,Y)$ the class of all (continuous linear) operators from X into Y, and $S(X)$ the class of all infinite-dimensional (closed) subspaces of X.

An operator $T \in L(X,Y)$ is said to be *upper semi-Fredholm* if its range is closed and its kernel is finite-dimensional, and it is said to be *strictly singular* if no restriction of T to $M \in S(X)$ is an isomorphism. Denoting by

$$n(T) := \|T\|$$

the norm of $T \in L(X,Y)$, Schechter [13] (with a different notation) defined

$$in(T) := \inf\{n(TJ_M) : M \in S(X)\},$$
$$sin(T) := \sup\{in(TJ_M) : M \in S(X)\},$$

where J_M stands for the canonical inclusion of M into X, and proved that

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$$T \text{ is upper semi-Fredholm if and only if } in(T) > 0,$$
$$T \text{ is strictly singular if and only if } sin(T) = 0,$$

and for $K, T \in L(X, Y)$,

$$sin(K) < in(T) \Rightarrow T + K \text{ is upper semi-Fredholm.}$$

In particular, the last result unifies and improves previous results about the stability of upper semi-Fredholm operators under perturbation by small-norm and strictly singular operators (see [4]).

Analogous results have been obtained for operational quantities derived from the *injection modulus*

(1) $$j(T) := \inf\{\|Tx\| : x \in X,\ \|x\| = 1\},$$

or from the *Hausdorff measure of noncompactness*

(2) $$h(T) := \inf\{\varepsilon > 0 : TB_X \subset F + \varepsilon B_Y \text{ for some finite subset } F \subset Y\},$$

where B_X stands for the closed unit ball of X, and also for other operational quantities.

Here an operational quantity will be a procedure which determines, for every pair X, Y of infinite-dimensional Banach spaces, a map from $L(X, Y)$ into the nonnegative numbers.

Given two operational quantities a and b we will write $a \le \alpha b$, for $\alpha > 0$, if for any infinite-dimensional Banach spaces X, Y and $T \in L(X, Y)$ we have $a(T) \le \alpha b(T)$. We will say that a and b are *comparable* if $\alpha a \le b$ or $\alpha b \le a$ for some $\alpha > 0$; and we will say that they are *equivalent* if $\alpha a \le b \le \beta a$ for some $\beta > \alpha > 0$.

In this paper we will show that the operational quantities which have appeared in the literature characterizing upper semi-Fredholm operators can be divided into three classes, in such a way that two quantities are equivalent if they belong to the same class, and are comparable but not equivalent if they belong to different classes. We observe that, since the class $SF_+(X, Y)$ of upper semi-Fredholm operators is open in $L(X, Y)$, the distance $d(T)$ of $T \in L(X, Y)$ to the complement of $SF_+(X, Y)$ can also be used to characterize upper semi-Fredholm operators. All the quantities we consider are less than or equal to d, and we do not know if any of them is equivalent to d. An analogous classification will be given for operational quantities characterizing lower semi-Fredholm operators.

NOTATION. Throughout, X and Y will be infinite-dimensional Banach spaces, X^* the dual space of X, B_X the closed unit ball of X, $L(X, Y)$ the class of all (continuous linear) operators from X into Y, J_M the canonical inclusion of the subspace M of X into X, Q_M the quotient map from X onto X/M and $T^* \in L(Y^*, X^*)$ the conjugate operator of $T \in L(X, Y)$.

2. Operational quantities derived from the injection modulus. We will consider the following families of (closed) subspaces of X:

$$S(X) := \{M \subset X : M \text{ is an infinite-dimensional subspace of } X\},$$
$$S^*(X) := \{M \subset X : M \text{ is a finite-codimensional subspace of } X\}.$$

First we give the definitions of two operational quantities.

DEFINITION 2.1. For $T \in L(X, Y)$ the quantities s^*j and j_{Co} are defined by

$$s^*j(T) := \sup\{j(TJ_M) : M \in S^*(X)\},$$
$$j_{\mathrm{Co}}(T) := \sup\{\varepsilon > 0 : \exists Z, \exists K \in \mathrm{Co}(X, Z), \forall x \in X,\ \varepsilon\|x\| \le \|Tx\| + \|Kx\|\},$$

where Z is a Banach space and $\mathrm{Co}(X, Z)$ is the class of all compact operators from X into Z.

The quantity s^*j was introduced by Schechter [13], denoted by ν. In [19] s^*j was denoted by B because of its relation with the Bernstein numbers. The quantity j_{Co} was defined by Förster and Liebetrau [3].

We have [13, Lemma 2.13]

$$T \text{ is upper semi-Fredholm if and only if } s^*j(T) > 0.$$

Next we show that the quantities s^*j and j_{Co} coincide.

THEOREM 2.2. *For* $T \in L(X, Y)$,

$$s^*j(T) = j_{\mathrm{Co}}(T) = \sup\{\varepsilon > 0 : \exists K \in \mathrm{Co}(X, Y), \forall x \in X,\ \varepsilon\|x\| \le \|Tx\| + \|Kx\|\}.$$

Proof. Define

$$g(T) := \sup\{\varepsilon > 0 : \exists K \in \mathrm{Co}(X, Y), \forall x \in X,\ \varepsilon\|x\| \le \|Tx\| + \|Kx\|\}.$$

It is enough to prove the following chain of inequalities:

$$j_{\mathrm{Co}}(T) \le s^*j(T) \le g(T).$$

(a) $j_{\mathrm{Co}}(T) \le s^*j(T)$. Assume $\varepsilon < j_{\mathrm{Co}}(T)$. There exist a Banach space Z and an operator $K \in \mathrm{Co}(X, Z)$ such that $\varepsilon \le \|Tx\| + \|Kx\|$ for every $x \in X$ with $\|x\| = 1$. Moreover, given $\delta > 0$, since K is compact, there exists $M \in S^*(X)$ such that $n(KJ_M) < \delta$ [4, Theorem III.2.3]. Then we obtain

$$\begin{aligned}\varepsilon &\le \inf\{\|Tx\| + \|Kx\| : x \in M,\ \|x\| = 1\}\\ &\le \inf\{\|Tx\| : x \in M,\ \|x\| = 1\} + \sup\{\|Kx\| : x \in M,\ \|x\| = 1\}\\ &= j(TJ_M) + n(KJ_M) < s^*j(T) + \delta.\end{aligned}$$

Hence $\varepsilon \le s^*j(T)$.

(b) $s^*j(T) \le g(T)$. If $s^*j(T) = 0$, then the inequality is clear. Assume $s^*j(T) > 0$ and take $\varepsilon < s^*j(T)$. There exists a subspace $M \in S^*(X)$ such that

$$\varepsilon < j(TJ_M) \le s^*j(T),$$

hence for every $m \in M$ we have $\varepsilon\|m\| \le \|Tm\|$. We take a finite-dimensional subspace N of X such that $X = M \oplus N$. For each $x \in X$, let $x = m + n$ be the decomposition of x associated with the direct sum $M \oplus N$. We take an operator $A \in L(N, Y)$ such that $j(A) \ge n(T) + \varepsilon$; hence

$$(n(T) + \varepsilon)\|n\| \le \|An\| \quad \text{for every } n \in N,$$

and consider the operator $K : X \to Y$ given by

$$Kx := K(m + n) = An,$$

which is compact. For every $x \in X$ we obtain

$$\begin{aligned}\varepsilon\|x\| &\le \varepsilon\|m\| + \varepsilon\|n\| \le \|Tm\| + \varepsilon\|n\| \le \|Tx\| + \|Tn\| + \varepsilon\|n\| \\ &\le \|Tx\| + \|An\| = \|Tx\| + \|Kx\|.\end{aligned}$$

Hence $\varepsilon \le g(T)$ and the result is proved. ∎

The *Hausdorff* or *ball measure of noncompactness* of a bounded subset D of X is defined in the following way:

$$h(D) := \inf\{\varepsilon > 0 : D \subset F + \varepsilon B_X \text{ for some finite subset } F \subset X\}.$$

It is clear that the Hausdorff measure of noncompactness for operators defined by (2) satisfies

$$h(T) = h(TB_X). \tag{3}$$

From the Hausdorff measure of noncompactness for bounded subsets, the following operational quantities have been derived.

DEFINITION 2.3. For $T \in L(X, Y)$ the quantities h_{b} and h_{cb} are defined by

$$h_{\mathrm{b}}(T) := \inf\{h(TD) : D \subset X \text{ bounded}, \ h(D) = 1\},$$
$$h_{\mathrm{cb}}(T) := \inf\{h(TD) : D \subset X \text{ bounded countable}, \ h(D) = 1\}.$$

The quantity h_{b} has been considered in [2], [8], [17], [19]; and h_{cb} in [2] and [19]. These quantities are equivalent to s^*j, hence they characterize the upper semi-Fredholm operators.

PROPOSITION 2.4. *The quantities s^*j, h_{b} and h_{cb} are equivalent:*

$$h_{\mathrm{b}} \le h_{\mathrm{cb}} \le 2h_{\mathrm{b}}, \qquad (1/2)h_{\mathrm{cb}} \le s^*j \le 2h_{\mathrm{cb}}.$$

Proof. See [2, Propositions 4, 5]. ∎

Remark 2.5. (a) The quantity $h_{\mathrm{cb}}(T)$ coincides with the injection modulus of a certain operator $\widehat{T}$ associated with T. Consider $\ell_\infty(X)$, the Banach

space of all bounded sequences on X, with the supremum norm, and $\mathrm{rc}(X)$ the closed subspace of $\ell_\infty(X)$ of all sequences with relatively compact range. Given $T \in L(X, Y)$ we consider the operator

$$\widehat{T} : \ell_\infty(X)/\mathrm{rc}(X) \to \ell_\infty(Y)/\mathrm{rc}(Y)$$

given by

$$\widehat{T}((x_n) + \mathrm{rc}(X)) := (Tx_n) + \mathrm{rc}(Y).$$

We have [2, proof of Proposition 4]

$$h_{\mathrm{cb}}(T) = j(\widehat{T}).$$

(b) If we consider the *Kuratowski* or *set measure of noncompactness* k, instead of the Hausdorff measure of noncompactness h, we can obtain some other equivalent quantities, because $h \le k \le 2h$.

DEFINITION 2.6. For $T \in L(X, Y)$ the quantities sj and isj are defined by

$$sj(T) := \sup\{j(TJ_M) : M \in S(X)\} \qquad [13],$$
$$isj(T) := \inf\{sj(TJ_M) : M \in S(X)\} \qquad [9] \text{ (see also [6])}.$$

We have [9], [6]

$$T \text{ is upper semi-Fredholm if and only if } isj(T) > 0.$$

Moreover,

$$T \text{ is strictly singular if and only if } sj(T) = 0.$$

THEOREM 2.7. *For every $T \in L(X, Y)$,*

$$s^*j(T) \le isj(T);$$

however, the quantities s^*j and isj are not equivalent.

Proof. For every $M \in S(X)$ we have

$$s^*j(T) \le s^*j(TJ_M) \le sj(TJ_M).$$

Consequently, $s^*j(T) \le isj(T)$.

In order to show that s^*j and isj are not equivalent we take a Banach space X such that there exists a strictly singular operator $A \in L(X, X)$ which is not compact; for example, $X = L_1[0, 1]$ (see [4, Example III.3.10]). Note that for any $\alpha > 0$ we can choose A in such a way that

$$\alpha < i^*n(A) := \inf\{n(AJ_M) : M \in S^*(X)\}.$$

(The operational quantity i^*n was defined by Lebow and Schechter [8] and by Sedaev [16], and it is a measure of noncompactness for operators.) Consider the space $X \oplus X$ with the norm $\|(x, y)\| := \|x\| + \|y\|$ and the operator

$T : X \oplus X \to X \oplus X$ defined by $T(x,y) := (0, Ax)$. Clearly T is strictly singular, hence [9] (also [6])

$$isj(I-T) = isj(I) = 1,$$

where I is the identity operator on $X \oplus X$. Consider the subspace

$$G := \{(x, Ax) : x \in X\} \in S(X \oplus X).$$

If $M \in S^*(X \oplus X)$, then $M \cap G \in S^*(G)$; hence there exists $N \in S^*(X)$ such that

$$M \cap G = \{(x, Ax) : x \in N\}.$$

Moreover, since $n(AJ_N) > \alpha$, for some $z \in N$, $\|z\| = 1$, we have

$$\frac{\|(I-T)(z, Az)\|}{\|(z, Az)\|} \le \frac{1}{1+\alpha};$$

then

$$j((I-T)J_M) \le \frac{1}{1+\alpha};$$

hence

$$s^*j(I-T) \le \frac{1}{1+\alpha} = \frac{1}{1+\alpha} isj(I-T).$$

Consequently, there is no $\beta > 0$ such that $isj \le \beta s^*j$. ∎

The example given in the proof of Theorem 2.7 is inspired by [14, Example 15].

3. Operational quantities derived from the norm. The following operational quantity was introduced by Gramsch (see [13]). In [13] it was denoted by Γ, and in [19] it was denoted by G because of its relation with the Gelfand numbers.

DEFINITION 3.1. For $T \in L(X,Y)$ the quantity in is defined by

$$in(T) := \inf\{n(TJ_M) : M \in S(X)\}.$$

Remark 3.2. In [21], for $T \in L(X,Y)$, the operational quantity

$$\Delta'(T) := \sup\{in(TJ_P) : P \in S^*(X)\}$$

was introduced. This quantity coincides with in. In fact, it is clear that $in \le \Delta'$; moreover, given $P \in S^*(X)$, for every $M \in S(X)$ we have $M \cap P \in S(P)$; then $in(T) = in(TJ_P)$, hence $in(T) = \Delta'(T)$.

Given $T \in L(X,Y)$, we have [13]:

T is upper semi-Fredholm if and only if $in(T) > 0$.

When comparing operational quantities, the notion of distortable Banach space, as given by Schlumprecht [15], will be useful.

DEFINITION 3.3. Given a number $\lambda > 1$, we will say that the infinite-dimensional Banach space $(X, \|\cdot\|)$ is *λ-distortable* if there exists an equivalent norm $|\cdot|$ on X such that for each subspace $M \in S(X)$ we have

$$\sup\left\{\frac{|x|}{|y|} : x, y \in M,\ \|x\| = \|y\| = 1\right\} \ge \lambda.$$

We will say that X is *arbitrarily distortable* if it is λ-distortable for any $\lambda > 1$.

A *λ-distortion* of X will be an isomorphism A of X onto a Banach space Z such that for every $M \in S(X)$ we have $\lambda j(AJ_M) \le n(AJ_M)$.

James [7] proved that c_0 and ℓ_1 are λ-distortable for no $\lambda > 1$, and the long-standing open question if the spaces ℓ_p $(1 < p < \infty)$ are distortable has recently been solved by Odell and Schlumprecht [10], showing that they are arbitrarily distortable.

THEOREM 3.4. *For an infinite-dimensional Banach space X and $\lambda > 1$, the following assertions are equivalent*:

(a) *The space X is λ-distortable.*

(b) *There exists a λ-distortion of X.*

(c) *There exists an isomorphism A from X onto a Banach space Z such that for every $M \in S(X)$ we have $\lambda isj(AJ_M) \le in(AJ_M)$.*

Proof. (a)⇒(b). Let Z be the space X endowed with the norm $|\cdot|$ equivalent to $\|\cdot\|$ and satisfying (4). The identity operator

$$A : X \to Z, \qquad Ax := x,$$

is an isomorphism such that for every $M \in S(X)$ we have

$$\lambda \le \sup\left\{\frac{|Ax|}{|Ay|} : x, y \in M,\ \|x\| = \|y\| = 1\right\}$$
$$= \frac{\sup\{|Ax| : x \in M,\ \|x\| = 1\}}{\inf\{|Ay| : y \in M,\ \|y\| = 1\}} = \frac{n(AJ_M)}{j(AJ_M)}.$$

Hence A is a λ-distortion of X.

(b)⇒(c). Given $\varepsilon > 0$, for each $M \in S(X)$ there exists $N \in S(M)$ such that $n(AJ_N) < in(AJ_M) + \varepsilon$. We choose $P \in S(N)$ such that $j(AJ_P) + \varepsilon > sj(AJ_N)$. Then

$$\lambda isj(AJ_M) \le \lambda sj(AJ_N) < \lambda j(AJ_P) + \lambda\varepsilon$$
$$\le n(AJ_P) + \lambda\varepsilon < in(AJ_M) + \varepsilon + \lambda\varepsilon.$$

Consequently, $\lambda isj(AJ_M) \le in(AJ_M)$.

(c)⇒(a). Let A be an isomorphism from X onto a Banach space Z such that for every $M \in S(X)$ we have $\lambda isj(AJ_M) \le in(AJ_M)$. Then $|x| := \|Ax\|$

defines an equivalent norm on X, and for every $M \in S(X)$, we have

$$\lambda j(AJ_M) \le \lambda isj(AJ_M) \le in(AJ_M) \le n(AJ_M),$$

and consequently,

$$\sup\left\{\frac{|x|}{|y|} : x, y \in M,\ \|x\| = \|y\| = 1\right\}$$

$$= \sup\left\{\frac{\|Ax\|}{\|Ay\|} : x, y \in M,\ \|x\| = \|y\| = 1\right\} = \frac{n(AJ_M)}{j(AJ_M)} \ge \lambda.$$

That is, X is λ-distortable. ∎

COROLLARY 3.5. *For every* $T \in L(X,Y)$,

$$isj(T) \le in(T);$$

however, the operational quantities isj *and* in *are not equivalent.*

Proof. Clearly $sj \le n$; hence for every $M \in S(X)$ we obtain $sj(TJ_M) \le n(TJ_M)$; consequently, $isj(T) \le in(T)$.

On the other hand, given an arbitrarily distortable Banach space X (for example ℓ_2), for every $\lambda > 1$, there exist Z_λ and $A_\lambda \in L(X, Z_\lambda)$ such that

$$\lambda isj(A_\lambda) \le in(A_\lambda),$$

hence the quantities isj and in are not equivalent. ∎

Remark 3.6. It follows from Theorem 2.7 and Corollary 3.5 that the operational quantities s^*j and in (that is, the quantities B and G associated with the Bernstein numbers and Gelfand numbers [19]) are not equivalent. ∎

Now we relate the operational quantity in with another quantity derived from the Hausdorff measure of noncompactness for operators (2), (3).

DEFINITION 3.7. For $T \in L(X,Y)$ the quantity ih is defined by

$$ih(T) := \inf\{h(TJ_M) : M \in S(X)\}.$$

The quantity ih was introduced (independently) by Rakočević [12] and Tylli [17]. We show that the operational quantities in and ih are equivalent, hence ih characterizes the upper semi-Fredholm operators. We need the following lemma.

LEMMA 3.8. *Let* Y *be an infinite-dimensional Banach space and let* $U \subset Y$ *be a finite-dimensional subspace. For every* $\varepsilon > 0$ *there exists a finite-codimensional closed subspace* $V \subset Y$ *such that* $U \cap V = \{0\}$ *and* $\|u\| \le (1+\varepsilon)\|u+v\|$ *for every* $u \in U$ *and* $v \in V$.

Proof. See [5, Lemma 2(a)]. ∎

PROPOSITION 3.9. *For every* $T \in L(X,Y)$,

$$ih(T) \le in(T) \le 2ih(T).$$

Proof. It is obvious that $ih(T) \le in(T)$. Let $M \in S(X)$ and $\varepsilon > 0$. We have [1, Theorem II.3.9], [3, Proposition 1]

$$h(T) = \inf\{n(Q_U T) : U \subset Y \text{ is a finite-dimensional subspace}\},$$

where Q_U denotes the quotient map from Y onto Y/U. Then there exists $U \subset Y$ finite-dimensional such that

$$n(Q_U TJ_M) < h(TJ_M) + \varepsilon.$$

Using Lemma 3.8, we can find a finite-codimensional closed subspace $V \subset Y$ such that $U \cap V = \{0\}$ and for any $u \in U, v \in V$,

$$\|v\| \le \|u\| + \|u+v\| \le (2+\varepsilon)\|u+v\|.$$

Define $P := T^{-1}(V) \cap M \in S(X)$. Then

$$\begin{aligned} h(TJ_M) + \varepsilon > n(Q_U TJ_M) &\ge n(Q_U TJ_P) \\ &= \sup\{\inf\{\|u + Tx\| : u \in U\} : x \in P,\ \|x\| = 1\} \\ &\ge \sup\{(2+\varepsilon)^{-1}\|Tx\| : x \in P,\ \|x\| = 1\} = (2+\varepsilon)^{-1} n(TJ_P) \\ &\ge (2+\varepsilon)^{-1} in(T). \end{aligned}$$

Consequently, $in(T) \le 2h(TJ_M)$ for every $M \in S(X)$. Therefore $in(T) \le 2ih(T)$. ∎

Recall that the quantity d is the distance of an operator to the class of non-upper-semi-Fredholm operators and we have $T \in SF_+$ if and only if $d(T) > 0$. In the following diagram we summarize the relations between the operational quantities which characterize the upper semi-Fredholm operators. The symbol $\updownarrow$ means "equivalent" and $\to$ means "$\le$ and comparable not equivalent":

$$\begin{array}{ccccc} s^*j & \to isj \to & in & \le d \\ \updownarrow & & \updownarrow & \\ h_{\rm b} & & ih & \\ \updownarrow & & & \\ h_{\rm cb} & & & \end{array}$$

Remark 3.10. We do not know if the quantities in and d are equivalent.

Remark 3.11. It is well known that

(5) $$\lim(a(T^n)^{1/n}) = \inf\{|\lambda| : \lambda I - T \notin SF_+\},$$

for $a = in, s^*j$ [19], [17]. Consequently, (5) holds for $a = isj, ih, h_{\rm b}, h_{\rm cb}$. That is, the operational quantities which characterize the upper semi-Fredholm operators have the same asymptotic behaviour.

We observe that the asymptotic behaviour of the distance to ∂SF_+ is given by (5) (cf. [20]).

Remark 3.12. All the quantities characterizing the upper semi-Fredholm operators appearing in the literature are comparable. However, it is not difficult to define other quantities which are not comparable. Consider the operational quantities a and b defined by

$$a(T) := s^*j(T)in(T) \quad \text{and} \quad b(T) := isj(T)^2.$$

Using the example introduced in the proof of Theorem 2.7, for each $\alpha > 0$, there exists an operator T such that $isj(T) = in(T) = 1$ and $s^*j(T) \le 1/(1+\alpha)$; that is,

$$a(T) \le \frac{1}{1+\alpha} \quad \text{and} \quad b(T) = 1.$$

Hence, there is no $\beta > 0$ such that $\beta a \le b$.

In [15, Theorem 3] a Banach space $(X, \|\cdot\|)$ is considered such that, for any $n = 1, 2, \ldots$, there is an equivalent norm $|\cdot|_n$ such that, for any $x \in X$,

$$\frac{1}{\log_2(n+1)}\|x\| \le |x|_n \le \|x\|,$$

and for every $\varepsilon > 0$ and each $M \in S(X)$ there exist $x, y \in M$, $\|x\| = \|y\| = 1$, with

$$|x|_n > 1 - \varepsilon \quad \text{and} \quad |y|_n \le \frac{1+\varepsilon}{\log_2(n+1)}.$$

Clearly, the isomorphism

$$T : (X, \|\cdot\|) \to (X, |\cdot|_n), \qquad Tx := x,$$

satisfies $isj(T) = s^*j(T) = 1/\log_2(n+1)$ and $in(T) = 1$; that is,

$$a(T) = \frac{1}{\log_2(n+1)} \quad \text{and} \quad b(T) = \frac{1}{\log_2(n+1)^2}.$$

Hence, there is no $\beta > 0$ such that $\beta b \le a$.

Consequently, the operational quantities a and b are not comparable.

4. Lower semi-Fredholm operators. An operator $T \in L(X,Y)$ is said to be *lower semi-Fredholm* if its range is finite-codimensional (hence closed). In this section we classify operational quantities characterizing the lower semi-Fredholm operators, in a similar way to that in the previous section for upper semi-Fredholm operators.

If Y is an infinite-dimensional Banach space, consider the following families of (closed) subspaces of Y with associated infinite-dimensional quotient:

$$Q(Y) := \{U \subset Y : Y/U \text{ is infinite-dimensional}\},$$
$$Q_*(Y) := \{U \subset Y : U \text{ is a finite-dimensional subspace of } Y\}.$$

We denote by Q_U the quotient map of Y onto Y/U.

First we consider some operational quantities derived from the *surjection modulus*

$$q(T) := \sup\{\varepsilon > 0 : \varepsilon B_Y \subset TB_X\}$$

of $T \in L(X,Y)$.

DEFINITION 4.1. For $T \in L(X,Y)$ the quantities s_*q', sq' and isq' are defined by

$$s_*q'(T) := \sup\{q(Q_U T) : U \in Q_*(Y)\},$$
$$sq'(T) := \sup\{q(Q_U T) : U \in Q(Y)\},$$
$$isq'(T) := \inf\{sq'(Q_U T) : U \in Q(Y)\}.$$

The quantities s_*q' and sq' were introduced by Zemánek [19], and s_*q' was denoted by M because of its relation with the Mityagin numbers; isq' was introduced in [9], [6]. We have [19], [9], [6]:

$$T \text{ is lower semi-Fredholm} \Leftrightarrow s_*q'(T) > 0 \Leftrightarrow isq'(T) > 0.$$

Also, $sq'(T) = 0$ if and only if T is *strictly cosingular*; that is, $Q_U T$ is not a surjection for any $U \in Q(Y)$.

Förster and Liebetrau [3] defined, for $T \in L(X,Y)$,

$$q_{\mathrm{Co}}(T) := \sup\{\varepsilon \ge 0 : \exists Z,\ \exists K \in \mathrm{Co}(Z,Y),\ \varepsilon B_Y \subset TB_X + KB_Z\}.$$

PROPOSITION 4.2. *For* $T \in L(X,Y)$,

$$\begin{aligned} s_*q'(T) &= q_{\mathrm{Co}}(T) \\ &= \sup\{\varepsilon \ge 0 : \exists K \in \mathrm{Co}(X,Y),\ \varepsilon B_Y \subset TB_X + KB_X\}. \end{aligned}$$

Proof. It is enough to prove the following chain of inequalities:

$$\begin{aligned} q_{\mathrm{Co}}(T) &\le s_*q'(T) \\ &\le g'(T) := \sup\{\varepsilon \ge 0 : \exists K \in \mathrm{Co}(X,Y),\ \varepsilon B_Y \subset TB_X + KB_X\}. \end{aligned}$$

(a) $q_{\mathrm{Co}}(T) \le s_*q'(T)$. Let $\delta > 0$. If $q_{\mathrm{Co}}(T) - \delta < \alpha < q_{\mathrm{Co}}(T)$, then there exist a Banach space Z and a compact operator $K : Z \to Y$ such that

$$\alpha B_Y \subset TB_X + KB_Z.$$

Since K is compact, there exists $U \in Q_*(Y)$ such that $n(Q_U K) < \delta$. Consider the space $X \oplus Z$ with the norm $\|(x,z)\| := \max\{\|x\|, \|z\|\}$ and the operators

$$T_0 : X \oplus Z \to Y, \qquad T_0(x,z) := Tx,$$
$$K_0 : X \oplus Z \to Y, \qquad K_0(x,z) := Kz.$$

Note that $q(Q_U T_0) = q(Q_U T)$ and $n(Q_U K_0) = n(Q_U K)$. We have

$$\alpha B_{Y/U} \subset Q_U TB_X + Q_U KB_Z = Q_U(T_0 + K_0)B_{X\oplus Z},$$

Consequently,

$$\alpha \le q(Q_U(T_0+K_0)) \le q(Q_UT_0) + n(Q_UK_0) \le q(Q_UT) + \delta;$$

that is, $q(Q_UT) \ge \alpha - \delta$. Since δ is arbitrary, $s_*q'(T) \ge \alpha$; hence $q_{\mathrm{Co}}(T) \le s_*q'(T)$.

(b) $s_*q'(T) \le g'(T)$. If $s_*q'(T) = 0$, then the result follows from (a). Assume $s_*q'(T) > 0$. For each $\varepsilon < s_*q'(T)$ there exists $U \in Q_*(Y)$ such that $\varepsilon < q(Q_UT)$; that is, $\varepsilon B_{Y/U} \subset Q_UTB_X$. Take $V \in Q_*(X)$ such that U and V have equal dimension and let $A \in L(V,U)$ be an isomorphism from V onto U satisfying $q(A) \ge n(T)+\varepsilon$; that is,

$$(n(T)+\varepsilon)B_U \subset AB_V.$$

Let W be a complement of V. For each $x \in X$ let $x = w + v$ be the decomposition of x associated with the direct sum $W \oplus V$. Now the operator $K : X \to Y$ given by $Kx := (\varepsilon + n(T))Av$ is compact. Moreover, for every $y \in B_Y$, there is $x \in B_X$ such that

$$\varepsilon y + U = Tx + U \in B_{Y/U}.$$

We have $\varepsilon y - Tx \in U$ and $\|\varepsilon y - Tx\| \le n(T)+\varepsilon$. Then there is $v \in B_V$ such that $\varepsilon y - Tx = Av = Kv$, hence $\varepsilon y \in TB_X + KB_X$ and $\varepsilon \le g'(T)$. ■

THEOREM 4.3. *For every $T \in L(X,Y)$,*

$$s_*q'(T) \le isq'(T);$$

however, the operational quantities s_*q' and isq' are not equivalent.

Proof. Obviously $s_*q' \le sq'$; hence, for every $U \in Q(Y)$,

$$s_*q'(T) \le s_*q'(Q_UT) \le sq'(Q_UT).$$

Consequently, $s_*q'(T) \le isq'(T)$.

Now we show that s_*q' and isq' are not equivalent. We take a Banach space X such that there exists a strictly cosingular noncompact operator $A \in L(X,X)$; for example, we can take $X = L_1[0,1]$ (cf. [Remark III.3.11]). We define

$$T : X \oplus X \to X \oplus X, \quad T(x,y) := (Ay, 0).$$

In $X \oplus X$ we consider the norm $\|(x,y)\| := \max\{\|x\|, \|y\|\}$. Since T is strictly cosingular, $isq'(I-\delta T) = isq'(I) = 1$ for any $\delta > 0$ [9], [6]. We will show that $s_*q'(I-\delta T) \le (\alpha\delta)^{-1}$, where

$$\alpha := h(A) = \inf\{n(Q_VA) : V \in Q_*(X)\} > 0,$$

because A is noncompact (see the proof of Proposition 3.9).

We fix a finite-dimensional subspace $U \subset X \oplus X$. Then we take finite-dimensional subspaces U_1, U_2 of X such that $U \subset U_1 \oplus U_2$, and we let $V := U_1 + AU_2$. We have $n(Q_VA) \ge \alpha$. Then we can find $x \in X$ such that $\|x\| = 1$ and $\|Q_VAx\| = \mathrm{dist}(Ax, V) \ge \alpha$.

Assume that $(0,x)+U$ belongs to the range of $Q_U(I-T)$ (notice that $\|(0,x)\| = 1$). Then

$$(0,x) = (u_1,u_2) + (I-T)(y,z) = (u_1,u_2) + (y - Az, z),$$

with $y,z \in X$, $u_1 \in U_1$, $u_2 \in U_2$. So $x = u_2 + z$ and

$$0 = u_1 + y - Az = u_1 + y - Ax + Au_2.$$

Then $y = Ax - u_1 - Au_2$; hence

$$\|y\| \ge \mathrm{dist}(Ax, V) \ge \alpha.$$

In this way we conclude that $Q_U(I-T)(y,z) = (0,x)$ implies $\|(y,z)\| \ge \alpha$, and so $q(Q_U(I-T)) \le \alpha^{-1}$.

Since U is an arbitrary finite-dimensional subspace, $s_*q'(I-T) \le \alpha^{-1}$. In an analogous way, we obtain $s_*q'(I-\delta T) \le (\alpha\delta)^{-1}$. ■

Now we consider an operational quantity derived from the norm.

DEFINITION 4.4. For $T \in L(X,Y)$ the quantity in' is defined by

$$in'(T) := \inf\{n(Q_UT) : U \in Q(Y)\}.$$

Weis [18] introduced the quantity in' and proved that

$$T \text{ is lower semi-Fredholm if and only if } in'(T) > 0.$$

In [19], in' is denoted by K because of its relation with the Kolmogorov numbers.

THEOREM 4.5. *For every $T \in L(X,Y)$,*

$$isq'(T) \le in'(T);$$

however, the quantities isq' and in' are not equivalent.

Proof. From $q \le n$ we obtain $sq'(Q_UT) \le n(Q_UT)$ for every $U \in Q(Y)$, hence $isq'(T) \le in'(T)$.

Now, in order to show that isq' and in' are not equivalent, we consider a reflexive arbitrarily distortable Banach space X ($X = \ell_2$ for example; cf. [10]). For every $\lambda > 1$, let $A_\lambda : X \to Z_\lambda$ be a λ-distortion of X. Given any $M \in S(X)$ we have $\lambda j(A_\lambda J_M) \le n(A_\lambda J_M)$. As $j(B) = q(B^*)$ for every operator B (see [11, Proposition B.3.8]), we obtain

$$\lambda q(Q_UA_\lambda^*) \le n(Q_UA_\lambda^*) \tag{6}$$

with $U := M^\perp$, the annihilator of M. Since X is reflexive, any infinite-codimensional subspace U of X^* can be written in the form $M^\perp$ for a suitable infinite-dimensional subspace M of X. Consequently, (6) is true for any $U \in Q(X^*)$.

Let $\varepsilon > 0$. There exists $U \in Q(X^*)$ such that $n(Q_UA_\lambda^*) < in'(A_\lambda^*) + \varepsilon$. Moreover, there is $V \in Q(X^*)$, $U \subset V$, satisfying $q(Q_VA_\lambda^*)+\varepsilon > sq'(Q_UA_\lambda^*)$.

Then

$$\lambda isq'(A^*_\lambda) \le \lambda sq'(Q_U A^*_\lambda) < \lambda q(Q_V A^*_\lambda) + \lambda\varepsilon$$
$$\le n(Q_V A^*_\lambda) + \lambda\varepsilon < in'(A^*_\lambda) + \varepsilon + \lambda\varepsilon.$$

Consequently, $\lambda isq'(A^*_\lambda) \le in'(A^*_\lambda)$. Hence there is no $\beta > 0$ such that $in'(T) \le \beta isq'(T)$ for every operator T acting between infinite-dimensional Banach spaces. ■

Remark 4.6. It follows from Theorems 4.3 and 4.5 that the operational quantities $s_*q' = M$ and $in' = K$, associated with the Mityagin numbers and Kolmogorov numbers [19], are not equivalent.

Tylli [17] proved that the following operational quantity derived from the Hausdorff measure of noncompactness h,

$$ih'(T) := \inf\{h(Q_U T) : U \in S'(Y)\},$$

coincides with in'.

Remark 4.7. The class $SF_-(X,Y)$ of all lower semi-Fredholm operators from X into Y is an open subset of $L(X,Y)$, and $in'(T)$ is smaller than or equal to the distance of T to the boundary of $SF_-(X,Y)$ (denoted by $\partial SF_-(X,Y)$). Then the question arises whether in' and the distance to $\partial SF_-(X,Y)$ coincide, or are equivalent.

Remark 4.8. For $a = in'$, s_*q' we have [19]

$$\lim(a(T^n)^{1/n}) = \inf\{|\lambda| : \lambda I - T \notin SF_-\}. \tag{7}$$

Hence (7) also holds for $a = isq'$.

Note that (7) holds for the distance to ∂SF_- (cf. [20]).

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