Representing non-weakly compact operators

by

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Abstract. For each \( S \in L(E) \) (with \( E \) a Banach space) the operator \( R(S) \in L(E^{**}/E) \) is defined by \( R(S)(x^{**} + E) = S^{**}x^{**} + E, \) \( x^{**} \in E^{**} \). We study mapping properties of the correspondence \( S \mapsto R(S) \), which provides a representation \( R \) of the weak Calkin algebra \( L(E)/W(E) \) (where \( W(E) \) denotes the weakly compact operators on \( E \)). Our results display strongly varying behaviour of \( R \). For instance, there are no non-zero compact operators in \( \text{Im}(R) \) in the case of \( L^1 \) and \( C(0,1) \), but \( R(L(E)/W(E)) \) identifies isometrically with the class of lattice regular operators on \( L^2 \) for \( E = L^2(J) \) (here \( J \) is James' space). Accordingly, there is an operator \( T \in L(E^{**}(J)) \) such that \( R(T) \) is invertible but \( T \) fails to be invertible modulo \( W(L^2(J)) \).

Introduction. Suppose that \( E \) and \( F \) are Banach spaces and let \( L(E,F) \) stand for the bounded linear operators from \( E \) to \( F \). The operator \( T : E \to F \) is weakly compact, denoted \( T \in W(E,F) \), if the image \( TB_E \) of the closed unit ball \( B_E \) of \( E \) is relatively weakly compact in \( F \). The quotient space \( L(E,F)/W(E,F) \) equipped with the norm \( \|S\|_w = \text{dist}(S,W(E,F)) \) is a complicated object and there is a need for useful representations of the elements \( S + W(E,F) \). A fundamental result due to Davis et al. [DFJP] provides for any \( S \in L(E,F) \) a factorization \( S = BA \) through a Banach space \( X \) so that \( X \) is reflexive if and only if \( S \in W(E,F) \). However, this construction is not adapted to the quotient space since the intermediate space \( X \) depends on \( S \).

We consider here the following natural concept: any \( S \in L(E,F) \) induces an operator \( R(S) : E^{**}/E \to F^{**}/F \) by

\[
R(S)(x^{**} + E) = S^{**}x^{**} + F, \quad x^{**} \in E^{**},
\]

where any Banach space is taken to be canonically embedded in its bidual (the inclusion \( E \to E^{**} \) is denoted by \( K_E \) if required). We have \( R(S) = 0 \).

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if and only if $S \in W(E, F)$ since $S \in W(E, F)$ precisely when $S^*E^* \subset F$ (see [DS, VI.4.2]). The induced map $S + W(E, F) \to R(S)$ gives an injective contraction from $L(E, F)/W(E, F)$ into $L(E^*/F, F^*/F)$. Moreover,

$$R(Id_E) = Id_E \cdot R, \quad R(ST) = R(S)R(T)$$

whenever $ST$ is defined. Hence $S + W(E) \to R(S)$ provides a representation of the weak Calkin algebra $W(E) = L(E)/W(E)$ and its image $\{R(S) : S \in L(E)\}$ is a subalgebra of $L(E^*/F)$ containing the identity. Some basic properties of $R$ are found in [Y1] and [Y2], whereas this representation was used to discuss invertibility modulo the weakly compact operators. It was employed in [Re], [LW] to exhibit discontinuous derivations on $L(E)$ and infinite-dimensional commutative quotient algebras of $L(E)$ for some Banach spaces $E$. Applications to tauberian operators appear in [AG]. A concrete interpretation of $R(S)$ for operators $S$ on $L^1(0, 1)$ was obtained in [WW].

This paper studies the mapping properties of the map $R$. We discuss the size of the image $\text{Im}(R)$ for concrete non-reflexive Banach spaces and the question whether $\text{Im}(R)$ is closed. We compare for this purpose in Section 1 some properties of the norm $\|R(\cdot)\|$, that measures the deviation of an operator from weak compactness, to those of other seminorms of this kind. Section 2 focuses on several results and examples displaying radically varying behaviour of $R(W(E))$. For instance, we establish that $\text{Im}(R)$ does not contain non-zero inessential operators in the case of many concrete spaces, such as $L^1(0, 1)$ or $C(0, 1)$. We also exhibit Banach spaces $X$ and $Y$ so that $X^{**}/X$ and $Y^{**}/Y$ are isomorphic to $\ell^2$ and $R$ is a surjection on $W(X)$, but $R(W(Y))$ is not even closed. Our main result (Theorem 2.6) identifies $\text{Im}(R)$ with the lattice regular operators on $\ell^2$ in the case of the countable $\ell^2$-sum $\ell^2(J)$ of James' space $J$. We also discuss some applications. An operator $S \in L(E)$ is called weak Fredholm if $S + W(E)$ is invertible in $L(E)/W(E)$. It remains unclear whether the weak Fredholm operators admit any geometric characterizations analogous to those of the Fredholm operators. Theorem 2.6 is applied to exhibit an operator $S \in L(\ell^2(J))$ so that $R(S)$ is invertible, but $S$ fails to be invertible modulo the weakly compact operators. Proposition 2.5 solves the following “inverse” problem: given a reflexive Banach space $E$ there is $X$ such that $X^{**}/X \cong E$ and $R : L(X) \to L(E)$ is onto.

1. Duality properties. This preliminary section compares $\|R(\cdot)\|$ with other measures of weak non-compactness. This determines whether the map $R$ has closed range or not, but quantities associated with weak compactness also have other applications and our results illustrate the quite delicate properties of such quantities (cf. [AT] and its references).

We will use standard Banach space terminology and notation in accordance with [LT2]. Let $E$ be a Banach space. Set $E_1 = \ell^1(B_E)$, $E_\infty = \ell^\infty(B_E)$ and let $Q_1 : E_1 \to E$ stand for the surjection $Q_1(\alpha)_{x \in B_E} = \sum_{x \in B_E} \alpha(x)x$ and $J_\infty : E \to E_\infty$ for the isometric embedding $J_\infty(x) = (x^*(e))_{e \in B_{E^*}}$. We refer to [Re] for the definition and examples of operator ideals. Let $I$ be a closed operator ideal in the sense that $I(E, F)$ is closed in the operator norm for all Banach spaces $E$ and $F$. Set

$$\gamma_I(S) = \inf \{ \varepsilon > 0 : SB_E \subset RB_E + \varepsilon B_F \}$$

for some Banach space $Z$ and $R \in I(Z, F)$,

$$\beta_I(S) = \inf \{ \varepsilon > 0 : \text{there is a Banach space } Z \text{ and } R \in I(E, Z) \text{ so that } \|Sx\| \leq \|Rx\| + \varepsilon \|x\|, \quad x \in E \}$$

for $S \in L(E, F)$, following [A] and [T2]. Then $\gamma_I$ and $\beta_I$ are seminorms in $L(E, F)$, and $\gamma_I(S) = 0$ if and only if there is a sequence $(S_n)$ in $I(E_1, F)$ so that $\lim_{n \to \infty} \|SQ_1 - S_n\| = 0$, while $\beta_I(S) = 0$ if and only if there is a sequence $(S_n)$ in $I(E, F_\infty)$ so that $\lim_{n \to \infty} \|J_\infty S - S_n\| = 0$ (see [A, 3.5], [T2, 1.1]).

Recall two consequences of the geometric Hahn–Banach theorem.

**Lemma 1.1** ([R, 2.1 and 2.2].) Let $E$, $F$, $G$ and $H$ be Banach spaces and suppose that $S \in L(E, F^*), T \in L(G, F), R \in L(H, F^*)$ and $\varepsilon > 0$.

(i) $\|Sx\| \leq \|Tx\| + \varepsilon \|x\|$ for all $x \in E$ if and only if $S^*B_{F^*} \subset T^*B_{G^*} + \varepsilon B_{F^*}$.

(ii) $\|S^*x^*\| \leq \|R^*x^*\| + \varepsilon \|x^*\|$ for all $x^* \in F^*$ if and only if $SB_E \subset RB_H + \varepsilon B_F$.

Define the adjoint ideal $I^*$ of the operator ideal $I$ by $I^*(E, F) = \{ S \in L(E, F^*) : S^* \in I(F^*, E^*) \}$ for Banach spaces $E$ and $F$. Recall that $I$ is injective if $I(E, F) = \{ S \in L(E, F) : J_\infty S \in I(E, F_\infty) \}$ for all $E$ and $F$. Our first duality result is quite general.

**Proposition 1.2.** Let $I$ be a closed injective operator ideal so that $S^* \in I(E^*, F^*)$ whenever $S \in I(E, F), E$ and $F$ Banach spaces. Then

$$\beta_I(S) = \gamma_I(S^*) = \beta_I(S^*)$$

for all $S \in L(E, F), E$ and $F$ Banach spaces.

**Proof.** Suppose that $\lambda > \beta_I(S)$ and take $R \in I(E, G)$ so that $\|Sx\| \leq \|Rx\| + \lambda \|x\|$ for all $x \in E$. Lemma 1.1(i) implies that $S^*B_{F^*} \subset R^*B_{G^*} + \lambda B_{F^*}$. Hence $\gamma_I(S^*) \leq \lambda$, since $R^* \in I^*(G^*, E^*)$ by the symmetry assumption on $I$. Thus $\gamma_I(S^*) \leq \beta_I(S)$.

Observe next that $\beta_I(T^*) \leq \gamma_I(T)$ for any $T \in L(E, F)$. In fact, assume that $\lambda > \gamma_I(T)$ and take $R \in I^*(G, F^*)$ so that $TB_E \subset RB_{E^*} + \lambda B_{F^*}$. Hence $\|T^*x^*\| \leq \|R^*x^*\| + \lambda \|x^*\|$ for all $x \in F^*$ by Lemma 1.1(ii) and we get
\( \beta_1(T^*) \leq \lambda \). The preceding facts imply
\[
\beta_1(S) = \beta_1(K_F S) = \beta_1(S^{**} K_F) \leq \beta_1(S^{**}) \leq \gamma_1(S^*),
\]
since \( \beta_1 \) is preserved by isometries. This proves the first equality in (1.1). Hence we see from [A, 5.1] that \( \beta_1(S^{**}) = \gamma_1(S^{**}) = \gamma_1(S^*) = \beta_1(S) \) for any \( S \in L(E, F) \).

The special case \( \beta_K(S) = \gamma_K(S^*) \) of (1.1) was verified in [GM, Thm. 2] by different means for the ideal \( K \) of compact operators. The customary notation \( \omega(S) = \gamma_W(S) \) for \( S \in L(E, F) \) will be used for the weakly compact operators \( W \). Thus \( \beta_K(S) = \omega(S^*) \) by (1.1), since \( W^* = W \) according to [DS, VI.4.8]. The example in [AT, Thm. 4] demonstrates that there are no uniform estimates between \( \omega(S) \) and \( \omega(S^*) \). We establish as a contrast that \( ||R(\cdot)|| \) is uniformly self-dual. Let \( \pi_{E^*} \) denote the canonical projection \( E^{**} \to E^* \) defined by \( \pi_{E^*}(u) = u_{E^*} \) for \( u \in E^{**} \) and set \( \varrho_{E^*} = I - \pi_{E^*} \).

**Proposition 1.3.** Let \( E \) and \( F \) be Banach spaces. Then
\[
(1.2) \quad \frac{1}{||\varrho_{E^*}||} ||R(S)|| \leq ||R(S^*)|| \leq ||\varrho_{E^*}|| \cdot ||R(S)||, \quad S \in L(E, F).
\]

**Proof.** The map \( \varrho_{E^*} \) is a projection onto \( E^{**} = \{u \in E^{**}: u_{E^*} = 0\} \) and \( \text{Ker}(\varrho_{E^*}) = E^* \). Thus \( \varrho_{E^*} \) induces the isomorphism \( \varrho_{E^*}: E^{**}/E^* \to E^{**} \) by \( \varrho_{E^*}(u + E^*) = \varrho_{E^*}u \) for \( u \in E^{**} \). We verify that
\[
(1.3) \quad \varrho_{E^*} R(S^*) = R(S)^* \varrho_{E^*}, \quad S \in L(E, F),
\]
where the standard identification \( (E^{**}/E^*)^* = E^* \) has been applied. Indeed, \( \varrho_{E^*} R(S^*)(u + F^*) = \varrho_{E^*} S^{**} u \) for \( u + F^* \in E^{**}/F^* \). On the other hand, if \( x + E \in E^{**}/E^* \), then
\[
(R(S)^* \varrho_{E^*}(u + F^*), x + E) = (\varrho_{E^*}u, S^{**}x + F) = (\varrho_{E^*}u, S^{**}x) = (S^{**}\varrho_{E^*}u, x + E).
\]
The last equality results by noting that \( S^{**} F^* \subseteq E^* \) and \( S^{**} F^* \subseteq E^* \). Finally, (1.2) follows from (1.3) and the fact that \( ||(\varrho_{E^*})^{-1}|| \leq 1 \) in view of \( ||u + E^*|| \leq ||u - u_{E^*}|| = ||\varrho_{E^*}(u + E^*)|| \) for \( u + E^* \in E^{**}/E^* \).

[Y1, 2.8] states that \( R(S^*) \) and \( R(S)^* \) are similar, but [1.3] was not made explicit there. The preceding proposition yields \( ||R(S)||/2 \leq ||R(S^*)|| \leq 2||R(S)|| \) for \( S \in L(E, F) \). It was observed in [Y1, 1.1] that
\[
(1.4) \quad ||R(S)|| \leq \omega(S)
\]
for any \( S \in L(E, F) \), \( E \) and \( F \) Banach spaces. We improve this below. A proof of the known fact (i) is included, since we need an estimate for the norm of the inverse map.

**Proposition 1.4.** Let \( E \) and \( F \) be Banach spaces and \( S \in L(E, F) \).

(i) Assume that \( M \) is a non-reflexive subspace of \( E \) such that the restriction \( S|_M \) is an embedding, where \( J : M \to E \) stands for the inclusion map. Then \( R(S|_M) \) embeds \( M^{**}/M \) into \( F^{**}/F \).

(ii) \( ||R(S)|| \leq \min\{\omega(S), 2\omega(S^*), 2\omega(S^{**})\} \).

**Proof.** (i) Standard duality and \( w^*-w^* \) continuity identifies \( M^{**} \) with \( M^{**} \), the \( w^* \)-closure of \( M \) in \( E^{**} \), and \( (S|_M)^{**}/M \) with \( (SM^{**})^{1-1} = SM^{**} \). Suppose that \( x^{**} \in M^{**} \) and \( \epsilon > 0 \). The Proposition of [V, pp. 107-108] yields an element \( y \in SM \) so that \( ||y + F^* - y^{**}|| \leq 2\text{dist}(S^{**}y^{**}, F) + \epsilon \). Set \( V = (SM^{**})^{1-1} : SM \to M \). We get
\[
||x^{**} + M|| = ||R(V)R(S|_M)(x^{**} + M)|| \leq ||R(V)|| \cdot ||S^{**}y^{**} + SM|| \leq ||R(V)|| \cdot ||S^{**}y^{**} + SM|| = 2||R(V)|| \text{dist}(S^{**}y^{**}, F^{**}) + \epsilon.
\]

(ii) (1.2) and (1.4) imply \( ||R(S)|| \leq 2||R(S^*)|| \leq 2\omega(S^*) \) for \( S \in L(E, F) \). Moreover, from the proof of part (i) and [A, 5.1] we get
\[
||R(S)|| \leq 2||R(K_F)R(S)|| \leq 2\omega(K_F S) \omega(S^{**}) \|
\]
and \( ||R(\cdot)|| \) is not uniformly comparable with any of the other quantities appearing in Proposition 1.4.(ii). Recall that a Banach space \( E \) has the Schur property if weakly convergent sequences of \( E \) are norm-convergent. \( \ell^1 \) is an example of a space with the Schur property.

**Example 1.5.** [AT, Theorem 4] constructs a separable \( c_0 \)-sum \( E = (\oplus_{n \in \mathbb{N}} (\ell^1, ||\cdot||_n))_n \), where \( (\ell^1, ||\cdot||_n) \) is a certain sequence of equivalent renormings of \( c_0 \), and operators \( S_n \subseteq L(\ell^1, \ell^1) \) so \( \omega(S_n) \leq 1/n \) but \( \omega(S_n^*) = 1 \) for all \( n \in \mathbb{N} \). Put \( T_n = S_n^* \in L(\ell^1, E^*) \), \( n \in \mathbb{N} \). Proposition 1.3 implies that \( ||R(T_n)|| \leq 2||R(S_n)|| \leq 2n \), but \( \omega(T_n^*) = \omega(T_n) = \omega(S_n) = 1 \) for all \( n \in \mathbb{N} \) according to [A, 5.1] and the construction. This yields that \( ||R(S)|| \) is not in general uniformly equivalent to any of \( \omega(S) \), \( \omega(S^*) \) or \( \omega(S^{**}) \).

The space \( E^* \) admits another property of relevance for Section 2: for all \( S \in L(E, E^*) \) and arbitrary Banach spaces \( Z \),
\[
(1.5) \quad ||S||_{w^*} \leq 2\omega(S).
\]
Indeed, \( E^* = (\oplus_{n \in \mathbb{N}} (\ell^1, ||\cdot||_n^*)_n \), has the metric approximation property, since \( E^* \) is a separable dual space having the approximation property (see [LT2, 1.6.15]). Hence [I8, 3.6] and the Schur property of \( E^* \) yield for \( S \in L(Z, E^*) \) that
\[
||S||_{w^*} = \text{dist}(S, K(Z, E^*)) \leq 2\inf\{\epsilon > 0 : SB_Z \subset D + \epsilon B_{E^*}, D \subset E^* \text{ is a finite set} \} = 2\omega(S).
\]
PROBLEM. It remains unknown whether there is \( c > 0 \) so that
\[
\omega(S^{**}) \geq \omega(S), \quad S \in L(E, F).
\]
One has \( \omega(S^{**}) = \omega(K_F S) \leq \omega(S) \) for any \( S \) by [A, 5.1], so this asks about the behaviour of \( \omega \) under \( K_F : F \to F^{**} \). We refer to [AT, p. 372] for a condition that ensures (1.6). The constant \( c = 1/2 \) is the best possible in (1.6) for operators \( S : E \to c_0 \) (see [A, 1.10] and [AT, p. 374]).

2. MAPPING PROPERTIES OF \( R \). This section focusses on the mapping properties of the correspondence \( S + W(E, F) \to R(S) \) from the quotient space \( L(E, F)/W(E, F) \) to \( L(E^{**}/E, F^{**}/F) \). Several examples demonstrate strongly varying behaviour of \( R(W(E)) \) in the algebraic case \( E = F \), where \( W(E) \) denotes the weak Calkin algebra \( L(E)/W(E) \). They indicate that the problem of identifying \( \text{Im}(R) \) is quite hard for given Banach spaces.

We first consider when \( R \) isometrically faithful in the sense that the image \( \text{Im}(R) \) is closed. It was pointed out in [T1, 1.2] that \( R(W(E)) \) is not always a closed subalgebra of \( L(E^{**}/E) \). The following two weakly approximate approximation properties of Banach spaces from [AT] and [T2] will yield further examples.

- The space \( F \) has property (P1) if there is \( c \geq 1 \), so that \( \text{inf}\{\|R-U\| : U \in W(F), \|I-U\| \leq c\} = 0 \) for all Banach spaces \( E \) and \( R \in L(W(E, F)) \).
- The space \( F \) has property (P2) if there is \( c \geq 1 \), so that \( \text{inf}\{\|R-U\| : U \in W(F), \|I-U\| \leq c\} = 0 \) for all Banach spaces \( E \) and \( R \in W(F, E) \).

We refer to [LT1, II.5.b] for the definition of the class of \( L^1 \)- and \( L^\infty \)-spaces, which contains the \( C(K) \)- and \( L^1(\mu) \)-spaces.

THEOREM 2.1. (i) Let \( E \) be an \( L^1 \)- or \( L^\infty \)-space. Then \( E \) has property (P1) if and only if \( E \) has the Schur property, and \( E \) has property (P2) if and only if \( E^* \) has the Schur property.

(ii) If \( \text{Im}(R) \) is closed in \( L(E^{**}/E, F^{**}/F) \) for all Banach spaces \( E \) then \( F \) has property (P1).

(iii) If \( \text{Im}(R) \) is closed in \( L(E^{**}/E, F^{**}/F) \) for all Banach spaces \( F \) then \( E \) has property (P1).

Proof. (i) See [AT, Cor. 3] and [T2, 3.5].

(ii) If the Banach space \( F \) does not satisfy (P1), then the proof of [AT, Thm. 4] yields a Banach space \( E \) and a sequence \( (S_n) \subset L(E, F) \) so that \( \|S_n\| = 1 \) and \( \omega(S_n) \leq 1/n \) for all \( n \in \mathbb{N} \). Hence (1.4) implies that \( \text{Im}(R) \) fails to be closed in \( L(E^{**}/E, F^{**}/F) \).

(iii) If the Banach space \( E \) does not satisfy (P2), then according to the proof of [T2, 1.2] there is a Banach space \( F \) and a sequence \( (S_n) \subset L(E, F) \) so that \( \|S_n\| = 1 \) and \( \beta_{W}(S_n) \leq 1/n \) for all \( n \in \mathbb{N} \). From (1.2) and Propositions 1.2 (applied to \( W \)) and 1.3 we get
\[
\|R(S_n)\| \leq 2\|R(S_n)\| \leq 2\omega(S_n) = 2\beta_{W}(S_n) \leq 2/n,
\]
for all \( n \in \mathbb{N} \). Thus \( \text{Im}(R) \) fails to be closed in \( L(E^{**}/E, F^{**}/F) \).

REMARKS. The converse implications to those of (ii) and (iii) above do not hold. To see this let \( E \) and \( (S_n) \subset L(E, c_0) \) be as in Example 1.5. The map \( H \) has closed range neither on \( L(E, c_0) \) nor on \( L(\ell^1, E^*) \), since \( \|S_n\| \geq \|S_n\| \geq \omega(S_n) = 1 \) for all \( n \) but \( R(S_n) \) and \( R(S_n^*) \) tend to 0 as \( n \to \infty \). One verifies that \( E^* \) satisfies (P1) and that \( E \) satisfies (P2) by using [T2, Remark (ii)] after Example 2.5 and the fact that \( E^* \) has the metric approximation property and the Schur property.

It turns out that \( R \) is not surjective for many classical non-reflexive Banach spaces (here we disregard pairs \( E, F \) of non-reflexive Banach spaces for which \( L(E, F) = W(E, F) \)). Recall that the operator \( S : E \to F \) is inessential, denoted \( S \in I(E, F) \), if \( \text{Ker}(I_{E^*}-US) \) is finite-dimensional and \( \text{Im}(I_{E^*}-US) \) has finite codimension in \( E \) for all \( U \in L(F, E) \). It is well known that \( I \) is a closed operator ideal so that \( K(E) \subset I(E, F) \) and that \( \text{Id}_E \in I(E) \) only if \( E \) is finite-dimensional.

THEOREM 2.2. Suppose that \( E \) is one of the spaces \( c_0, C(K) \) for a countable compact set \( K \), \( C(0, 1), \ell^1, L^1(0, 1), \ell^\infty \) or the analytic function spaces \( H^\infty \) and \( A(D) \). Then
\[
R(W(E)) \cap I(E^{**}/E) = \{0\}.
\]
In particular, \( R \) is not surjective. However, \( R(W(E)) \) is closed in \( L(E^{**}/E) \) if \( E \) is \( c_0 \) or \( \ell^1 \) or \( L^1(0, 1) \).

Proof. Suppose that \( E \) equals \( c_0 \) or \( \ell^1 \) and assume that \( S \not\in W(E) = K(E) \). It is well known that there are \( A, B \in L(E) \) so that \( \text{Id}_E = BSA \) (see [Pi, 5.1]). Hence
\[
\text{Id}_{E^{**}/E} = R(B)R(S)R(A).
\]
and \( R(S) \not\in I(E^{**}/E) \), since otherwise \( \text{Id}_{E^{**}/E} \in I \) but \( \text{dim}(E^{**}/E) = \infty \).

Factorization (2.2) is also valid for \( E = \ell^\infty \) and \( S \not\in W(\ell^\infty) \). Indeed, a result of Rosenthal [1LT2, 2.4A] gives a subspace \( M \subset \ell^\infty \), \( M \approx \ell^\infty \), so that the restriction \( S|_M \) defines an isomorphism \( M \to SM \). Since any \( \ell^\infty \)-copy is complemented there is a projection \( Q : \ell^\infty \to SM \) as well as an isomorphism \( A : \ell^\infty \to M \). Then (2.2) holds with \( H = A^{-1}S|_M^{-1}Q \).

If \( S \not\in W(C(0,1)) \), then there is a subspace \( M \subset C(0,1) \), \( M \approx c_0 \), so that the restriction \( S|M \) determines an isomorphism. Both \( M \) and \( SM \) are complemented in \( C(0,1) \) by Sobczyk’s theorem. We find as above operators...
A, B so that $BSA = \text{Id}_{c_0}$. A similar argument applies to all separable $C(K)$-spaces. Moreover, if $S \notin W(L^1(0,1))$, then there are operators $A, B$ with $BSA = \text{Id}_{c_0}$. The above facts are based on [P2, pp. 35 and 39]. We thus obtain (2.2) with $E = c_0$, respectively $E = ℓ^1$. Similarly, for $H^∞$ and $A(D)$ one applies [B, Thm. 1] and [K] in order to deduce (2.2) with $E = ℓ^∞$, respectively $E = c_0$.

Suppose next that $E$ is $c_0$ or $ℓ^1$. Then $\|R(S)\| = \text{dist}(S, K(E)) = \|S\|_{∞}$ for $S \in L(E)$. This follows from the uniqueness of submultiplicative norms in certain quotient algebras (see [M, Thm. 2]). Moreover, $\|R(S)\| = \|S\|_{∞}$ for $S \in L(L^1(0,1))$ by [WW, 3.1]. Thus $R$ has closed range in these cases.

Remarks. Actually, (2.2) implies that any non-zero $R(S)$ is large in the sense that $R(S)$ determines an isomorphism between complemented copies of $E^{**} / E$. It remains unclear to us whether $R(W(E))$ is closed if $E$ is $C(0,1)$ or $ℓ^∞$.

Theorem 2.2 expresses the fact that $\text{Im}(R)$ does not contain “small” operators, e.g. compact ones, for many concrete spaces. There are two general Banach space properties that allow a similar conclusion. This is the content of Theorem 2.3 below.

Let $R_{f}$ stand for the operator ideal of weakly conditionally compact operators: $S \in R_{f}(E, F)$ if $(s_{n})$ admits a weak Cauchy sequence for all bounded sequences $(x_{n})$ in $E$. A Banach space $E$ is weakly sequentially complete if any weak Cauchy sequence of $E$ converges weakly. Examples of weakly sequentially complete spaces are known to include all subspaces of $L^1(0,1)$ and $C_1$, the trace class operators on $ℓ^2$.

The operator $S : E \to F$ is unconditionally converging, denoted $S \in U(E, F)$, if $\sum_{n=1}^{∞} s_{n}$ is unconditionally convergent in $F$ whenever the formal series $\sum_{n=1}^{∞} s_{n}$ in $E$ satisfies $\sum_{n=1}^{∞} \|x^{*}(s_{n})\| < ∞$ for all $x^{*} \in E^{*}$. A Banach space $E$ has Pelczynski’s property (V) if $U(E, F) = W(E, F)$ for all Banach spaces $F$. Any $C(K)$-space, and more generally any $C^{*}$-algebra, has property (V) ([P1, Thm. 1] and [Pf, Cor. 6]) as well as any Banach space $E$ that is an M-ideal in $E^{**}$ (see [HWW, III.1 and III.3.4] for a list of examples).

Theorem 2.3. Let $E$ and $F$ be Banach spaces.

(i) If $S \in L(E, F)$ and $R(S) \in R_{f}(E^{**} / E, F^{**} / F)$, then we have $S^{**} \in R_{f}(E^{**} / E, F^{**} / F)$.

(ii) If $F$ is weakly sequentially complete, then we have $R(L(E, F)) \cap R_{f}(E^{**} / E, F^{**} / F) = \{0\}$.

(iii) If $E$ has property (V), then $R(L(E, F)) \cap U(E^{**} / E, F^{**} / F) = \{0\}$.

Proof. (i) [DFJP, pp. 313–314] produces for each $U \in L(E, F)$ a factorization $U = jA$ through a Banach space $Z$. The intermediate space $Z$ has the property

$$U \in R_{f}(E, F) \iff \text{and only if } ℓ^1 \text{ does not embed in } Z$$

(see [W, Satz 1]). The DFJP-factorization of $U^{**}$ and $R(U)$ can be obtained as $U^{**} = j^{**} A^{**}$ and $R(U) = (j)(R(A))$, through the intermediate spaces $Z^{**}$, respectively $Z^{**} / Z$, by [G, 1.5 and 1.6].

Suppose that $R(S) \in R_{f}(E^{**} / E, F^{**} / F)$. We claim that $S^{**}$ is weakly conditionally compact. It suffices to verify in view of (2.3) that $ℓ^1$ embeds in $Z^{**} / Z$ whenever $ℓ^1$ embeds in $Z^{**}$.

Case 1. Assume that $ℓ^1$ does not embed in $Z$. Let $M \subset Z^{**}$ be a subspace so that $M \approx ℓ^1$. Hence $Z$ and $M$ are totally incomparable and $M + Z$ is closed in $Z^{**}$. We may suppose that $M \cap Z = \{0\}$. This implies that $Q_{M}$ defines an embedding and $Q_{M} \approx ℓ^1$ in $Z^{**} / Z$, where $Q : Z^{**} \to Z^{**} / Z$ stands for the quotient map.

Case 2. Assume that $ℓ^1$ embeds in $Z$. Clearly $ℓ^1$ embeds in $(ℓ^1)^{**} / ℓ^1$ as this quotient is an $ℓ^1$-space. Thus $ℓ^1$ embeds in $Z^{**} / Z$, since $(ℓ^1)^{**} / ℓ^1$ is isomorphic to a subspace of $Z^{**} / Z$ by Proposition 1.4(i).

(ii) If $R(S) \in R_{f}(E^{**} / E, F^{**} / F)$, then part (i) implies that $S$ is weakly conditionally compact. Hence $S \in W(E, F)$ since $F$ is weakly sequentially complete.

(iii) We first verify that $S \in U(E, F)$ whenever $R(S)$ is unconditionally converging. In fact, if $S \notin U(E, F)$, then there is a subspace $M \subset E$, $M \approx c_0$, so that $S_{M}$ is an embedding [P2, p. 34]. Let $J : M \to E$ be the inclusion map. Proposition 1.4(i) yields that $R(S_{M})$ is an embedding on $M^{**} / M \approx ℓ^∞ / c_0$. This implies that $R(S)$ is not unconditionally converging as $c_0$ embeds in $ℓ^∞ / c_0$ (for instance by [LT2, 2.4.4]). If $E$ has property (V) and $R(S)$ is unconditionally converging, then the preceding observation yields that $S \in U(E, F) = W(E, F)$.

We next construct various examples where $R$ has quite different properties compared with Theorems 2.2 and 2.3. In these examples $\text{Im}(R)$ contains plenty of “small” operators and in some cases $R$ is even an isomorphism.

The quotient $E^{**} / E$ is quite unwieldy for most Banach spaces $E$, but if the space $Z$ is weakly compactly generated, then there is a Banach space $X$ so that $X^{**} / X$ is isomorphic to $Z$ (see [DFJP, p. 321]). We recall here a more restricted construction. The James sum of a Banach space $E$ is

$$J(E) = \{(x_{k}) : x_{k} \in E, \|x_{k}\| < ∞ \text{ and } \lim_{k \to ∞} x_{k} = 0\},$$

where $\|x_{k}\| = \sup_{1 \leq i_{1} < \ldots < i_{n}} \left(\sum_{i=1}^{n} \|x_{i_{1}+1} - x_{i_{1}}\|^{2}\right)^{1/2}$. The supremum is taken over all increasing sequences $1 \leq i_{1} < \ldots < i_{n}$ of natural numbers and $n \in \mathbb{N}$. It is known [Wo] that $J(E)^{**}$ is the space of all sequences $(x_{k})$ with $x_{k} \in E^{**}$ for which the above 2-variation norm is finite. If $E$ is reflexive,
then any \((x_k) \in J(E)^**\) can be written as \((x_k - x)_{k \in \mathbb{N}} + (x)_{k \in \mathbb{N}}\), where \(x = \lim_{n \to \infty} x_n\) (the limit clearly exists in \(E\)), and \((x_k + J(E)) \to \lim_{k \to \infty} x_k\) gives an isomorphism \(J(E)^** / J(E) \to E\).

A Banach space \(E\) is quasi-reflexive of order \(n\) if \(\dim(E^*/E) = n\) for some \(n \in \mathbb{N}\). In this case \(R(W(E))\) identifies with a subalgebra of the scalar-valued \(n \times n\)-matrices and there is \(c = c(E) > 0\) so that \(c||S||_w \leq ||R(S)||\) for all \(S \in L(E)\). We use \(J\) for \(J(R)\), the (real) James space, which is quasi-reflexive of order 1 (see [LT2, 1.d.2]). One has \(J^* = J \oplus \mathbb{R}f\), where \(f = (1,1,1,\ldots)\). The behaviour of \(R\) varies even within the class of quasi-reflexive spaces.

**Examples 2.4. (i) Let \(\ell^2_p(J) = J \oplus \cdots \oplus J\) \((n\) copies\) with the \(\ell^2_p\)-norm, whence \(\dim(\ell^2_p(J)^*/\ell^2_p(J)) = n\) for all \(n\). Then \(R: W(\ell^2_p(J)) \to L(\ell^2_p(J)^*/\ell^2_p(J))\) is a bijection. This follows from the fact that \(R(Id_J)\) identifies with the 1-dimensional operator taking \(f = (1,1,\ldots)\) to itself. It is computed below during the proof of Theorem 2.6 that \(\inf_{n \in \mathbb{N}} c(\ell^2_p(J)) = 0\).

(ii) Let \(J_p\) stand for the quasi-reflexive James space of order 1 defined using \(p\)-variation in the norm instead of 2-variation for \(1 < p < \infty\) (thus \(J_p = J\)). Suppose that \(1 < p_1 < \cdots < p_n < \infty\). Loy and Willis [LW, p. 345] observed for the quasi-reflexive space \(\bigoplus_{j=1}^n J_p\) of order \(n\) that the image of \(R\) coincides with the lower-triangular \(n \times n\)-matrices. This is based on the facts that, for \(1 < p < q < \infty\), any operator \(J_p \to J_q\) is compact while the formal identity \(J_p \to J_q\) is not weakly compact.

(iii) Leung [L, Prop. 6] constructed a quasi-reflexive Banach space \(F\) of order 1 so that \(L(F,F^*) = W(F,F^*)\) and \(L(F^*, F) = W(F^*, F)\). Then \(E = F \oplus F^*\) is quasi-reflexive of order 2, but \(\text{Im}(R)\) identifies with the class of diagonal \(2 \times 2\)-matrices.

In our next result \(X^**/X\) is infinite-dimensional, but \(R\) is surjective.

**Proposition 2.5.** Suppose that \(E\) is a reflexive infinite-dimensional Banach space and let \(J(E)\) be the corresponding James-sum. Then \(R\) is an isomorphism and \(R(W(J(E))) = L(J(E)^**/J(E))\), where \(J(E)^**/J(E) \cong E\).

**Proof.** Let \(\phi : J(E)^**/J(E) \to E\) stand for the isomorphism from \((x_k + J(E)) \to \lim_{k \to \infty} x_k\). It suffices to verify that any \(S \in L(E)\) belongs to the image of \(R\) under this identification. Suppose that \(S \in L(E)\) and let \(\tilde{S}\) be the bounded operator on \(J(E)\) defined by \(\tilde{S}(x_k) = (Sx_k)\) for \((x_k) \in J(E)^**\). One verifies using \(w^*\)-convergence that \(\tilde{S}(x_k) = (Sx_k)\) whenever \((x_k) \in J(E)^**\). Then \(R(\tilde{S}) = S\).

**Problem.** Is \(E^**/E\) always reflexive if \(R : W(E) \to L(E^**/E)\) is a bijection?

Let \(X = \ell^2(J)^*\) stand for the \(\ell^2\)-sum of a countable number of copies of James’ space \(J\). Thus \(\ell^2(J)^** = \ell^2(J^*)\) isometrically and it is not difficult to verify that \(X^*/X\) is isometric to \(\ell^2\) through \((x_k) \to (\ell^2(J)^* \to (\omega, \omega_2, \ldots)\), where \(\omega_k = \lim_{n \to \infty} x_n^{(k)}\) for \(x_k = (x_k^{(k)})_{j \in \mathbb{N}} \in J^**\). The lattice regular operators on \(\ell^2\) (with respect to the natural orthonormal basis) are defined by

\[
\text{Reg}(\ell^2) = \{ A = (a_{ij}) \in L(\ell^2) : |a_{ij}| \leq \text{||a_{ij}||} \text{ defines a bounded operator on } \ell^2 \}.
\]

Here \((a_{ij})\) is the matrix representation of \(A\). It is known that \(A \in \text{Reg}(\ell^2)\) if and only if \(A = U - V\), where \(U\) and \(V\) are operators having matrices with non-negative entries. The algebra \(\text{Reg}(\ell^2)\) is complete in the regular norm \(||A||_r = ||A||\) (see [AB, 15.2]) and \(||A|| \leq ||A||_r\), but \(\text{Reg}(\ell^2)\) is not a closed subalgebra of \(L(\ell^2)\). For instance, let \((A_n)\) be the \(2^n \times 2^n\) Walsh-Littlewood matrices

\[
A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix}
\]

for \(n \in \mathbb{N}\). Then \(||A_n||_r/||A_n|| = 2^{1/2}\) for all \(n\). Moreover, the Hilbert-Schmidt operators are included in \(\text{Reg}(\ell^2)\).

Let \((e_n)\) be the standard coordinate basis of \(J\). James’ space \(J\) also admits the Schauder basis \((f_k)\), where \(f_k = \sum_{j=1}^n e_j \) for \(k \in \mathbb{N}\). The norm in \(J\) is computed in \((f_k)\) as

\[
\sum_{k=1}^{\infty} \sum_{1 \leq i_1 < \cdots < i_{n+1} \leq \infty} \left| \sum_{j=1}^n (b_{i_1} + \cdots + b_{i_{n+1}}) \right|^2 \right)^{1/2}
\]

\[
\text{for } \sum_{k=1}^{\infty} \sum_{i_1 < \cdots < i_{n+1} \leq \infty} (b_{i_1} + \cdots + b_{i_{n+1}}) < \infty.
\]

The main result of this section identifies \(R(W(\ell^2(J)))\) with the algebra \(\text{Reg}(\ell^2)\) (note that \(\ell^2(J)^**/\ell^2(J)\) is isometric to \(\ell^2\) as above). This provides a concrete Banach space \(X\) so that \(\text{Reg}(\ell^2)\) and \(\text{Reg}_w\) fail to be comparable on \(L(X)\) (see also Theorem 2.1). The proof uses local properties of \(J\). Our result also settles a basic question concerning the representation \(R\) (Corollary 2.10).

**Theorem 2.6.** \(R\) is a weakly isometric isomorphism of \(W(\ell^2(J))\) onto \(\text{Reg}(\ell^2)\), \(\text{|||\cdot|||}_r\),

\[
||S||_w = ||R(S)||_r
\]

for all \(S \in L(\ell^2(J))\). Thus \(\text{Im}(R)\) is not closed in \(L(\ell^2)\).

**Proof.** We first verify that for any \(A \in \text{Reg}(\ell^2)\) there is \(\tilde{A} \in L(\ell^2(J))\) so that \(R(\tilde{A}) = A\) and \(||\tilde{A}||_w \leq ||A||_r\).

In our next result \(X^**/X\) is infinite-dimensional, but \(R\) is surjective.
Let $A = (a_{ij})$ be a bounded regular operator on $l^2$ and consider the formal operator $\hat{A}$ defined by the operator matrix $(a_{ij} I)$, where $I$ stands for the identity mapping on $J$.

Assume that $(x_r) \in l^2(J)$. We obtain

$$
\| \hat{A}(x_r) \|^2 = \sum_{i=1}^{\infty} \sum_{r=1}^{\infty} a_{ir} x_r = \sum_{i=1}^{\infty} \left( \sum_{r=1}^{\infty} |a_{ir}| \cdot \|x_r\| \right)^2 \\
= \| A(\|x_r\|) \|^2 \leq \| A \|^2 \sum_{r=1}^{\infty} \|x_r\|^2.
$$

Thus $\hat{A}$ defines a bounded operator on $l^2(J)$ and $\| \hat{A} \| \leq \| A \|$. One checks that $R(\hat{A} = A$, since $R(I)$ is the 1-dimensional identity taking $f = (1, 1, \ldots)$ to itself.

It remains to prove that $R(U) \in \text{Reg} (l^2)$ and $\| R(U) \|_r \leq \| U \|_m$ for $U \in L(l^2)$. Suppose that $S = (s_{ij})$ is a matrix so that $s_{ij} = 0$ whenever $i > n$ or $j > n$ for some $n \in \mathbb{N}$. Let $\tilde{S} = (a_{ij})$ stand for the corresponding vector-valued operator on $l^2(J)$. We claim that

$$
\| \tilde{S} - W \|_r \geq \| S \|_r
$$

for any operator-valued matrix $W = (W_{ij})$ on $l^2(J)$ so that $W_{ij} \in W(J)$ for all $i, j \in \mathbb{N}$ and $W_{ij} = 0$ whenever $i > n$ or $j > n$.

Before establishing the claim we indicate how (2.5), and thus the theorem, follows from (2.6) with the help of a simple cut-off argument. Assume that $U = (U_{ij}) \in L(l^2(J))$, where $(U_{ij})$ is the matrix representation of $U$. We may write $U_{ij} = s_{ij} I + W_{ij}$ with $W_{ij} \in W(J)$ for $i, j \in \mathbb{N}$ so that $R(U) = (s_{ij})$. Define for $n \in \mathbb{N}$ the cut-off $U_n = (a_{ij}^{(n)} U_{ij})$, where $a_{ij}^{(n)} = 1$ if $i, j \leq n$ and $a_{ij}^{(n)} = 0$ otherwise. (2.6) yields that

$$
\| U_n \|_r \geq \| (a_{ij}^{(n)} s_{ij}) \|_r.
$$

By letting $n \to \infty$ above we obtain $\| U \| \geq \| R(U) \|_r$. This implies the desired inequality $\| U \|_r \geq \| R(U) \|_r$, since $R(U)$ is invariant under weakly compact perturbations of $U$.

It remains to establish (2.6). The main ingredients of the argument are presented as independent lemmas in order to make the strategy of the proof more transparent.

**Lemma 2.7.** Let $S = (s_{ij})$ be an $n \times n$-matrix and define $\tilde{S} : l^2(l^2) \to l^2(l^2)$ by

$$
\tilde{S}(y_1, \ldots, y_n) = \left( \sum_{j=1}^{n} s_{ij} y_j, \ldots, \sum_{j=1}^{n} s_{nj} y_j \right) \quad \text{for } y_1, \ldots, y_n \in l^2.
$$

Then $\| \tilde{S} \| = \| S \|_r$.

**Proof.** We obtain $\| \tilde{S} \| = \| S \|$ as above. Choose $a = (a_1, \ldots, a_n) \in l^2$ so that $\| a \| = 1$ and $\| S a \| = \| S \|$. Let $\{h_1, \ldots, h_n\}$ be the unit vector basis of $l^2$. We get

$$
\| \tilde{S} \| \lesssim \| \tilde{S}(a_1 h_1, \ldots, a_n h_n) \|^2 = \sum_{l=1}^{n} \sum_{j=1}^{n} a_{lj} a_{l} h_j \| h_j \|^2 \\
= \sum_{l=1}^{n} \left( \sum_{j=1}^{n} |a_{lj}| \right) \| h_j \|^2 = \| S \|_r.
$$

The proof of the next two auxiliary results are momentarily postponed.

The first one establishes a joint “smallest” property for finite collections of weakly compact operators on $J$. This fact may have some independent interest. We remark that $U \in W(J)$ defined by $U f_l = f_l$, $U h_k = f_k - l$ for $k \geq 2$, demonstrates that a weakly compact operator on $J$ is not necessarily small between diagonal blocks of $(f_k)$. The second result records the technical fact that convex blocks of $(f_k)$ span isometric copies of $J$ in the norm considered here. A proof is included because we are not aware of a suitable reference.

**Proposition 2.8.** Suppose that $S_1, \ldots, S_r \in W(J)$. For any $\varepsilon > 0$ and $n \in \mathbb{N}$ there is a natural number $l$ and a sequence $(z_k)_{k=1}^{l}$ consisting of disjoint convex blocks of the basis $(f_k)$ so that each $z_k$ is supported after $l$ and for $M_n = \{z_1, \ldots, z_n\}$ we have

$$
\max_{1 \leq k \leq l} \| (I - R) S_j | M_n \| < \varepsilon.
$$

**Lemma 2.9.** Let $z_k = \sum_{i=n_k+1}^{n_k+1} c_j f_j$ be disjoint convex blocks of $(f_j)$, where the sequence $(n_k)$ is strictly increasing, $c_j \geq 0$ for all $j$ and $\sum_{j=n_k+1}^{n_k+1} c_j = 1$ for $k > 1$. Then $(z_k)$ is a basic sequence in $J$ that is isometrically equivalent to $(f_k)$:

$$
\| \sum_{k=1}^{\infty} b_k z_k \| = \| \sum_{k=1}^{\infty} b_k f_k \|
$$

for all $\sum_{k=1}^{\infty} b_k f_k \in J$.

**Proof of (2.9).** Let $S, W$ and $n$ be as in the claim. Suppose that $\delta > 0$. There is an integer $m$ so that $l^n$ embeds $(1+\delta)$-isomorphically in $(f_1, \ldots, f_m)$ (see [GJ, Thm. 4]). Proposition 2.8 provides an integer $l$ together with disjoint convex blocks $z_1, \ldots, z_m$ of $(f_k)$ so that the following properties are satisfied:

(i) $Q_l z_j = z_j$ for $j = 1, \ldots, m$, where $Q_l = I - P_l$,

(ii) $\sum_{j=1}^{m} \| Q_l W_{ij} | M_n \| < \delta$. Here $M_n = \{z_1, \ldots, z_n\}$.
According to Lemma 2.9, \( M_m \) is isometric to \([f_1, \ldots, f_m] \) and there is a subspace \( N \subset M_m \) so that \( N \) is \((1 + \delta)\)-isomorphic to \( \ell^2(J) \). Write \( \tilde{N} = \{ (x_k) \in \ell^2(J) \mid x_k \in N, k \leq n \text{ and } x_k = 0 \text{ otherwise} \} \). Let \( \tilde{Q} \in L(\ell^2(J)) \) be the norm-1 operator defined by \( \tilde{Q}f = (Q(x_k), f_1, \ldots, f_m) \). Observe that (ii) implies \( ||\tilde{Q}W_\delta|| < \delta \). Moreover, \( \tilde{Q}W_\delta = \tilde{N} \) and \( \tilde{S}\tilde{N} \subset \tilde{N} \), so that Lemma 2.7 yields
\[
||\tilde{Q}S|| \geq ||S|| \geq (1 + \delta)^{-2}||S||.
\]
Finally,
\[
||\tilde{S} - W|| \geq ||\tilde{Q}(\tilde{S} - W)|| \geq (1 + \delta)^{-2}||S|| - \delta.
\]
We get (2.6) by letting \( \delta \to 0 \) above. \( \blacksquare \)

Proof of Proposition 2.8. Observe that \( f_k \to f \) in \( J^{**} \) as \( k \to \infty \). Thus \( S_1f_k \to S_1f \) in \( J \) as \( k \to \infty \), since \( S_1 \) is weakly compact. Fix a natural number \( l_k \) such that \( ||(I - P_{l_k})S_1^*f|| < \epsilon/(2n) \). Mazur's theorem implies that \( S_1^*f \in \overline{\mathcal{V}}\{(S_1f_k : k \in \mathbb{N})\} \). One obtains by induction disjoint convex blocks \( u_k = \sum_{j \neq m_k} c_j f_j, \) where \( l_k \leq m_1 < m_2 < \ldots \) and \( S_1u_k = S_1^*f \) in norm as \( k \to \infty \). Notice that \( ||u_k|| = 1 \) for all \( k \) by (2.4). We may assume that \( ||S_1u_k - S_1^*f|| < \epsilon/(2n) \) whenever \( k \in \mathbb{N} \). Consequently,
\[
||(I - P_{l_k})S_1u_k|| \leq ||(I - P_{l_k})|| \cdot ||S_1u_k - S_1^*f|| + ||(I - P_{l_k})S_1^*f|| < \epsilon/n
\]
for all \( k \).

Observe that \( u_k \to f \) in \( J^{**} \) as \( k \to \infty \), since \( (u_k) \) converges coordinate-wise to \( f \) in the shrinking basis \( (e_k) \). Choose an integer \( l_2 \geq l_1 \) so that \( ||(I - P_{l_2})S_2^*f|| < \epsilon/(2n) \). Apply the preceding argument to \((S_2u_k)\) and recover as above disjoint convex blocks \( v_k = \sum_{j \neq m_k} c_j f_j \) of \( (u_k) \) that are supported after \( l_2 \) with respect to \( (f_j) \), so that \( ||S_2v_k - S_2^*f|| < \epsilon/(2n) \) for all \( k \). We deduce as before that \( ||(I - P_{l_2})S_2v_k|| < \epsilon/n \). Note further that \( (v_k) \) are disjoint convex blocks of \( f_k \) and
\[
||(I - P_{l_2})S_1u_k|| \leq ||(I - P_{l_2})S_1u_k|| \leq \sum_{j \neq m_k} ||(I - P_{l_2})S_1u_j|| < \epsilon/n
\]
for all \( k \).

These observations allow us to repeat the above procedure in order to find eventually an integer \( l \) and disjoint convex blocks \( s_k = \sum_{j \neq m_k} c_j f_j \) so that \( ||(I - P_l)s_k|| < \epsilon/n \) for any \( j = 1, \ldots, r \) and \( k \in \mathbb{N} \). This estimates clearly imply that \( ||(I - P_l)s_j|| < \epsilon/n \). This completes the proof of Proposition 2.8. \( \blacksquare \)

Proof of Lemma 2.9. By approximation there is no loss of generality in assuming that \( S_1f_k \) is finitely supported, \( b_k = 0 \) for \( k \geq m \) and some \( m \in \mathbb{N} \). According to (2.4) there are integers \( 1 = m_1 < m_2 < \ldots < m_t = m \) so that
\[
||(I - P_{m_1})S_1f_k||^2 = \left| \sum_{i = 1}^{m_1 - 1} b_i^2 \right|^2.
\]
Set \( d_i = c_i b_i \) if \( n_i \leq i < n_{i+1} \) for some \( 1 \leq k \leq t - 1 \), and \( d_i = 0 \) otherwise. Thus \( \sum b_k z_k = \sum d_i f_i \), where \( \sum_{m_i < m_i} b_k = \sum_{m_i < m_i} d_i \). Hence the right-hand side of (2.8) is a lower bound for \( ||\sum b_k z_k|| \) so that \( \sum_{m_i < m_i} b_k z_k \geq \sum_{m_i < m_i} d_i \).

In order to prove the reverse inequality let \( l \) and \( m_1, \ldots m_t \) be integers satisfying \( 1 = m_1 < m_2 < \ldots < m_t = m \). Put
\[
N(m_l) = \sum_{i = 1}^{m_{l-1}} d_i^2.
\]
for each \( m_l \). Assume now that \( (m_l) \) is chosen so that \( \sum_{m_l} b_k z_k \geq \sum_{m_l} d_i \) for some \( k \). Thus \( \sum_{m_l} b_k z_k \geq \sum_{m_l} d_i \). Hence the right-hand side of (2.8) is a lower bound for \( ||\sum b_k z_k|| \) so that \( \sum_{m_l} b_k z_k \geq \sum_{m_l} d_i \).

Clearly the convexity of the blocks and (2.4) together imply that \( N(m_r) \leq \sum b_k z_k \). This proves the lemma once \( (m_r) \) is found.

The argument proceeds as follows. Consider a fixed \( m_r \) and assume that \( n_k \leq m_r < n_{k+1} \) for some \( k \). Set
\[
u = \sum_{i = 1}^{m_r-1} d_i \text{ and } \mu = \sum_{i = 1}^{m_r+1-1} d_i.
\]
If \( \nu \geq 0 \), then \( \nu^2 \geq \nu \mu \) and \( N(m_1, \ldots, m_{r-1}, m_{r+1}, \ldots, m_1) \geq N(m_r) \). Simply discard \( m_r \) in this case.

In the case \( \nu < 0 \) we proceed differently. We may suppose by symmetry that \( \nu < 0 \) and \( \mu > 0 \). There are two possibilities.

Case 1. Suppose that \( b_k \geq 0 \). We have \( m_{r-1} < n_k \), since otherwise \( n_i \geq 0 \). Hence we get
\[
\sum_{j = m_{r-1}}^{m_r-1} d_j \leq \nu < 0 \quad \text{and} \quad \sum_{j = m_r}^{m_{r+1}-1} d_j \geq \mu > 0
\]
(there the fact that \( c_j \geq 0 \) for each \( j \) is used). This yields that \( N(m_1, \ldots, m_{r-1}, n_k, m_{r+1}, \ldots, m_1) \geq N(m_r) \). Replace \( m_r \) by \( n_k \).

Case 2. Suppose that \( b_k < 0 \). This implies that \( m_{r+1} > m_{r-1} \). Deduce as above that \( N(m_1, \ldots, m_{r-1}, m_{r+1}, m_{r-1}, \ldots, m_1) \geq N(m_r) \). Replace \( m_r \) by \( n_{k+1} \).
By repeating the above procedure a finite number of times one arrives at the desired sequence \((\mathcal{m}_k)\). This completes the proof of Lemma 2.9 and thus of Theorem 2.6.

We consider as an application weak analogues of the Fredholm operators. Let \(E\) be a Banach space and set
\[
\Phi_w(E) = \{S \in L(E) : S + W(E) \text{ is invertible in } L(E)/W(E)\},
\]

\[
\Phi_c(E) = \{S \in L(E) : R(S) \text{ is a bijection}\},
\]

so that \(\Phi_w(E) \subset \Phi_c(E)\). Yang [Y2, p. 522] states without citing examples that these concepts appear to be different. Theorem 2.6 gives rise to such examples. We refer to [T1] for additional motivation.

**Corollary 2.10.** Let \(J\) be the complex James space. Then \(\Phi_w(\ell^2(J)) \subset \Phi_c(\ell^2(J))\).

**Proof:** The proof of Theorem 2.6 carries through with some modifications in the case of complex scalars and (2.5) is replaced by the inequalities
\[
eq \|S\|_w \leq \|S\|_\sigma \}
\]

for convex blocks \((z_k)\) of \((f_k)\) (apply (2.7) separately to the real and complex parts).

- The complex spaces \(\ell^2_\sigma(C)\) embed with uniform constant in the complex linear span \([f_1, \ldots, f_m]\) for \(m \) large enough. Indeed, it suffices to check that \(\ell^2_\sigma(C)\) embeds uniformly in the complex James space, and this is easily deduced from the fact that \(\ell^2_\sigma(R)\) embeds \((1 + \delta)\)-isomorphically in the real James space [GJ, Thm. 4] for all \(\delta > 0\) and \(\eta \in \mathbb{N}\).

It follows that \(S \in \Phi_w(\ell^2(J))\) if and only if \(R(S)\) is an isomorphism and its inverse \(R(S)^{-1}\) is a regular operator. Also \(\sigma, R, \eta, \sigma, \sigma, \sigma\) one regular operator \(U\) on \(\ell^2\) so that its spectrum \(\sigma(U) \subset \sigma(U)\). Here \(\sigma(U)\) denotes the spectrum of \(U\) in \(\sigma(U)\). Lift \(U\) to an operator \(\tilde{U} \in L(\ell^2(J))\) so that \(R(\tilde{U}) = U\). Then \(\sigma(\tilde{U} + W(\ell^2(J))) \subset \sigma(R(\tilde{U}))\), which yields the claim.

**Problem:** The Yosida-Hewitt decomposition theorem implies that \((\ell^1)^* = \ell^1 \oplus c_0\) coincides with \((\ell^1)^* = c_0(\ell^1) = \sigma(\ell^1)\) \((\sigma = M_{\infty})\), where \(M_{\infty} = \{\mu \in \mu(2^n) : \mu \text{ is purely finitely additive}\}. \text{Find conditions on } \mu \in L(M_{\infty}) \text{ so that } U \text{ identifies with } R(S) \text{ for some } S \in L(\ell^1)\).

Buoni and Klein [BK] introduced a sequential representation of the quotient space \(L(E,F)/W(E,F)\) (see [AT] for some further properties). Let \(E\) be a Banach space, \(\ell^\infty(E) = \{x_k : (x_k) \text{ is bounded in } E\}\) equipped with the supremum norm and \(w(E)\) its closed subspace \(\{x_k \in \ell^\infty(E) : \}

\{x_k : k \in \mathbb{N}\}\) is relatively weakly compact in \(E\). Set \(Q(E) = \ell^\infty(E)/w(E)\) and consider \(Q(S) \in L(Q(E), Q(F))\) for \(S \in L(E,F)\), where

\[
Q(S)((x_k) + w(E)) = (Sx_k) + w(F), \quad (x_k) \in \ell^\infty(E).
\]

We have \(Q(S) = 0\) if and only if \(S \in W(E,F)\). 

- **References:**


Adjoint characterizations of unbounded weakly compact, weakly completely continuous and unconditionally converging operators

by

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Abstract. Characterizations are obtained for the following classes of unbounded linear operators between normed spaces: weakly compact, weakly completely continuous, and unconditionally converging operators. Examples of closed unbounded operators belonging to these classes are exhibited. A sufficient condition is obtained for the weak compactness of $T'$ to imply that of $T$.

1. Introduction and preliminaries. In this paper we shall be considering a linear operator $T : X \rightarrow D(T) \rightarrow Y$ where $X$ and $Y$ are normed spaces.

Let us first recall some facts about bounded operators. Let $T$ be bounded and everywhere defined and let $X$ and $Y$ be Banach spaces. Then $T$ is weakly compact if it transforms bounded sequences into sequences having a weakly convergent subsequence; $T$ is weakly completely continuous if it transforms weak Cauchy sequences into weakly convergent sequences; and $T$ is unconditionally converging if it transforms weakly unconditionally convergent series into unconditionally convergent series. In order to characterize these classes of operators we introduce, for a given normed space $E$, the following subsets of $E^*$:

$$K(E) = \{ e' \in E^* : \text{there exists a sequence } (e_n) \text{ in } E \text{ such that } e' = \sigma(E^*, E^*_w) - \text{lim } e_n \},$$

$$N(E) = \{ e' \in E^* : \text{there exists a weakly unconditionally Cauchy series } \sum e_i \text{ in } E \text{ such that } e' = \sigma(E^*, E^*_w) - \text{lim } \sum_{i=1}^\infty e_i \}.$$