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Adjoint characterisations of unbounded weakly compact, weakly completely continuous and unconditionally converging operators

by

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Abstract. Characterisations are obtained for the following classes of unbounded linear operators between normed spaces: weakly compact, weakly completely continuous, and unconditionally converging operators. Examples of closed unbounded operators belonging to these classes are exhibited. A sufficient condition is obtained for the weak compactness of T'' to imply that of T.

1. Introduction and preliminaries. In this paper we shall be considering a linear operator $T : X \supset D(T) \to Y$ where X and Y are normed spaces.

Let us first recall some facts about bounded operators. Let T be bounded and everywhere defined and let X and Y be Banach spaces. Then T is *weakly compact* if it transforms bounded sequences into sequences having a weakly convergent subsequence; T is *weakly completely continuous* if it transforms weak Cauchy sequences into weakly convergent sequences; and T is *unconditionally converging* if it transforms weakly unconditionally convergent series into unconditionally convergent series. In order to characterize these classes of operators we introduce, for a given normed space E, the following subsets of E'':

$$K(E) = \{e'' \in E'' : \text{there exists a sequence } (e_n) \text{ in } E \text{ such that } e'' = \sigma(E'', E')\text{-}\lim Je_n\},$$
$$N(E) = \{e'' \in E'' : \text{there exists a weakly unconditionally Cauchy series } \textstyle\sum e_i \text{ in } E \text{ such that } e'' = \sigma(E'', E')\text{-}\lim \sum_{i=1}^{n} Je_i\}$$

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(throughout this paper J denotes the canonical injection of a given normed space into its second dual). Clearly $J(\widetilde{E}) \subset N(E) \subset K(E) \subset E''$.

We have the following theorem of J. Howard and K. Melendez:

1.1. THEOREM [HM]. *Let $T : X \to Y$ be a bounded linear operator, where X and Y are Banach spaces, and let A denote X'' (resp. $K(X)$, $N(X)$). The following properties are equivalent:*

(i) *T is weakly compact (resp. weakly completely continuous, unconditionally converging),*

(ii) *$T''(A) \subset JY$,*

(iii) *T' is $\sigma(Y',Y)$-$\sigma(X',A)$-continuous.*

The unbounded analogues of these operators will now be defined.

Throughout the remainder of the paper T will denote a linear operator $T : X \supset D(T) \to Y$, where X and Y are normed spaces, unless otherwise specified. The domain, null space and range of T are denoted by $D(T)$, $N(T)$ and $R(T)$ respectively. The operator J_E^X (or simply J_E) denotes the natural injection of the subspace E into X.

The adjoint (or conjugate) of T is the operator T' defined by

$$D(T') = \{y' \in Y' : y'T \text{ is continuous on } D(T)\},$$
$$T' : Y' \supset D(T') \to D(T)', \quad (T'y')x' = y'(Tx) \quad (x \in D(T)).$$

Given a linear subspace M of X, J_M^X (or simply J_M) will denote the operator that is the natural injection of M into X. Then T' is the adjoint (or conjugate) of $TJ_{D(T)}$ in the usual sense (see e.g. [G, II.2.2]).

We write $B_X = \{x \in X : \|x\| \le 1\}$.

The operator T is called

- *weakly compact* if $TB_{D(T)}$ is relatively $\sigma(Y, D(T'))$-compact,
- *weakly completely continuous* if T transforms $\sigma(D(T), D(T)')$-Cauchy sequences into $\sigma(Y, D(T'))$-convergent sequences,
- *unconditionally converging* if whenever $\sum x_i$ is weakly unconditionally convergent in $D(T)$, each subseries $\sum Tx_{k_i}$ of $\sum Tx_i$ is $\sigma(Y, D(T'))$-convergent.

The corresponding classes of operators will be abbreviated $WC(X,Y)$, $WCC(X,Y)$ and $UC(X,Y)$ (or simply WC, WCC and UC). Evidently $WC \subset WCC \subset UC$.

Let Q denote the canonical quotient map of Y'' onto $Y''/D(T')^\perp$. Then, with the usual identification, $QY'' = D(T')'$ and thus the second adjoint of T presents as an operator $T'' : D(T)'' \supset D(T'') \to QY''$.

In Section 2 we shall obtain characterisations analogous to those of Theorem 1.1. The properties corresponding to (ii) and (iii) are:

(ii)′ $T''(A) \subset QJY$,

(iii)′ T' is $\sigma(D(T'),Y)$-$\sigma(D(T)',A)$-continuous, where $A = D(T)''$, $K(D(T))$ or $N(D(T))$. (However, other subsets of $D(T)''$ can be considered (see Proposition 2.2).)

It should be noted that the topology $\sigma(D(T'),Y)$ is Hausdorff if and only if T is closable (see [G, II.2.11]).

In Section 3 we describe examples and investigate special cases; in particular, the upper-semi-Fredholm operators (F_+-operators) (Definition 3.4), and the Tauberian operators (Definition 3.3).

Section 4 investigates the connection between weak compactness of T and that of T''.

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1.2. PROPOSITION. *If $x \in D(T)$ then $T''Jx = QJTx$.*

Proof. For $y' \in D(T')$ and $y'' \in Y''$, we have $(Qy'')(y') = (y''|D(T'))(y') = y''y'$. Hence

$$Qy'' = y''J_{D(T')}^{Y'}. \tag{1.2.1}$$

Now let $x \in D(T)$ and $y' \in D(T')$. Then $JxT'y' = y'Tx = (JT)y' = (JTx)J_{D(T')}^{Y'}y' = (QJTx)y'$ by (1.2.1). Therefore $(Jx)T'$ is continuous on $D(T')$, whence $Jx \in D(T'')$ and $T''Jx = QJTx$. ∎

1.3. PROPOSITION. *T'' is the only $\sigma(D(T''),D(T)')$-$\sigma(D(T')',D(T'))$-continuous operator from $D(T'')$ into $D(T')$ such that $T''J_{D(T)}^{D(T)''} = QJ_Y^{Y''}T$.*

Proof. Suppose $S : D(T)'' \supset D(T'') \to D(T')'$ is a $\sigma(D(T''),D(T)')$-$\sigma(D(T')',D(T'))$-continuous operator satisfying $SJ_{D(T)}^{D(T)''} = QJ_YT$. Let $x'' \in D(T'')$. By Goldstine's theorem there is a net (x_α) in $D(T)$ such that $\sigma(D(T)'',D(T)')$-$\lim Jx_\alpha = x''$. By assumption $Jx_\alpha \to Sx''$ with respect to $\sigma(D(T')',D(T'))$. Thus $Sx'' = \sigma(D(T')',D(T'))$-$\lim Jx_\alpha = \sigma(D(T')',D(T'))$-$\lim QJ_Y^{Y''}Tx_\alpha = \sigma(D(T')',D(T'))$-$\lim T''Jx_\alpha = T''x''$. ∎

2. Characterisations

2.1. PROPOSITION. *Let T' be continuous and let $T''(A) \subset QJY$, where $A = D(T)''$ (resp. $K(D(T))$, $N(D(T))$). Then $T \in WC$ (resp. WCC, UC).*

Proof. Since T' is continuous, we have $D(T'') = D(T)''$ [G, II.2.8].

Case 1: $T''(D(T)'') \subset QJY$. Let (x_α) be a net in $B_{D(T)}$. Then (Jx_α) is a net in $B_{D(T)''}$ which is $\sigma(D(T)'', D(T)')$-compact and accordingly has a subnet, assumed for simplicity to be itself, which is $\sigma(D(T)'', D(T)')$-convergent to some point $x'' \in B_{D(T)''}$. Now by the $\sigma(D(T''), D(T)')$-$\sigma(D(T')', D(T'))$-continuity of T'' (Proposition 1.3) we have

$$T''x'' = \sigma(D(T')', D(T'))\text{-}\lim T''Jx_\alpha.$$

By hypothesis $T''x'' = QJY$ where $y \in Y$. But $T''Jx_\alpha = QJTx_\alpha$ by Proposition 1.2. Hence $QJy = \sigma(D(T')', D(T'))$-$\lim QJTx_\alpha$. Therefore $y = \sigma(Y, D(T'))$-$\lim Tx_\alpha$, showing that $T \in WC$.

Case 2: $T''(K) \subset QJY$ (where $K = K(D(T))$). Let (x_n) be a $\sigma(D(T), D(T)')$-Cauchy sequence. Then by the uniform boundedness principle, the sequence of norms $(\|x_n\|)$ is bounded. The rest of the proof is now similar to that of the preceding case.

Case 3: $T''(N) \subset QJY$ (where $N = N(D(T))$). Let $\sum x_i$ be a weakly unconditionally convergent series in $D(T)$. Then $\sum |Jx_i(x')| < \infty$ for each $x' \in D(T)'$. It follows that the sequence of partial sums of $\sum x_i$ is norm bounded. The proof now proceeds as in the previous two cases. ■

2.2. Proposition. *Let B be a linear subspace of $D(T'')$. Then the following properties are equivalent*:

(i) $T''(B) \subset QJY$,
(ii) T' *is* $\sigma(D(T'), QJY)$-$\sigma(D(T)', B)$-*continuous.*

Proof. The proof is exactly analogous to that of the bounded case [HM], making the appropriate changes for the general case.

(i)⇒(ii). Assume $T''(B) \subset QJY$ and let y'_α be a net in $D(T')$ which is $\sigma(D(T'), QJY)$-convergent to y'. For $b \in B$ we have $T''b \in QJY$ and thus $(T''b)(y') = \lim(T''b)(y'_\alpha)$, i.e. $b(T'y') = \lim b(T'y'_\alpha)$. Thus $T'y' = \sigma(D(T)', B)$-$\lim T'y'_\alpha$. Hence T' is $\sigma(D(T'), QJY)$-$\sigma(D(T)', B)$-continuous.

(ii)⇒(i). Assume (ii) and let $b \in B$. We shall verify that the linear functional $T''b$ is $\sigma(D(T'), QJY)$-continuous. Let (y'_α) be a net in $D(T')$ which is $\sigma(D(T'), QJY)$-convergent to y'. Then by (ii) we have $b(T'y') = \lim b(T'y'_\alpha)$, i.e. $(T''b)(y') = \lim(T''b)(y'_\alpha)$, as required. Hence $T''(B) \subset QJY$. ■

2.3. Proposition. *Let $S : Y \supset D(S) \to Z$ be a continuous operator, let $T \in WC(X, Y)$ (resp. $WCC(X, Y)$, $UC(X, Y)$) and let $R(T) \subset D(S)$. Then $ST \in WC$ (resp. WCC, UC).*

Proof. Case 1: $T \in WC$. Let (x_α) be a bounded net in $D(ST) = D(T)$. Then (Tx_α) has a subnet, for simplicity assumed to be itself, which is $\sigma(Y, D(T'))$-convergent. Evidently $(ST)' = T'S'$, by the hypothesis on S. Let $z' \in D((ST)') = D(T'S')$. Then $z' \in D(S')$ and $S'z' \in D(T')$ whence

$$\lim z'(STx_\alpha) = \lim S'z'(Tx_\alpha) = S'z'y = z'(Sy)$$

for some $y \in Y$. Therefore $ST \in WC$.

Case 2: $T \in WCC$. Let (x_n) be a $\sigma(D(ST), D(ST)')$-Cauchy sequence. Then (x_n) is $\sigma(D(T), D(T)')$-Cauchy and hence (Tx_n) is $\sigma(Y, D(T'))$-convergent. The proof now proceeds as in the previous case.

Case 3: $T \in UC$. Let $\sum x_i$ be a weakly unconditionally convergent series in $D(ST)$, hence in $D(T)$. Then for each subseries $\sum Tx_{k_i}$ there exists $y \in Y$ such that $y = \sigma(Y, D(T'))$-$\lim \sum_{i=1}^n Tx_{k_i}$. The rest of the proof now follows as in the previous two cases. ■

2.4. Theorem. *Let T be a closable unconditionally converging operator and let Y be complete. Then T is continuous.*

Proof. We may clearly suppose that X is complete and that T is densely defined and non-zero. Since T is closable, $D(T')$ is a total subset of Y' [G, II.2.11] and accordingly the functional

$$\|y\|_1 = \sup_{y' \in D(T')} \frac{|y'y|}{\|y'\| + \|T'y'\|} \quad (y \in Y)$$

defines a renorming of Y. Write $Y_1 = (Y, \|\cdot\|_1)$ and let J_1 denote the identity from Y into Y_1. Write $T_1 = J_1 T$. Then T_1 is a continuous operator. Let $\widetilde{T}_1$ be the closure of T_1. By Proposition 2.3, $T_1 \in UC$. Let W denote the injective operator $J_{Y_1}^{\widetilde{Y}_1} J_1$. We shall verify that $R(\widetilde{T}_1) \subset J_{Y_1} Y_1$. Indeed, let $x \in X = D(\widetilde{T}_1)$ and select a sequence (x_n) in $D(T)$ such that $\|x - \sum_{i=1}^n x_i\| \to 0$. Then $\sum x_i$ is weakly unconditionally convergent. Since $T_1 \in UC$ there exists $y_1 \in Y_1$ such that $y'_1 \sum_{i=1}^n T_1 x_i \to y'_1 y_1$ for every $y'_1 \in Y'_1$. But $\widetilde{T}_1(\sum x_i) = \widetilde{T}_1 x$. Therefore $J_{Y_1} y_1 = \widetilde{T}_1 x$ as required. The proof is completed by showing that $D(W^{-1}\widetilde{T}_1) = X$ for then, since $W^{-1}\widetilde{T}_1$ is evidently a closed operator, $W^{-1}\widetilde{T}_1$ will be bounded by the Closed Graph Theorem and the continuity of T then follows since $T = W^{-1}\widetilde{T}_1|D(T)$. We have $D(W^{-1}\widetilde{T}_1) = \widetilde{T}_1^{-1}(D(W^{-1})) = \widetilde{T}_1^{-1}(J_{Y_1}Y_1) = D(\widetilde{T}_1) = X$ since $R(\widetilde{T}_1) \subset J_{Y_1}Y_1$, as required. ■

Examples 3.13 show that completeness of Y is essential in Theorem 2.4.

An example of a naturally arising closable operator with incomplete range space is the following: Let T be the inverse of a continuous or closable operator $S : Y \subset D(S) \to X$ (where Y may be complete). Then the operator $T_0 : X \supset D(T) \to R(T)$ defined by $T_0 x = Tx$ $(x \in D(T))$ is closable.

2.5. Corollary. *If T is an unconditionally converging operator then T' is continuous.*

Proof. We may clearly suppose that Y is complete. Then if S is the regular contraction of T (see [K]), defined by $S = Q_{D(T')_\perp} T$, it follows from Theorem 2.4 that S' is continuous. Hence T' is continuous (see e.g. [C5, 3.3]). ■

2.6. LEMMA. *$T''B_{D(T'')}$ is contained in the $\sigma(D(T')', D(T'))$-closure of $QJTB_{D(T)}$.*

Proof. Let $x'' \in B_{D(T'')}$. By Goldstine's theorem there is a net (x_α) in $B_{D(T)}$ such that $Jx_\alpha \to x''$ in the $\sigma(D(T)'', D(T)')$-topology. Since T'' is $\sigma(D(T''), D(T)')$-$\sigma(D(')', D(T'))$-continuous, $T''x'' = \lim T''Jx_\alpha = \lim QJTx_\alpha$, convergence being in the $\sigma(D(T')', D(T'))$-topology. ■

2.7. THEOREM. *Let $A = D(T)''$ (resp. $K(D(T))$, $N(D(T))$). Then the following properties are equivalent:*

(i) *T is weakly compact (resp. weakly completely continuous, unconditionally converging).*

(ii) *T' is continuous and $T''(A) \subset QJY$.*

(iii) *T' is both norm-norm continuous and $\sigma(D(T'), QJY)$-$\sigma(D(T)', A)$-continuous.*

Proof. (i)⇒(ii). Case 1: $A = D(T)''$. Let T be weakly compact. Then $J(TB_{D(T)})$ is relatively $\sigma(JY, D(T'))$-compact. Let $x'' \in D(T'')$. By Lemma 2.6 there is a net (x_α) in $B_{D(T)}$ such that $Q(JTx_\alpha) \to T''x''$ with respect to $\sigma(D(T')', D(T'))$. By hypothesis (x_α) has a subnet, which for simplicity we assume to be itself, such that $JTx_\alpha \to Jy$ with respect to $\sigma(JY, D(T'))$ for some $y \in Y$. Hence $QJTx_\alpha \to QJy$ with respect to $\sigma(D(T')', D(T'))$. Since $\sigma(D(T')', D(T'))$ is a Hausdorff topology, it follows that $T''x'' = QJy \in QJY$.

Case 2: $A = K(D(T))$. Assume $T \in WCC$. Let $x'' \in K(D(T))$ and let (x_n) be a sequence in $D(T)$ such that $x'' = \sigma(D(T)'', D(T)')$-$\lim Jx_n$. Since (x_n) is thus $\sigma(D(T), D(T)')$-Cauchy, (Tx_n) is $\sigma(Y, D(T'))$-convergent. Let $y = \sigma(Y, D(T'))$-$\lim Tx_n$. We have

$$(Jx_n)x' \to x''(x') \quad (x' \in D(T)'). \tag{2.7.1}$$

Hence by the continuity of T'', $D(T'') = D(T)''$ and

$$(T''Jx_n)y' = (QJTx_n)y' \to (T''x'')y' \quad (y' \in D(T')) \tag{2.7.2}$$

since $x'' \in D(T'')$. But since

$$(Tx_n)y' \to y'y \quad (y' \in D(T')) \tag{2.7.3}$$

it follows from (2.7.2) and (2.7.3) that

$$(T''x'')y' = (QJy)y' \quad (y' \in D(T')).$$

Since the topology $\sigma(D(T')', D(T'))$ is evidently Hausdorff we have $T''x'' = QJY$. Hence (ii) follows.

Case 3: $A = N(D(T))$. Assume $T \in UC$ and let $x'' \in N(D(T))$. Then there exists a weakly unconditionally convergent series $\sum x_i$ in $D(T)$ such that $x'' = \sigma(D(T)'', D(T)')$-$\lim \sum_{i=1}^n Jx_i$. Since $T \in UC$, for each subseries $\sum Tx_{k_i}$ there exists $y \in Y$ such that $y = \sigma(Y, D(T'))$-$\lim \sum Tx_{k_i}$. The rest of the proof now proceeds as for the previous case.

By Corollary 2.5, T' is continuous for each of the three cases considered. The proof of the implication (i)⇒(ii) is now complete.

The remaining implications are covered by Propositions 2.1 and 2.2 since the continuity of T' ensures that $D(T'') = D(T)''$ (see [G, II.2.8]). ■

In the particular case of weakly compact operators, V. Fonf [F] has observed that condition (iii) of Theorem 2.7 can be simplified as follows:

(iii)′ *T' is $\sigma(D(T'), Y)$-$\sigma(D(T)', D(T)'')$-continuous* (see Corollary 4.7).

2.8. COROLLARY [F]. *Let K denote the class of operators T such that T' is continuous, and let Y be reflexive. Then $WC = WCC = UC = K$.*

3. Examples of weakly compact, weakly completely continuous and unconditionally converging operators. Let X be a Banach space. It is shown in [HM] that X is weakly sequentially complete if and only if $K(X) = JX$, and that X contains no isomorphic copies of c_0 if and only if $N(X) = JX$. Combining this observation with Proposition 2.1 yields immediately the following:

3.1. PROPOSITION. *Let $T : X \supset D(T) \to Y$ be a linear operator.*

(a) *If $D(T)$ is reflexive then T is weakly compact.*

(b) *If $D(T)$ is weakly sequentially complete then T is weakly completely continuous.*

(c) *If $D(T)$ is a Banach space containing no isomorphic copies of c_0 then T is unconditionally converging.*

We may relax the completeness condition on $D(T)$ in Proposition 3.1, substituting an alternative assumption. Thus we have:

3.2. PROPOSITION. *Let T' be continuous and let Y be a Banach space.*

(a) *If $\widetilde{D}(T)$ is reflexive then T is weakly compact.*

(b) *If $\widetilde{D}(T)$ is weakly complete then T is weakly completely continuous.*

(c) *If $\widetilde{D}(T)$ contains no isomorphic copies of c_0 then T is unconditionally converging.*

Proof. (a) Since T' is continuous we have $D(T'') = D(T)''$. Let $\widetilde{D}(T)$ be reflexive. Then $D(T'') = J\widetilde{D}(T)$. Hence $R(T'') = T''((JD(T))^\sim) \subset (T''JD(T))^\sim$ (by the continuity of T'') $= (QJR(T))^\sim = QJ\widetilde{R}(T) \subset QJ\widetilde{Y} = QJY$. Therefore T is weakly compact by Proposition 2.1.

(b) Let $\widetilde{D}(T)$ we weakly complete. Then $K(\widetilde{D}(T)) = J\widetilde{D}(T)$ (see [HM]) and hence $T''K(D(T)) \subset T''(J\widetilde{D}(T)) \subset QJY$ as in the proof of (a). Therefore T is weakly completely continuous by Proposition 2.1.

(c) Let $\widetilde{D}(T)$ contain no isomorphic copies of c_0. Then $N(\widetilde{D}(T)) = J(\widetilde{D}(T))$ (see [HM]) and the proof proceeds as in (a) and (b) above. ■

We recall the definitions of unbounded Tauberian and upper semi-Fredholm operators:

3.3. Definition [C5, C6]. The operator $T : X \supset D(T) \to Y$ is called *Tauberian* if $(T'')^{-1}(QJY) \subset J\widetilde{D}(T)$.

3.4. Definition [C1, C2]. The operator $T : X \supset D(T) \to Y$ is called *upper semi-Fredholm* (or an *F_+-operator*) if there exists a finite-codimensional subspace E of X (or of $D(T)$) for which $(T|E)^{-1}$ exists and is continuous.

It was shown in [C3] that F_+-operators are Tauberian. We shall obtain partial converses to Propositions 3.1 and 3.2.

3.5. Proposition. *Let T be a Tauberian operator.*

(a) *If T is weakly compact then $\widetilde{D}(T)$ is reflexive.*

(b) *If T is weakly completely continuous then $\widetilde{D}(T)$ is weakly complete.*

(c) *If T is unconditionally converging then $\widetilde{D}(T)$ contains no isomorphic copies of c_0.*

Proof. (a) Let T be weakly compact. Then $T''(D(T'')) \subset QJY$ by Theorem 2.7. Since T' is continuous, $D(T'') = D(T)''$. The Tauberian property now implies that

$$J\widetilde{D}(T) \subset D(T'') = D(T)'' \subset J\widetilde{D}(T),$$

whence $D(T)'' = J\widetilde{D}(T)$, as required.

(b) Let $T \in WCC$. Then $T''K(D(T)) \subset QJY$. Now $J\widetilde{D}(T) \subset K(D(T)) \subset J\widetilde{D}(T)$, whence $(KD(T))^{\sim} = K(\widetilde{D}(T)) = J\widetilde{D}(T)$. Therefore $\widetilde{D}(T)$ is weakly complete by [HM, 1.4].

(c) Similar to (b) above, using [HM, 1.3]. ■

As an immediate consequence of Proposition 3.5(a) it follows that the spectrum (and in fact the essential spectrum) of a weakly compact operator with non-prereflexive domain contains the zero vector.

Proposition 3.5(a) also shows that a weakly compact operator defined on a normed space with non-reflexive completion must have an infinite-dimensional precompact restriction [C1, 2.2].

The operator T is called *nowhere continuous* [C1] if there is no infinite-dimensional subspace M of its domain for which the restriction $T|M$ is continuous.

3.6. Example. On every separable reflexive Banach space X there exists an everywhere defined weakly compact and nowhere continuous operator from X into ℓ_1.

This follows immediately from Theorem 2.7 combined with [BKS, Proposition 5].

On the other hand, if Y is reflexive, $T \in L(\ell_1, Y)$ and $D(T) = \ell_1$, then T is weakly compact, and a dense subspace E of ℓ_1 exists such that $T|E$ is continuous, by the proof of [BKS, Proposition 2].

3.7. Example. A weakly compact operator between Banach spaces having a non-weakly compact restriction:

Let $X = L_1[0,1]$, $Y = L_q[0,1]$, $1 \le q \le \infty$, and define T by $D(T) = \{f \in L_1[0,1] : f' \text{ exists almost everywhere and } f' \in L_q[0,1]\}$, $Tf = f'$ $(f \in D(T))$, where f' is the derivative of f. Then $D(T') = (0)$ (see e.g. [CL2]). Hence T is weakly compact. Now let M be the dense subspace of $L_1[0,1]$ consisting of the absolutely continuous functions. Then, as is well known, $T|M$ is a closed Fredholm operator and hence not weakly compact by Proposition 3.5.

While F_+-operators are Tauberian, Proposition 3.5(b) can be sharpened in this special case (see Proposition 3.12). First we prove a lemma.

3.8. Lemma. *Let E be a closed finite-codimensional subspace of the normed space X. Then $D(T') = D((T|E)')$.*

Proof. Let P be a bounded projection defined on X with range E, and let $y' \in D((T|E)')$. Then

$$(*) \qquad |y'Te| \le \|y'T|E\|\|e\| < \infty \qquad (e \in E).$$

We have, for $x \in D(T)$,

$$\begin{aligned} |y'Tx| &\le |y'TPx| + |y'T(I-P)x| \\ &\le \|y'T|E\|\|Px\| + \|y'T|(I-P)X\|\|(I-P)x\| \quad \text{by } (*) \\ &\le \|y'T|E\|(\|x\| + \|(I-P)x\|) + \|y'T|(I-P)X\|(\|x\| + \|Px\|) \\ &\le \|y'T|E\|(1 + \|I-P\|)\|x\| + \|y'T|(I-PX)\|(1 + \|P\|)\|x\| \\ &< c\|x\| \end{aligned}$$

for some $c > 0$ depending on y', since $\dim(I-P)X < \infty$. Hence we have $D((T|E)') \subset D(T')$. Obviously $D(T') \subset D((T|E)')$. Therefore $D(T') = D((T|E)')$. ■

3.9. Corollary. *Let $T \in WC$ (resp. WCC, UC) and let E be a closed finite-codimensional subspace of $D(T)$. Then $T|E$ is WC (resp. WCC, UC).*

3.10. Definition. We say that a subclass $\mathbf{A}$ of the class of all Banach spaces has the *three-space property* if it satisfies the following condition: If M is a closed subspace of a Banach space X such that $X/M \in \mathbf{A}$ and $M \in \mathbf{A}$, then $X \in \mathbf{A}$.

3.11. PROPOSITION [AO]. *The classes of reflexive spaces, weakly sequentially complete spaces and Banach spaces containing no isomorphic copy of* c_0, *are spaces with the three-space property.*

3.12. PROPOSITION. *Let* T *be a weakly completely continuous* F_+-*operator. Then* $D(T)$ *is weakly complete.*

Proof. There exists a closed finite-codimensional subspace E of $D(T)$ such that $(T|E)^{-1}$ exists and is continuous [C1, 2.2]. We write $I_E = (T|E)^{-1}(T|E)$.

Let $T \in WCC$. Then $T|E \in WCC$ by Corollary 3.9. Hence $I_E \in WCC$ by Proposition 2.3. Consequently, every $\sigma(E, E')$-Cauchy sequence in E is $\sigma(E, E')$-convergent, i.e. E is weakly complete; in particular, E is complete. It follows that $D(T)$ is complete and hence by the three-space property (Proposition 3.11) weakly complete. ∎

3.13. EXAMPLES. We construct a closed unbounded operator defined everywhere on a Banach space of each of the following types:

(i) weakly compact,
(ii) weakly completely continuous but not weakly compact,
(iii) unconditionally converging but not weakly completely continuous.

Let X be an infinite-dimensional Banach space. Let T_0 be a restriction of the identity on X to a dense subspace of codimension one. Put $Y = R(T_0)$ and select a point $x_0 \in X \setminus D(T_0)$. Define $T \in L(X, Y)$ by $Tx = T_0 x$ $(x \in D(T_0))$ and $Tx_0 = 0$. Then T is an everywhere defined unbounded F_+-operator with one-dimensional null space. We shall verify that T is closable, hence closed. Let P, Q be complementary linear projections defined on X such that $R(P) = \mathrm{sp}\{x_0\}$, $R(Q) = D(T_0)$. Then $Tx = Qx$. Let $(x_n, Tx_n) \to (0, y)$ $(y \in Y = QX)$. Then $(x_n, x_n - Px_n) \to (0, y)$, whence $Px_n \to -y \in PX$. But $PX \cap QX = (0)$. Hence $y = 0$ whence T is closable as claimed. By Propositions 3.1 and 3.5, T has the property (i), (ii) or (iii) stated above for the following respective choices of X:

(i) X a reflexive space,
(ii) X a weakly complete non-reflexive space (e.g. ℓ_1 or L_1),
(iii) X a Banach space containing no isomorphic copy of c_0 which is not weakly complete (e.g. the James space [LT, p. 25]).

4. Adjoint weakly compact operators. Here we investigate the relationship between a weakly compact operator and its adjoint or preadjoint. We find (Corollary 4.3 below) that Gantmacher's theorem [DS, p. 485] holds for the Köthe "regular contraction" of $J_Y^{\check{Y}} T$, described in the proof of Corollary 2.5.

4.1. PROPOSITION. *The following statements are equivalent*:

(i) T *is continuous*,
(ii) T' *is continuous and* $D(T')_{\perp \check{Y}} = 0$.

Proof. Recall that T' is continuous if and only if $D(T')$ is $\sigma(Y', \widetilde{Y})$-closed ([L]; see also [C5]). Assume (i). Then T' is continuous, and since $J_Y T$ is continuous and thus closable, $D(T')$ is $\sigma(T', \widetilde{Y})$-dense [G, II.2.11] as $(J_Y T)' = T'$. Hence $D(T')_{\perp \check{Y}} = 0$. Thus (i)⇒(ii). Conversely, assume (ii). Then $D(T')$ is both $\sigma(Y', \widetilde{Y})$-dense and closed in Y', and thus $D(T') = Y'$. Therefore T is continuous by [G, II.2.8]. ∎

The following corollary can be viewed as an extension of the Closed Graph Theorem for Banach spaces. The operator T will be called *completely closable* if $J_Y^{\check{Y}} T$ is closable.

4.2. COROLLARY. *Let* T *be completely closable. Then* T *is continuous if and only if* T' *is continuous.*

An equivalent restatement of Corollary 4.2 is the following: For an arbitrary operator T, T' is continuous if and only if the regular contraction of $J_Y T$ is continuous.

4.3. COROLLARY. *Let* Y *be complete and* T *closable. Then* T *is weakly compact if and only if* T' *is weakly compact* (*and then* T *is continuous*).

Proof. Let T be weakly compact. Then T is continuous by Theorem 2.4 and hence T' is weakly compact by standard results. Conversely, let T' be weakly compact. Then T' is continuous by Corollary 2.5. Hence T is continuous by Corollary 4.2. Now the continuous extension $\widetilde{T}$ of T in $L(\widetilde{D}(T)Y)$ is continuous and has weakly compact adjoint T'. Therefore T is weakly compact by Theorem 1.1. ∎

4.4. COROLLARY. *If* T *is weakly compact then so is* T'.

Proof. Consider the regular contraction of T. ∎

Suppose that T is an F_--operator [CL2] (i.e. T' is a ϕ_+-operator) having a continuous adjoint. Then T cannot be weakly compact unless $D(T')$ is reflexive. This follows from combining Proposition 3.5 with Corollary 4.4. For example, if T is a linear surjection of ℓ_2 onto c_0 then $D(T')$ is finite-dimensional; in particular, T cannot be partially continuous [CL1, Cor. 4].

Further conditions for the implication T' weakly compact ⇒ T weakly compact to hold will now be derived. We shall do this through first generalising the notion of weak compactness by replacing the topology $\sigma(T, D(T'))$ by $\sigma(E, D(T'))$ for an arbitrary linear subspace E satisfying $JY \subset E \subset Y''$ (cf. [C6]). Our results combine those of the thesis [Gv] with earlier sections of the present paper.

4.5. PROPOSITION. *Let $JY \subset E \subset Y''$. Then the following properties are equivalent*:

(i) $T''D(T'') \subset QE$,
(ii) T' *is* $\sigma(D(T'),E)$-$\sigma(D(T)',D(T''))$-*continuous*.

Proof. (i)⇒(ii). Assume (i) and let (y'_α) be a net in $D(T')$ such that $y'_\alpha \to y'$ with respect to $\sigma(D(T'),E)$. Let $x'' \in D(T'')$. By assumption there exists $y'' \in E$ such that $T''x'' = Qy''$. Now $x''T'y' = T''x''y' = y''y' = \lim y''y'_\alpha = \lim T''x''y'_\alpha = \lim x''T'y'_\alpha$. Hence $T'y'_\alpha \to T'y'$ in $\sigma(D(T)', D(T''))$.

(ii)⇒(i). Suppose T' is $\sigma(D(T'),E)$-$\sigma(D(T)',D(T''))$-continuous. Let $x'' \in B_{D(T'')}$. By assumption there exists a finite subset $F \subset E$ such that $F_0 \subset (T')^{-1}(\{x''\}_0)$. We claim that

$$\bigcap_{z' \in QF} \ker(z') \subset \ker(T''x''). \tag{4.5.1}$$

Let $y' \in D(T') \setminus \ker(T''x'')$. Put $w' = 2(T''x''y')^{-1}y'$. Then $|x''T'w'| = |T''x''w'| = 2 > 1$ so $T'w' \notin \{x''\}_0$. Thus $w' \notin (T')^{-1}(\{x''\}_0) \supset F_0 \supset \bigcap_{y'' \in F} \ker(y'')$. Hence $y' \notin \bigcap_{y'' \in F} \ker(y'')$. Consequently, there exists $y'' \in F$ for which $y''y' \neq 0$. Since $y' \in D(T')$, this means that $Qy''y' \neq 0$, proving (4.5.1). It now follows from (4.5.1) that $T''x'' \in \operatorname{sp}\{QF\} \subset QE$. ∎

4.6. COROLLARY. *Let $JY \subset E \subset Y''$. Then the following properties are equivalent*:

(i) $T''D(T'') \subset QE$ *and T' is continuous*,
(ii) T' *is* $\sigma(D(T'),E)$-$\sigma(D(T)',D(T)'')$-*continuous*.

Proof. Since $D(T'') = D(T)''$ if and only if T' is continuous [G, II.2.8], by Proposition 4.5 it only remains to verify that (ii) implies $D(T'') = D(T)''$. Let $x'' \in D(T)''$ and let $\|y'_n - y'\| \to 0$ where $y'_n, y' \in D(T')$. Assuming (ii), we have $\sigma(D(T'),E)$-$\lim y'_n = y'$, whence $x''T'y'_n \to x''T'y'$. Therefore $x''T' \in D(T')'$, i.e. $x'' \in D(T'')$ as required. ∎

4.7. COROLLARY. *The following properties are equivalent*:

(i) T *is weakly compact*,
(ii) $T''D(T'') \subset QJY$ *and T' is continuous*,
(iii) T' *is* $\sigma(D(T'),Y)$-$\sigma(D(T)',D(T)'')$-*continuous*.

Proof. Put $E = JY$ in Corollary 4.6 and use Theorem 2.7. ∎

4.8. PROPOSITION. *Let $JY \subset E \subset Y''$. Consider the following properties*:

(i) $JTB_{D(T)}$ *is relatively* $\sigma(E,D(T'))$-*compact*,
(ii) $T''D(T'') \subset QE$.

Then (i)⇒(ii), *and if T' is continuous then* (i) *and* (ii) *are equivalent*.

Proof. Suppose $JTB_{D(T)}$ is relatively $\sigma(E,D(T'))$-compact. Let $x'' \in B_{D(T'')}$. By Lemma 2.6 there exists a net (x_α) in $B_{D(T)}$ such that $QJTx_\alpha \to T''x''$ in the $\sigma(D(T')',D(T'))$-topology. By assumption (x_α) has a subnet, for simplicity assumed to be itself, such that $JTx_\alpha \to y''$ with respect to $\sigma(E,D(T'))$ for some $y'' \in E$. Now $QJTx_\alpha \to Qy''$ with respect to $\sigma(D(T')',D(T'))$. Since $\sigma(D(T')',D(T'))$ is Hausdorff, $T''x'' = Qy''$. Hence (i)⇒(ii).

Now suppose that T' is continuous (so that $D(T'') = D(T)''$) and that $T''D(T'') \subset QE$. Let (x_α) be a net in $B_{D(T)}$. Then (Jx_α) is a net in $B_{D(T'')}$ and since $B_{D(T)''}$ is $\sigma(D(T)'',D(T)')$-compact, (x_α) has a subnet, assumed for simplicity to be itself, such that $Jx_\alpha \to x''$ with respect to $\sigma(D(T)'',D(T)')$ for some $x'' \in B_{D(T)''}$. By hypothesis there exists $y'' \in E$ such that $T''x'' = Qy''$. Now $TJx_\alpha \to T''x''$ in the $\sigma(D(T')',D(T'))$-topology. For all $y' \in D(T')$ we have $y''y' = Qy''y' = \lim(T''Jx_\alpha)y' = \lim(JTx_\alpha)y'$. Thus $JTx_\alpha \to y''$ with respect to $\sigma(E,D(T'))$. ∎

4.9. DEFINITION. On Y'' define the seminorm $\| \|_{D(T')}$ by

$$\|y''\|_{D(T')} = \sup\{|y''y'| : y' \in D(T')\}.$$

Note that $\|y''\|_{D(T')} \le \|y''\|$ $(y'' \in Y'')$. For the $D(T')$-seminorm topology to coincide on JY with the norm topology it is necessary and sufficient for $D(T')$ to have positive Dixmier characteristic ([D]; see e.g. [C4]).

4.10. NOTATION. Let E be a subspace of Y'' and F a subspace of Y'.

$\overline{E}^{D(T')}$ will denote the $\| \|_{D(T')}$ seminorm closure of E.

$JF|E$ will denote the set of functionals $\{Jy'|E : y' \in F\}$,

$E_{D(T')}$ will denote E equipped with the $\| \|_{D(T')}$ seminorm restricted to E.

4.11. PROPOSITION. *Let E be a subspace of Y''. Then $J\overline{D(T')}|E \subset (E_{D(T')})'$.*

Proof. Let $y' \in \overline{D(T')}$, $y' \neq 0$. Let (y'_n) be a sequence in $D(T')$ such that $y'_n \neq 0$ and $y'_n \to y'$. For each $y'' \in Y''$ we have $(Jy'/\|y'\|)y'' = \lim y''(y'_n/\|y'_n\|) \le \|y''\|_{D(T')}$. Thus for all $y'' \in E$, $|(Jy')y''| \le \|y'\| \|y''\|_{D(T')}$, so $Jy'|E \in (E_{D(T')})'$. ∎

4.12. PROPOSITION. *Let $JY \subset E \subset Y''$.*

(a) *If $JD(T')|E = (E_{D(T')})'$ then T' is continuous.*

(b) *Let $JY \subset E \subset \overline{JY}^{D(T')}$. Then $JD(T')|E = (E_{D(T')})'$ if and only if T' is continuous.*

Proof. (a) Suppose $JD(T')|E = (E_{D(T')})'$. By Proposition 4.11, $JD(T')|E = J\overline{D(T')}|E$. Since $JY \subset E$, $D(T') = \overline{D(T')}$ and so T' is continuous [G, II.2.15].

(b) Let $JY \subset E \subset \overline{JY}^{D(T')}$. Necessity follows from (a). For the converse, assume that T' is continuous. Let $z' \in (E_{D(T')})'$ and define y' by $y'y = z'Jy$ ($y \in Y$). For $y \in Y$ we have $|y'y| = |z'Jy| \le \|z'\|\,\|Jy\|_{D(T')} \le \|z'\|\,\|y\|$, so $y' \in Y'$. For each $x \in D(T)$, $|y'Tx| \le \|z'\|\,\|JTx\|_{D(T')} = \|z'\| \sup\{|y'Tx| : y' \in B_{D(T')}\} \le \|z'\|\,\|T'\|\,\|x\|$, so $y' \in D(T')$. Now let $y'' \in E$. Since $y'' \in \overline{JY}^{D(T')}$, there is a sequence (y_n) in Y such that $\|Jy_n - y''\|_{D(T')} \to 0$. Now $y''y' = \lim y'y_n = \lim z'Jy_n = z'y''$. Therefore $z' = Jy'|E$. Hence $(E_{D(T')})' \subset JD(T')|E$. The reverse inclusion is clear. ■

4.13. PROPOSITION. *Let $JY \subset E \subset \overline{JY}^{D(T')}$ and let T' be continuous. Then $B_{D(T')}$ is $\sigma(D(T'), E)$-compact.*

Proof. By Proposition 4.12, $(E_{D(T')})' = JD(T')|E$. We first show that $B_{(E_{D(T')})'} = JB_{D(T')}|E$. Let $z' \in B_{(E_{D(T')})'}$. Then $\|z'\| \le 1$ and there exists $y' \in D(T')$ such that $z' = Jy'|E$. Let $y \in B_Y$. Then $\|Jy\|_{D(T')} \le 1$ so $|y'y| = |z'Jy| \le 1$. Thus $y' \in B_{D(T')}$. Let $z' \in JB_{D(T')}|E$. Then there exists $y' \in B_{D(T')}$ such that $z' = Jy'|E$. For each $y'' \in B_{D(T')}$, $|z'y''| = |y''y'| \le 1$, so $\|z'\| \le 1$. Thus $B_{(E_{D(T')})'} = JB_{D(T')}|E$ as claimed.

By the Banach–Alaoglu theorem, $B_{(E_{D(T')})'}$ is $\sigma((E_{D(T')})', E_{D(T')})'$-compact. From the preceding discussion, $B_{D(T')}$ is $\sigma(D(T'), E)$-compact. ■

4.14. COROLLARY. *Let $JY \subset E \subset \overline{JY}^{D(T')}$. If T' is $\sigma(D(T'), E)$-$\sigma(D(T)', D(T)'')$-continuous then $T'B_{D(T')}$ is $\sigma(D(T)', D(T)'')$-compact.*

Proof. By Corollary 4.6, T' is continuous. Hence $B_{D(T')}$ is $\sigma(D(T'), E)$-compact by Proposition 4.13, and the result follows. ■

4.15. THEOREM. *Let E denote the subspace $\overline{JY}^{D(T')}$. Then the following properties are equivalent*:

(i) *$JTB_{D(T)}$ is relatively $\sigma(E, D(T'))$-compact and T' is continuous*,

(ii) *$T''D(T'') \subset QE$ and T' is continuous*,

(iii) *T' is $\sigma(D(T'), E)$-$\sigma(D(T)', D(T)'')$-continuous*,

(iv) *$T'B_{D(T')}$ is $\sigma(D(T)', D(T)'')$-compact.*

Proof. We have (i) ⇒ (ii) (Proposition 4.8) ⇒ (iii) (Corollary 4.6) ⇒ (iv) (Corollary 4.14). It only remains to prove that (iv)⇒(i). Assume (iv). Write $S = T'$ and $E_1 = JD(T)'$. By Theorem 2.4, S is continuous, and then by Theorem 2.7, $S''D(S'') \subset E_1$. By Proposition 4.5, S' is $\sigma(D(S''), E_1)$-$\sigma(D(S)', D(S)'')$-continuous. By Corollary 4.14, $S'B_{D(S')}$ is $\sigma(D(S)', D(S)'')$-compact, i.e. $T''B_{D(T'')}$ is $\sigma(D(T')', D(T')'')$-compact. Let (x_α) be a net in $B_{D(T)}$. Then there exists a subnet of (x_α), assumed for simplicity to be itself, and a point $x'' \in B_{D(T'')}$ such that $T''Jx_\alpha \to T''x''$ with respect to $\sigma(D(T')', D(T')'')$. Now $T''x''$ is thus in the $\sigma(D(T')', D(T')'')$-closure of QJY, which coincides with its norm closure. Hence there is a sequence (y_n) in Y such that $\|QJy_n \to T''x''\| \to 0$. Let $y'' \in Y''$ be a Hahn–Banach extension of $T''x''$. We have $\|Jy_n - y''\|_{D(T')} = \sup_{y' \in B_{D(T')}} \|y'y_n - y''y'\| = \|QJy_n - Qy''\| \to 0$. Hence $y'' \in E$. Now for $y' \in D(T')$ we have $y''y' = T''x''y' = \lim JTx_\alpha y$, so $y'' = \sigma(E, D(T'))$-$\lim JTx_\alpha$. Therefore $JTB_{D(T)}$ is relatively $\sigma(E, D(T'))$-compact, showing that (iv)⇒(i). ■

4.16. COROLLARY. *Let JY be $D(T')$-seminorm closed. Then T is weakly compact if and only if T' is weakly compact.*

We cite two cases in which JY is $D(T')$-seminorm closed:

(a) if Y is reflexive,

(b) if $D(T')$ has positive Dixmier characteristics [D] and Y is complete.

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