STUDIA MATHEMATICA 113 (3) (1995)

Representing non-weakly compact operators

by

MANUEL GONZÁLEZ (Santander)
EERO SAKSMAN (Helsinki)
and HANS-OLAV TYLLI (Helsinki)

Abstract. For each $S \in L(E)$ (with E a Banach space) the operator $R(S) \in L(E^{**}/E)$ is defined by $R(S)(x^{**}+E) = S^{**}x^{**}+E$ ($x^{**} \in E^{**}$). We study mapping properties of the correspondence $S \to R(S)$, which provides a representation R of the weak Calkin algebra $L(E)/W(E)$ (here $W(E)$ denotes the weakly compact operators on E). Our results display strongly varying behaviour of R. For instance, there are no non-zero compact operators in $\mathrm{Im}(R)$ in the case of L^1 and $C(0,1)$, but $R(L(E)/W(E))$ identifies isometrically with the class of lattice regular operators on ℓ^2 for $E = \ell^2(J)$ (here J is James' space). Accordingly, there is an operator $T \in L(\ell^2(J))$ such that $R(T)$ is invertible but T fails to be invertible modulo $W(\ell^2(J))$.

Introduction. Suppose that E and F are Banach spaces and let $L(E,F)$ stand for the bounded linear operators from E to F. The operator $T : E \to F$ is *weakly compact*, denoted $T \in W(E,F)$, if the image TB_E of the closed unit ball B_E of E is relatively weakly compact in F. The quotient space $L(E,F)/W(E,F)$ equipped with the norm $\|S\|_w = \mathrm{dist}(S, W(E,F))$ is a complicated object and there is a need for useful representations of the elements $S + W(E,F)$. A fundamental result due to Davis *et al.* [DFJP] provides for any $S \in L(E,F)$ a factorization $S = BA$ through a Banach space X so that X is reflexive if and only if $S \in W(E,F)$. However, this construction is not adapted to the quotient space since the intermediate space X depends on S.

We consider here the following natural concept: any $S \in L(E,F)$ induces an operator $R(S) : E^{**}/E \to F^{**}/F$ by

$$R(S)(x^{**}+E) = S^{**}x^{**}+F, \qquad x^{**} \in E^{**},$$

where any Banach space is taken to be canonically embedded in its bidual (the inclusion $E \to E^{**}$ is denoted by K_E if required). We have $R(S) = 0$

1991 *Mathematics Subject Classification*: Primary 47D30; Secondary 46B28, 47A67.
Research of the first author supported in part by DGICYT Grant PB 91–0307 (Spain).
Research of the second author supported by the Academy of Finland.

if and only if $S \in W(E,F)$ since $S \in W(E,F)$ precisely when $S^{**}E^{**} \subset F$ (see [DS, VI.4.2]). The induced map $S+W(E,F) \to R(S)$ gives an injective contraction from $L(E,F)/W(E,F)$ into $L(E^{**}/E, F^{**}/F)$. Moreover,

$$R(\mathrm{Id}_E) = \mathrm{Id}_{E^{**}/E}, \quad R(ST) = R(S)R(T)$$

whenever ST is defined. Hence $S+W(E) \to R(S)$ provides a representation of the weak Calkin algebra $\mathcal{W}(E) = L(E)/W(E)$ and its image $\{R(S) : S \in L(E)\}$ is a subalgebra of $L(E^{**}/E)$ containing the identity. Some basic properties of R are found in [Y1] and [Y2], where this representation was used to discuss invertibility modulo the weakly compact operators. It was employed in [Re], [LW] to exhibit discontinuous derivations on $L(E)$ and infinite-dimensional commutative quotient algebras of $L(E)$ for some Banach spaces E. Applications to tauberian operators appear in [AG]. A concrete interpretation of $R(S)$ for operators S on $L^1(0,1)$ was obtained in [WW].

This paper studies the mapping properties of the map R. We discuss the size of the image $\mathrm{Im}(R)$ for concrete non-reflexive Banach spaces and the question whether $\mathrm{Im}(R)$ is closed. We compare for this purpose in Section 1 some properties of the norm $\|R(\cdot)\|$, that measures the deviation of an operator from weak compactness, to those of other seminorms of this kind. Section 2 focusses on several results and examples displaying radically varying behaviour of $R(\mathcal{W}(E))$. For instance, we establish that $\mathrm{Im}(R)$ does not contain non-zero inessential operators in the case of many concrete spaces, such as $L^1(0,1)$ or $C(0,1)$. We also exhibit Banach spaces X and Y so that X^{**}/X and Y^{**}/Y are isomorphic to ℓ^2 and R is a surjection on $\mathcal{W}(X)$, but $R(\mathcal{W}(Y))$ is not even closed. Our main result (Theorem 2.6) identifies $\mathrm{Im}(R)$ with the lattice regular operators on ℓ^2 in the case of the countable ℓ^2-sum $\ell^2(J)$ of James' space J. We also discuss some applications. An operator $S \in L(E)$ is called *weak Fredholm* if $S+W(E)$ is invertible in $L(E)/W(E)$. It remains unclear whether the weak Fredholm operators admit any geometric characterizations analogous to those of the Fredholm operators. Theorem 2.6 is applied to exhibit an operator $S \in L(\ell^2(J))$ so that $R(S)$ is invertible, but S fails to be invertible modulo the weakly compact operators. Proposition 2.5 solves the following "inverse" problem: given a reflexive Banach space E there is X such that $X^{**}/X \approx E$ and $R : L(X) \to L(E)$ is onto.

1. Duality properties. This preliminary section compares $\|R(\cdot)\|$ with other measures of weak non-compactness. This determines whether the map R has closed range or not, but quantities associated with weak compactness also have other applications and our results illustrate the quite delicate properties of such quantities (cf. [AT] and its references).

We will use standard Banach space terminology and notation in accordance with [LT2]. Let E be a Banach space. Set $E_1 = \ell^1(B_E)$, $E_\infty = \ell^\infty(B_{E^*})$ and let $Q_1 : E_1 \to E$ stand for the surjection $Q_1((a_x)_{x\in B_E}) = \sum_{x\in B_E} a_x x$ and $J_\infty : E \to E_\infty$ for the isometric embedding $J_\infty(x) = (x^*(x))_{x^*\in B_{E^*}}$. We refer to [Pi] for the definition and examples of operator ideals. Let I be a closed operator ideal in the sense that $I(E,F)$ is closed in the operator norm for all Banach spaces E and F. Set

$$\gamma_I(S) = \inf\{\varepsilon > 0 : SB_E \subset RB_Z + \varepsilon B_F \text{ for some Banach space } Z \text{ and } R \in I(Z,F)\},$$

$$\beta_I(S) = \inf\{\varepsilon > 0 : \text{there is a Banach space } Z \text{ and } R \in I(E,Z) \text{ so that } \|Sx\| \le \|Rx\| + \varepsilon\|x\|,\ x \in E\}$$

for $S \in L(E,F)$, following [A] and [T2]. Then γ_I and β_I are seminorms in $L(E,F)$, and $\gamma_I(S) = 0$ if and only if there is a sequence (S_n) in $I(E_1,F)$ so that $\lim_{n\to\infty}\|SQ_1 - S_n\| = 0$, while $\beta_I(S) = 0$ if and only if there is a sequence (S_n) in $I(E,F_\infty)$ so that $\lim_{n\to\infty}\|J_\infty S - S_n\| = 0$ (see [A, 3.5], [T2, 1.1]).

Recall two consequences of the geometric Hahn–Banach theorem.

LEMMA 1.1 [R, 2.1 and 2.2]. *Let E, F, G and H be Banach spaces and suppose that $S \in L(E,F)$, $T \in L(E,G)$, $R \in L(H,F)$ and $\varepsilon > 0$.*

(i) *$\|Sx\| \le \|Tx\| + \varepsilon\|x\|$ for all $x \in E$ if and only if $S^*B_{F^*} \subset T^*B_{G^*} + \varepsilon B_{E^*}$.*

(ii) *$\|S^*x^*\| \le \|R^*x^*\| + \varepsilon\|x^*\|$ for all $x^* \in F^*$ if and only if $SB_E \subset \overline{RB_H + \varepsilon B_F}$.*

Define the adjoint ideal I^* of the operator ideal I by $I^*(E,F) = \{S \in L(E,F) : S^* \in I(F^*,E^*)\}$ for Banach spaces E and F. Recall that I is injective if $I(E,F) = \{S \in L(E,F) : J_\infty S \in I(E,F_\infty)\}$ for all E and F. Our first duality result is quite general.

PROPOSITION 1.2. *Let I be a closed injective operator ideal so that $S^{**} \in I(E^{**},F^{**})$ whenever $S \in I(E,F)$, E and F Banach spaces. Then*

$$\beta_I(S) = \gamma_{I^*}(S^*) = \beta_I(S^{**}) \tag{1.1}$$

for all $S \in L(E,F)$, E and F Banach spaces.

Proof. Suppose that $\lambda > \beta_I(S)$ and take $R \in I(E,G)$ so that $\|Sx\| \le \|Rx\| + \lambda\|x\|$ for all $x \in E$. Lemma 1.1(i) implies that $S^*B_{F^*} \subset R^*B_{G^*} + \lambda B_{E^*}$. Hence $\gamma_{I^*}(S^*) \le \lambda$, since $R^* \in I^*(G^*,E^*)$ by the symmetry assumption on I. Thus $\gamma_{I^*}(S^*) \le \beta_I(S)$.

Observe next that $\beta_I(T^*) \le \gamma_{I^*}(T)$ for any $T \in L(E,F)$. In fact, assume that $\lambda > \gamma_{I^*}(T)$ and take $R \in I^*(G,F)$ so that $TB_E \subset RB_G + \lambda B_F$. Hence $\|T^*x^*\| \le \|R^*x^*\| + \lambda\|x^*\|$ for all $x^* \in F^*$ by Lemma 1.1(ii) and we get

$\beta_I(T^*) \le \lambda$. The preceding facts imply

$$\beta_I(S) = \beta_I(K_F S) = \beta_I(S^{**}K_E) \le \beta_I(S^{**}) \le \gamma_{I^*}(S^*),$$

since β_I is preserved by isometries. This proves the first equality in (1.1). Hence we see from [A, 5.1] that $\beta_I(S^{**}) = \gamma_{I^*}(S^{***}) = \gamma_{I^*}(S^*) = \beta_I(S)$ for any $S \in L(E,F)$. ■

The special case $\beta_K(S) = \gamma_K(S^*)$ of (1.1) was verified in [GM, Thm. 2] by different means for the ideal K of compact operators. The customary notation $\omega(S) = \gamma_W(S)$ for $S \in L(E,F)$ will be used for the weakly compact operators W. Thus $\beta_W(S) = \omega(S^*)$ by (1.1), since $W^* = W$ according to [DS, VI.4.8]. The example in [AT, Thm. 4] demonstrates that there are no uniform estimates between $\omega(S)$ and $\omega(S^*)$. We establish as a contrast that $\|R(\cdot)\|$ is uniformly self-dual. Let π_{E^*} denote the canonical projection $E^{***} \to E^*$ defined by $\pi_{E^*}(u) = u_{|E}$ for $u \in E^{***}$ and set $\varrho_{E^*} = I - \pi_{E^*}$.

PROPOSITION 1.3. *Let E and F be Banach spaces. Then*

$$\frac{1}{\|\varrho_{E^*}\|}\,\|R(S)\| \le \|R(S^*)\| \le \|\varrho_{F^*}\|\cdot\|R(S)\|, \qquad S \in L(E,F). \tag{1.2}$$

Proof. The map ϱ_{E^*} is a projection onto $E^\perp = \{v \in E^{***} : v_{|E} = 0\}$ and $\mathrm{Ker}(\varrho_{E^*}) = E^*$. Thus ϱ_{E^*} induces the isomorphism $\widehat{\varrho}_{E^*} : E^{***}/E^* \to E^\perp$ by $\widehat{\varrho}_{E^*}(u + E^*) = \varrho_{E^*}u$ for $u \in E^{***}$. We verify that

$$\widehat{\varrho}_{E^*}R(S^*) = R(S)^*\widehat{\varrho}_{F^*}, \qquad S \in L(E,F), \tag{1.3}$$

where the standard identification $(E^{**}/E)^* = E^\perp$ has been applied. Indeed, $\widehat{\varrho}_{E^*}R(S^*)(u + F^*) = \varrho_{E^*}S^{***}u$ for $u + F^* \in F^{***}/F^*$. On the other hand, if $x + E \in E^{**}/E$, then

$$\begin{aligned}\langle R(S)^*\widehat{\varrho}_{F^*}(u+F^*), x+E\rangle &= \langle \varrho_{F^*}u, S^{**}x + F\rangle = \langle \varrho_{F^*}u, S^{**}x\rangle \\ &= \langle S^{***}\varrho_{F^*}u, x\rangle = \langle \varrho_{E^*}S^{***}u, x + E\rangle.\end{aligned}$$

The last equality results by noting that $S^{***}F^\perp \subset E^\perp$ and $S^{***}F^* \subset E^*$. Finally, (1.2) follows from (1.3) and the fact that $\|(\widehat{\varrho}_{E^*})^{-1}\| \le 1$ in view of $\|u + E^*\| \le \|u - u_{|E}\| = \|\widehat{\varrho}_{E^*}(u + E^*)\|$ for $u + E^* \in E^{***}/E^*$. ■

[Y1, 2.8] states that $R(S^*)$ and $R(S)^*$ are similar, but (1.3) was not made explicit there. The preceding proposition yields $\|R(S)\|/2 \le \|R(S^*)\| \le 2\|R(S)\|$ for $S \in L(E,F)$. It was observed in [T1, 1.1] that

$$\|R(S)\| \le \omega(S) \tag{1.4}$$

for any $S \in L(E,F)$, E and F Banach spaces. We improve this below. A proof of the known fact (i) is included, since we need an estimate for the norm of the inverse map.

PROPOSITION 1.4. *Let E and F be Banach spaces and $S \in L(E,F)$.*

(i) *Assume that M is a non-reflexive subspace of E such that the restriction SJ is an embedding, where $J : M \to E$ stands for the inclusion map. Then $R(SJ)$ embeds M^{**}/M into F^{**}/F.*

(ii) $$\|R(S)\| \le \min\{\omega(S), 2\omega(S^*), 2\omega(S^{**})\}.$$

Proof. (i) Standard duality and w^*-w^* continuity identifies M^{**} with $M^{\perp\perp} = \overline{M}^*$, the w^*-closure of M in E^{**}, and $(SJ)^{**}M^{**}$ with $(SM)^{\perp\perp} = \overline{SM}^*$. Suppose that $x^{**} \in M^{**}$ and $\varepsilon > 0$. The Proposition of [V, pp. 107–108] yields an element $y \in SM$ so that $\|S^{**}J^{**}x^{**} - y\| \le 2(\mathrm{dist}(S^{**}J^{**}x^{**}, F) + \varepsilon)$. Set $V = (S_{|M})^{-1} : SM \to M$. We get

$$\begin{aligned}\|x^{**} + M\| &= \|R(V)R(S_{|M})(x^{**} + M)\| \le \|R(V)\|\cdot\|S^{**}J^{**}x^{**} + SM\| \\ &\le \|R(V)\|\cdot\|S^{**}J^{**}x^{**} - y\| \\ &\le 2\|R(V)\|(\mathrm{dist}(S^{**}J^{**}x^{**}, F) + \varepsilon).\end{aligned}$$

(ii) (1.2) and (1.4) imply $\|R(S)\| \le 2\|R(S^*)\| \le 2\omega(S^*)$ for $S \in L(E,F)$. Moreover, from the proof of part (i) and [A, 5.1] we get

$$\|R(S)\| \le 2\|R(K_F)R(S)\| \le 2\omega(K_F S) = 2\omega(S^{**}). \ ■$$

$\|R(\cdot)\|$ is not uniformly comparable with any of the other quantities appearing in Proposition 1.4(ii). Recall that a Banach space E has the *Schur property* if weakly convergent sequences of E are norm-convergent. ℓ^1 is a standard example of a space with the Schur property.

EXAMPLE 1.5. [AT, Theorem 4] constructs a separable c_0-sum $E = (\bigoplus_{n\in\mathbb{N}}(c_0, |\cdot|_n))_{c_0}$, where $(c_0, |\cdot|_n)$ is a certain sequence of equivalent renormings of c_0, and operators $(S_n) \subset L(E, c_0)$ so that $\omega(S_n) \le 1/n$ but $\omega(S_n^*) = 1$ for all $n \in \mathbb{N}$. Put $T_n = S_n^* \in L(\ell^1, E^*)$, $n \in \mathbb{N}$. Proposition 1.3 implies that $\|R(T_n)\| \le 2\|R(S_n)\| \le 2/n$, but $\omega(T_n^{**}) = \omega(T_n) = \omega(S_n^*) = 1$ for all $n \in \mathbb{N}$ according to [A, 5.1] and the construction. This yields that $\|R(S)\|$ is not in general uniformly equivalent to any of $\omega(S)$, $\omega(S^*)$ or $\omega(S^{**})$.

The space E^* admits another property of relevance for Section 2: for all $S \in L(Z, E^*)$ and arbitrary Banach spaces Z,

$$\|S\|_w \le 2\omega(S). \tag{1.5}$$

Indeed, $E^* = (\bigoplus_{n\in\mathbb{N}}(\ell^1, |\cdot|_n^*))_{\ell^1}$ has the metric approximation property, since E^* is a separable dual space having the approximation property (see [LT2, 1.e.15]). Hence [LS, 3.6] and the Schur property of E^* yield for $S \in L(Z, E^*)$ that

$$\begin{aligned}\|S\|_w &= \mathrm{dist}(S, K(Z, E^*)) \\ &\le 2\inf\{\varepsilon > 0 : SB_Z \subset D + \varepsilon B_{E^*},\ D \subset E^* \text{ is a finite set}\} = 2\omega(S).\end{aligned}$$

PROBLEM. It remains unknown whether there is $c>0$ so that

$$\omega(S^{**}) \geq c\omega(S), \qquad S\in L(E,F). \tag{1.6}$$

One has $\omega(S^{**}) = \omega(K_F S) \leq \omega(S)$ for any S by [A, 5.1], so this asks about the behaviour of ω under $K_F : F \to F^{**}$. We refer to [AT, p. 372] for a condition that ensures (1.6). The constant $c = 1/2$ is the best possible in (1.6) for operators $S : E \to c_0$ (see [A, 1.10] and [AT, p. 374]).

2. Mapping properties of R. This section focusses on the mapping properties of the correspondence $S + W(E,F) \to R(S)$ from the quotient space $(L(E,F)/W(E,F), \|\cdot\|_w)$ to $L(E^{**}/E, F^{**}/F)$. Several examples demonstrate strongly varying behaviour of $R(\mathcal{W}(E))$ in the algebra case $E = F$, where $\mathcal{W}(E)$ denotes the weak Calkin algebra $L(E)/W(E)$. They indicate that the problem of identifying $\mathrm{Im}(R)$ is quite hard for given Banach spaces.

We first consider when R is metrically faithful in the sense that the image $\mathrm{Im}(R)$ is closed. It was pointed out in [T1, 1.2] that $R(\mathcal{W}(E))$ is not always a closed subalgebra of $L(E^{**}/E)$. The following two weakly compact approximation properties of Banach spaces from [AT] and [T2] will yield further examples.

- The space F has property (P1) if there is $c \geq 1$ so that $\inf\{\|R - UR\| : U \in W(F), \|I - U\| \leq c\} = 0$ for all Banach spaces E and $R \in W(E,F)$.
- The space F has property (P2) if there is $c \geq 1$ so that $\inf\{\|R - RU\| : U \in W(F), \|I - U\| \leq c\} = 0$ for all Banach spaces E and $R \in W(F,E)$.

We refer to [LT1, II.5.b] for the definition of the class of $\mathcal{L}^1$- and $\mathcal{L}^\infty$-spaces, which contains the $C(K)$- and $L^1(\mu)$-spaces.

THEOREM 2.1. (i) *Let E be an $\mathcal{L}^1$- or an $\mathcal{L}^\infty$-space. Then E has property* (P1) *if and only if E has the Schur property, and E has property* (P2) *if and only if E^* has the Schur property.*

(ii) *If $\mathrm{Im}(R)$ is closed in $L(E^{**}/E, F^{**}/F)$ for all Banach spaces E then F has property* (P1).

(iii) *If $\mathrm{Im}(R)$ is closed in $L(E^{**}/E, F^{**}/F)$ for all Banach spaces F then E has property* (P2).

Proof. (i) See [AT, Cor. 3] and [T2, 3.5].

(ii) If the Banach space F does not satisfy (P1), then the proof of [AT, Thm. 4] yields a Banach space E and a sequence $(S_n) \subset L(E,F)$ so that $\|S_n\|_w = 1$ and $\omega(S_n) \leq 1/n$ for all $n \in \mathbb{N}$. Hence (1.4) implies that $\mathrm{Im}(R)$ fails to be closed in $L(E^{**}/E, F^{**}/F)$.

(iii) If the Banach space E does not satisfy (P2), then according to the proof of [T2, 1.2] there is a Banach space F and a sequence $(S_n) \subset L(E,F)$ so that $\|S_n\|_w = 1$ and $\beta_W(S_n) \leq 1/n$ for all $n \in \mathbb{N}$. From (1.2) and Propositions 1.2 (applied to W) and 1.3 we get

$$\|R(S_n)\| \leq 2\|R(S_n^*)\| \leq 2\omega(S_n^*) = 2\beta_W(S_n) \leq 2/n,$$

for all $n \in \mathbb{N}$. Thus $\mathrm{Im}(R)$ fails to be closed in $L(E^{**}/E, F^{**}/F)$. ■

Remarks. The converse implications to those of (ii) and (iii) above do not hold. To see this let E and $(S_n) \subset L(E, c_0)$ be as in Example 1.5. The map R has closed range neither on $L(E, c_0)$ nor on $L(\ell^1, E^*)$, since $\|S_n\|_w \geq \|S_n^*\|_w \geq \omega(S_n^*) = 1$ for all n but $R(S_n)$ and $R(S_n^*)$ tend to 0 as $n \to \infty$. One verifies that E^* satisfies (P1) and that E satisfies (P2) by using [T2, Remark (ii) after Example 2.5] and the fact that E^* has the metric approximation property and the Schur property.

It turns out that R is not surjective for many classical non-reflexive Banach spaces (here we disregard pairs E, F of non-reflexive Banach spaces for which $L(E,F) = W(E,F)$). Recall that the operator $S : E \to F$ is *inessential*, denoted $S \in I(E,F)$, if $\mathrm{Ker}(\mathrm{Id}_E - US)$ is finite-dimensional and $\mathrm{Im}(\mathrm{Id}_E - US)$ has finite codimension in E for all $U \in L(F,E)$. It is well known that I is a closed operator ideal so that $K(E,F) \subset I(E,F)$ and that $\mathrm{Id}_E \in I(E)$ only if E is finite-dimensional.

THEOREM 2.2. *Suppose that E is one of the spaces c_0, $C(\mathbf{K})$ for a countable compact set $\mathbf{K}$, $C(0,1)$, ℓ^1, $L^1(0,1)$, ℓ^∞ or the analytic function spaces H^∞ and $A(D)$. Then*

$$R(\mathcal{W}(E)) \cap I(E^{**}/E) = \{0\}. \tag{2.1}$$

In particular, R is not surjective. However, $R(\mathcal{W}(E))$ is closed in $L(E^{**}/E)$ if E is c_0, ℓ^1 or $L^1(0,1)$.

Proof. Suppose that E equals c_0 or ℓ^1 and assume that $S \notin W(E) = K(E)$. It is well known that there are $A, B \in L(E)$ so that $\mathrm{Id}_E = BSA$ (see [Pi, 5.1]). Hence

$$\mathrm{Id}_{E^{**}/E} = R(B)R(S)R(A) \tag{2.2}$$

and $R(S) \notin I(E^{**}/E)$, since otherwise $\mathrm{Id}_{E^{**}/E} \in I$ but $\dim(E^{**}/E) = \infty$.

Factorization (2.2) is also valid for $E = \ell^\infty$ and $S \notin W(\ell^\infty)$. Indeed, a result of Rosenthal [LT2, 2.f.4] gives a subspace $M \subset \ell^\infty$, $M \approx \ell^\infty$, so that the restriction $S_{|M}$ defines an isomorphism $M \to SM$. Since any ℓ^∞-copy is complemented there is a projection $Q : \ell^\infty \to SM$ as well as an isomorphism $A : \ell^\infty \to M$. Then (2.2) holds with $B = A^{-1}(S_{|M})^{-1}Q$.

If $S \notin W(C(0,1))$, then there is a subspace $M \subset C(0,1)$, $M \approx c_0$, so that the restriction $S_{|M}$ determines an isomorphism. Both M and SM are complemented in $C(0,1)$ by Sobczyk's theorem. We find as above operators

A, B so that $BSA = \mathrm{Id}_{c_0}$. A similar argument applies to all separable $C(\mathbf{K})$-spaces. Moreover, if $S \notin W(L^1(0,1))$, then there are operators A, B with $BSA = \mathrm{Id}_{\ell^1}$. The above facts are based on [P2, pp. 35 and 39]. We thus obtain (2.2) with $E = c_0$, respectively $E = \ell^1$. Similarly, for H^∞ and $A(D)$ one applies [B, Thm. 1] and [K] in order to deduce (2.2) with $E = \ell^\infty$, respectively $E = c_0$.

Suppose next that E is c_0 or ℓ^1. Then $\|R(S)\| = \mathrm{dist}(S, K(E)) = \|S\|_w$ for $S \in L(E)$. This follows from the uniqueness of submultiplicative norms in certain quotient algebras (see [M, Thm. 2]). Moreover, $\|R(S)\| = \|S\|_w$ for $S \in L(L^1(0,1))$ by [WW, 3.1]. Thus R has closed range in these cases. ■

Remarks. Actually, (2.2) implies that any non-zero $R(S)$ is large in the sense that $R(S)$ determines an isomorphism between complemented copies of E^{**}/E. It remains unclear to us whether $R(\mathcal{W}(E))$ is closed if E is $C(0,1)$ or ℓ^∞.

Theorem 2.2 expresses the fact that $\mathrm{Im}(R)$ does not contain "small" operators, e.g. compact ones, for many concrete spaces. There are two general Banach space properties that allow a similar conclusion. This is the content of Theorem 2.3 below.

Let Ro stand for the operator ideal of *weakly conditionally compact operators*: $S \in Ro(E,F)$ if (Sx_n) admits a weak Cauchy subsequence for all bounded sequences (x_n) in E. A Banach space E is *weakly sequentially complete* if any weak Cauchy sequence of E converges weakly. Examples of weakly sequentially complete spaces are known to include all subspaces of $L^1(0,1)$ and C_1, the trace class operators on ℓ^2.

The operator $S : E \to F$ is *unconditionally converging*, denoted $S \in U(E,F)$, if $\sum_{n=1}^\infty Sx_n$ is unconditionally convergent in F whenever the formal series $\sum_{n=1}^\infty x_n$ in E satisfies $\sum_{n=1}^\infty |x^*(x_n)| < \infty$ for all $x^* \in E^*$. A Banach space E has *Pełczyński's property* (V) if $U(E,F) = W(E,F)$ for all Banach spaces F. Any $C(\mathbf{K})$-space, and more generally any C^*-algebra, has property (V) ([P1, Thm. 1] and [Pf, Cor. 6]) as well as any Banach space E that is an M-ideal in E^{**} (see [HWW, III.1 and III.3.4] for a list of examples).

THEOREM 2.3. *Let E and F be Banach spaces.*

(i) *If $S \in L(E,F)$ and $R(S) \in Ro(E^{**}/E, F^{**}/F)$, then we have $S^{**} \in Ro(E^{**}, F^{**})$.*

(ii) *If F is weakly sequentially complete, then we have $R(L(E,F)) \cap Ro(E^{**}/E, F^{**}/F) = \{0\}$.*

(iii) *If E has property* (V), *then $R(L(E,F)) \cap U(E^{**}/E, F^{**}/F) = \{0\}$.*

Proof. (i) [DFJP, pp. 313–314] produces for each $U \in L(E,F)$ a factorization $U = jA$ through a Banach space Z. The intermediate space Z has the property

(2.3) $\quad U \in Ro(E,F)$ if and only if ℓ^1 does not embed in Z

(see [W, Satz 1]). The DFJP-factorization of U^{**} and $R(U)$ can be obtained as $U^{**} = j^{**}A^{**}$ and $R(U) = R(j)R(A)$, through the intermediate spaces Z^{**}, respectively Z^{**}/Z, by [G, 1.5 and 1.6].

Suppose that $R(S) \in Ro(E^{**}/E, F^{**}/F)$. We claim that S^{**} is weakly conditionally compact. It suffices to verify in view of (2.3) that ℓ^1 embeds in Z^{**}/Z whenever ℓ^1 embeds in Z^{**}.

Case 1. Assume that ℓ^1 does not embed in Z. Let $M \subset Z^{**}$ be a subspace so that $M \approx \ell^1$. Hence Z and M are totally incomparable and $M+Z$ is closed in Z^{**}. We may suppose that $M \cap Z = \{0\}$. This implies that $Q_{|M}$ defines an embedding and $QM \approx \ell^1$ in Z^{**}/Z, where $Q : Z^{**} \to Z^{**}/Z$ stands for the quotient map.

Case 2. Assume that ℓ^1 embeds in Z. Clearly ℓ^1 embeds in $(\ell^1)^{**}/\ell^1$ as this quotient is an $\mathcal{L}^1$-space. Thus ℓ^1 embeds in Z^{**}/Z, since $(\ell^1)^{**}/\ell^1$ is isomorphic to a subspace of Z^{**}/Z by Proposition 1.4(i).

(ii) If $R(S) \in Ro(E^{**}/E, F^{**}/F)$, then part (i) implies that S is weakly conditionally compact. Hence $S \in W(E,F)$ since F is weakly sequentially complete.

(iii) We first verify that $S \in U(E,F)$ whenever $R(S)$ is unconditionally converging. In fact, if $S \notin U(E,F)$, then there is a subspace $M \subset E$, $M \approx c_0$, so that $S_{|M}$ is an embedding [P2, p. 34]. Let $J : M \to E$ be the inclusion map. Proposition 1.4(i) yields that $R(SJ)$ is an embedding on $M^{**}/M \approx \ell^\infty/c_0$. This implies that $R(S)$ is not unconditionally converging as c_0 embeds in ℓ^∞/c_0 (for instance by [LT2, 2.f.4]). If E has property (V) and $R(S)$ is unconditionally converging, then the preceding observation yields that $S \in U(E,F) = W(E,F)$. ■

We next construct various examples where R has quite different properties compared with Theorems 2.2 and 2.3. In these examples $\mathrm{Im}(R)$ contains plenty of "small" operators and in some cases R is even an isomorphism.

The quotient E^{**}/E is quite unwieldy for most Banach spaces E, but if the space Z is weakly compactly generated, then there is a Banach space X so that X^{**}/X is isomorphic to Z (see [DFJP, p. 321]). We recall here a more restricted construction. The *James sum* of a Banach space E is

$$J(E) = \{(x_k) : x_k \in E, \|(x_k)\| < \infty \text{ and } \lim_{k\to\infty} x_k = 0\},$$

where $\|(x_k)\| = \sup_{i_1<\ldots<i_{n+1}} (\sum_{k=1}^n \|x_{i_{k+1}} - x_{i_k}\|^2)^{1/2}$. The supremum is taken over all increasing sequences $1 \le i_1 < \ldots < i_{n+1}$ of natural numbers and $n \in \mathbb{N}$. It is known [Wo] that $J(E)^{**}$ is the space of all sequences (x_k) with $x_k \in E^{**}$ for which the above 2-variation norm is finite. If E is reflexive, then any $(x_k) \in J(E)^{**}$ can be written as $(x_k - x)_{k\in\mathbb{N}} + (x)_{k\in\mathbb{N}}$, where $x = \lim_{n\to\infty} x_n$ (the limit clearly exists in E), and $(x_k)+J(E) \to \lim_{k\to\infty} x_k$ gives an isomorphism $J(E)^{**}/J(E) \to E$.

A Banach space E is *quasi-reflexive of order* n if $\dim(E^{**}/E) = n$ for some $n \in \mathbb{N}$. In this case $R(\mathcal{W}(E))$ identifies with a subalgebra of the scalar-valued $n \times n$-matrices and there is $c = c(E) > 0$ so that $c\|S\|_w \le \|R(S)\|$ for all $S \in L(E)$. We use J for $J(\mathbb{R})$, the (real) James space, which is quasi-reflexive of order 1 (see [LT2, 1.d.2]). One has $J^{**} = J \oplus \mathbb{R}f$, where $f = (1,1,\ldots)$. The behaviour of R varies even within the class of quasi-reflexive spaces.

EXAMPLES 2.4. (i) Let $\ell_2^n(J) = J \oplus \ldots \oplus J$ (n copies) with the ℓ_2^n-norm, whence $\dim(\ell_2^n(J)^{**}/\ell_2^n(J)) = n$ for all n. Then $R : \mathcal{W}(\ell_2^n(J)) \to L(\ell_2^n(J)^{**}/\ell_2^n(J))$ is a bijection. This follows from the fact that $R(\mathrm{Id}_J)$ identifies with the 1-dimensional operator taking $f = (1,1,\ldots)$ to itself. It is computed below during the proof of Theorem 2.6 that $\inf_{n\in\mathbb{N}} c(\ell_2^n(J)) = 0$.

(ii) Let J_p stand for the quasi-reflexive James space of order 1 defined using p-variation in the norm instead of 2-variation for $1 < p < \infty$ (thus $J_2 = J$). Suppose that $1 < p_1 < \ldots < p_n < \infty$. Loy and Willis [LW, p. 345] observed for the quasi-reflexive space $\bigoplus_{i=1}^n J_{p_i}$ of order n that the image of R coincides with the lower-triangular $n \times n$-matrices. This is based on the facts that, for $1 < p < q < \infty$, any operator $J_q \to J_p$ is compact while the formal identity $J_p \to J_q$ is not weakly compact.

(iii) Leung [L, Prop. 6] constructed a quasi-reflexive Banach space F of order 1 so that $L(F,F^*) = W(F,F^*)$ and $L(F^*,F) = W(F^*,F)$. Then $E = F \oplus F^*$ is quasi-reflexive of order 2, but $\mathrm{Im}(R)$ identifies with the class of diagonal 2×2-matrices.

In our next result X^{**}/X is infinite-dimensional, but R is surjective.

PROPOSITION 2.5. *Suppose that E is a reflexive infinite-dimensional Banach space and let $J(E)$ be the corresponding James-sum. Then R is an isomorphism and $R(\mathcal{W}(J(E))) = L(J(E)^{**}/J(E))$, where $J(E)^{**}/J(E) \approx E$.*

Proof. Let $\phi : J(E)^{**}/J(E) \to E$ stand for the isomorphism $(x_k) + J(E) \to \lim_{k\to\infty} x_k$. It suffices to verify that any $S \in L(E)$ belongs to the image of R under this identification. Suppose that $S \in L(E)$ and let $\widehat{S}$ be the bounded operator on $J(E)$ defined by $\widehat{S}(x_k) = (Sx_k)$ for $(x_k) \in J(E)$. One verifies using w^*-convergence that $\widehat{S}^{**}(x_k) = (Sx_k)$ whenever $(x_k) \in J(E)^{**}$. Then $R(\widehat{S})$ equals S as

$$\phi R(\widehat{S})((x_k) + J(E)) = \lim_{k\to\infty} Sx_k = S(\phi((x_k) + J(E))). \ \blacksquare$$

PROBLEM. Is E^{**}/E always reflexive if $R : \mathcal{W}(E) \to L(E^{**}/E)$ is a bijection?

Let $X = \ell^2(J)$ stand for the ℓ^2-sum of a countable number of copies of James' space J. Thus $\ell^2(J)^{**} = \ell^2(J^{**})$ isometrically and it is not difficult to verify that X^{**}/X is isometric to ℓ^2 through $(x_k) + \ell^2(J) \to (w_1, w_2, \ldots)$, where $w_k = \lim_{j\to\infty} x_j^{(k)}$ for $x_k = (x_j^{(k)})_{j\in\mathbb{N}} \in J^{**}$. The *lattice regular operators* on ℓ^2 (with respect to the natural orthonormal basis) are defined by

$$\mathrm{Reg}(\ell^2) = \{A = (a_{ij}) \in L(\ell^2) : |A| = (|a_{ij}|) \text{ defines a bounded operator on } \ell^2\}.$$

Here (a_{ij}) is the matrix representation of A. It is known that $A \in \mathrm{Reg}(\ell^2)$ if and only if $A = U - V$, where U and V are operators having matrices with non-negative entries. The algebra $\mathrm{Reg}(\ell^2)$ is complete in the regular norm $\|A\|_r = \|\,|A|\,\|$ (see [AB, 15.2]) and $\|A\| \le \|A\|_r$, but $\mathrm{Reg}(\ell^2)$ is not a closed subalgebra of $L(\ell^2)$. For instance, let (A_n) be the $2^n \times 2^n$ Walsh–Littlewood matrices,

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix}$$

for $n \in \mathbb{N}$. Then $\|A_n\|_r/\|A_n\| = 2^{n/2}$ for all n. Moreover, the Hilbert–Schmidt operators are included in $\mathrm{Reg}(\ell^2)$.

Let (e_n) be the standard coordinate basis of J. James' space J also admits the Schauder basis (f_k), where $f_k = \sum_{j=1}^k e_j$ for $k \in \mathbb{N}$. The norm in J is computed in (f_k) as

$$\Big\|\sum_{k=1}^\infty b_k f_k\Big\| = \sup_{1\le i_1<\ldots<i_{n+1}} \Big(\sum_{j=1}^n |b_{i_j} + \ldots + b_{i_{j+1}-1}|^2\Big)^{1/2} \tag{2.4}$$

for $\sum_{k=1}^\infty b_k f_k \in J$. Let $P_n : J \to [f_1,\ldots,f_n]$ be the basis projections. It follows from (2.4) that $\|P_n\| = \|I - P_n\| = 1$ for all $n \in \mathbb{N}$.

The main result of this section identifies $R(\mathcal{W}(\ell^2(J)))$ with the algebra $\mathrm{Reg}(\ell^2)$ (note that $\ell^2(J)^{**}/\ell^2(J)$ is isometric to ℓ^2 as above). This provides a concrete Banach space X so that $\|R(\cdot)\|$ and $\|\ \|_w$ fail to be comparable on $L(X)$ (see also Theorem 2.1). The proof uses local properties of J. Our result also settles a basic question concerning the representation R (Corollary 2.10).

THEOREM 2.6. *R is an algebra isometry of $\mathcal{W}(\ell^2(J))$ onto $(\mathrm{Reg}(\ell^2), \|\cdot\|_r)$,*

$$\|S\|_w = \|R(S)\|_r \tag{2.5}$$

for all $S \in L(\ell^2(J))$. Thus $\mathrm{Im}(R)$ is not closed in $L(\ell^2)$.

Proof. We first verify that for any $A \in \mathrm{Reg}(\ell^2)$ there is $\widehat{A} \in L(\ell^2(J))$ so that $R(\widehat{A}) = A$ and $\|\widehat{A}\|_w \le \|A\|_r$.

Let $A = (a_{ij})$ be a bounded regular operator on ℓ^2 and consider the formal operator $\widehat{A}$ defined by the operator matrix $(a_{ij}I)$, where I stands for the identity mapping on J.

Assume that $(x_r) \in \ell^2(J)$. We obtain

$$\|\widehat{A}(x_r)\|^2 = \sum_{l=1}^{\infty} \Big\| \sum_{r=1}^{\infty} a_{lr} x_r \Big\|^2 \le \sum_{l=1}^{\infty} \Big(\sum_{r=1}^{\infty} |a_{lr}| \cdot \|x_r\| \Big)^2$$
$$= \| \, |A|(\|x_r\|)\|^2 \le \|A\|_r^2 \sum_{r=1}^{\infty} \|x_r\|^2.$$

Thus $\widehat{A}$ defines a bounded operator on $\ell^2(J)$ and $\|\widehat{A}\| \le \|A\|_r$. One checks that $R(\widehat{A}) = A$, since $R(I)$ is the 1-dimensional identity taking $f = (1, 1, \ldots)$ to itself.

It remains to prove that $R(U) \in \mathrm{Reg}(\ell^2)$ and $\|R(U)\|_r \le \|U\|_w$ for $U \in L(\ell^2(J))$.

Suppose that $S = (s_{ij})$ is a matrix so that $s_{ij} = 0$ whenever $i > n$ or $j > n$ for some $n \in \mathbb{N}$. Let $\widehat{S} = (s_{ij}I)$ stand for the corresponding vector-valued operator on $\ell^2(J)$. We claim that

$$\|\widehat{S} - W\| \ge \|S\|_r \tag{2.6}$$

for any operator-valued matrix $W = (W_{ij})$ on $\ell^2(J)$ so that $W_{ij} \in W(J)$ for all $i, j \in \mathbb{N}$ and $W_{ij} = 0$ whenever $i > n$ or $j > n$.

Before establishing the claim we indicate how (2.5), and thus the theorem, follows from (2.6) with the help of a simple cut-off argument. Assume that $U = (U_{ij}) \in L(\ell^2(J))$, where (U_{ij}) is the matrix representation of U. We may write $U_{ij} = s_{ij}I + W_{ij}$ with $W_{ij} \in W(J)$ for $i, j \in \mathbb{N}$ so that $R(U) = (s_{ij})$. Define for $n \in \mathbb{N}$ the cut-off $U_n = (a_{ij}^{(n)} U_{ij})$, where $a_{ij}^{(n)} = 1$ if $i, j \le n$ and $a_{ij}^{(n)} = 0$ otherwise. (2.6) yields that

$$\|U_n\| \ge \|(a_{ij}^{(n)} s_{ij})\|_r.$$

By letting $n \to \infty$ above we obtain $\|U\| \ge \|R(U)\|_r$. This implies the desired inequality $\|U\|_w \ge \|R(U)\|_r$ since $R(U)$ is invariant under weakly compact perturbations of U.

It remains to establish (2.6). The main ingredients of the argument are presented as independent lemmas in order to make the strategy of the proof more transparent.

LEMMA 2.7. *Let $S = (s_{ij})$ be an $n \times n$-matrix and define $\widetilde{S} : \ell_2^n(\ell_1^n) \to \ell_2^n(\ell_1^n)$ by*

$$\widetilde{S}(y_1, \ldots, y_n) = \Big(\sum_{j=1}^{n} s_{1j} y_j, \ldots, \sum_{j=1}^{n} s_{nj} y_j \Big) \quad \textit{for } y_1, \ldots, y_n \in \ell_1^n.$$

Then $\|\widetilde{S}\| = \|S\|_r$.

Proof. We obtain $\|\widetilde{S}\| \le \|S\|_r$ as above. Choose $a = (a_1, \ldots, a_n) \in \ell_2^n$ so that $\|a\| = 1$ and $\|S\|_r = \| \, |S| a\|$. Let $\{h_1, \ldots, h_n\}$ be the unit vector basis of ℓ_1^n. We get

$$\|\widetilde{S}\|^2 \ge \|\widetilde{S}(a_1 h_1, \ldots, a_n h_n)\|^2 = \sum_{l=1}^{n} \Big\| \sum_{j=1}^{n} s_{lj} a_j h_j \Big\|^2$$
$$= \sum_{l=1}^{n} \Big(\sum_{j=1}^{n} |a_j| \, |s_{lj}| \Big)^2 = \|S\|_r^2. \ \blacksquare$$

The proofs of the next two auxiliary results are momentarily postponed. The first one establishes a joint "smallness" property for finite collections of weakly compact operators on J. This fact may have some independent interest. We remark that $U \in W(J)$ defined by $Uf_1 = f_1$, $Uf_k = f_{k-1} - f_k$ for $k \ge 2$, demonstrates that a weakly compact operator on J is not necessarily small between diagonal blocks of (f_k). The second result records the technical fact that convex blocks of (f_k) span isometric copies of J in the norm considered here. A proof is included because we are not aware of a suitable reference.

PROPOSITION 2.8. *Suppose that $S_1, \ldots, S_r \in W(J)$. For any $\varepsilon > 0$ and $n \in \mathbb{N}$ there is a natural number l and a sequence $(z_k)_{k=1}^n$ consisting of disjoint convex blocks of the basis (f_k) so that each z_k is supported after l and for $M_n = [z_1, \ldots, z_n]$ we have*

$$\max_{1 \le j \le r} \|(I - P_l) S_{j|M_n}\| < \varepsilon.$$

LEMMA 2.9. *Let $z_k = \sum_{j=n_k}^{n_{k+1}-1} c_j f_j$ be disjoint convex blocks of (f_j), where the sequence (n_i) is strictly increasing, $c_j \ge 0$ for all j and $\sum_{j=n_k}^{n_{k+1}-1} c_j = 1$ for all $k \ge 1$. Then (z_k) is a basic sequence in J that is isometrically equivalent to (f_k):*

$$\Big\| \sum_{k=1}^{\infty} b_k z_k \Big\| = \Big\| \sum_{k=1}^{\infty} b_k f_k \Big\| \tag{2.7}$$

for all $\sum_{k=1}^{\infty} b_k f_k \in J$.

Proof of (2.6). Let S, W and n be as in the claim. Suppose that $\delta > 0$. There is an integer m so that ℓ_1^n embeds $(1+\delta)$-isomorphically in $[f_1, \ldots, f_m]$ (see [GJ, Thm. 4]). Proposition 2.8 provides an integer l together with disjoint convex blocks $z_1, \ldots, z_m$ of (f_k) so that the following properties are satisfied:

(i) $Q_l z_j = z_j$ for $j = 1, \ldots, m$, where $Q_l = I - P_l$,
(ii) $\sum_{i,j=1}^{n} \|Q_l W_{ij|M_m}\| < \delta$. Here $M_m = [z_1, \ldots, z_m]$.

According to Lemma 2.9, M_m is isometric to $[f_1, \ldots, f_m]$ and there is a subspace $N \subset M_m$ so that N is $(1+\delta)$-isomorphic to ℓ_1^n. Write $\widehat{N} = \{(z_k) \in \ell^2(J) : z_k \in N,\ k \le n \text{ and } z_k = 0 \text{ otherwise}\}$. Let $\widehat{Q}_l \in L(\ell^2(J))$ be the norm-1 operator defined by $\widehat{Q}_l(x_r) = (Q_l x_r)$ for $(x_r) \in \ell^2(J)$. Observe that (ii) implies $\|\widehat{Q}_l W_{|\widehat{N}}\| < \delta$. Moreover, $\widehat{Q}_{l|\widehat{N}} = \mathrm{Id}_{|\widehat{N}}$ and $\widehat{S}\widehat{N} \subset \widehat{N}$, so that Lemma 2.7 yields

$$\|\widehat{Q}_l S_{|\widehat{N}}\| = \|S_{|\widehat{N}}\| \ge (1+\delta)^{-2}\|S\|_r.$$

Finally,

$$\|\widehat{S} - W\| \ge \|\widehat{Q}_l(\widehat{S} - W)_{|\widehat{N}}\| \ge (1+\delta)^{-2}\|S\|_r - \delta.$$

We get (2.6) by letting $\delta \to 0$ above. ■

Proof of Proposition 2.8. Observe that $f_k \xrightarrow{w^*} f = (1, 1, \ldots) \in J^{**}$ as $k \to \infty$. Thus $S_1 f_k \xrightarrow{w} S_1^{**} f \in J$ as $k \to \infty$, since S_1 is weakly compact. Fix a natural number l_1 such that $\|(I - P_{l_1}) S_1^{**} f\| < \varepsilon/(2n)$. Mazur's theorem implies that $S_1^{**} f \in \overline{\mathrm{co}}\{S_1 f_k : k \in \mathbb{N}\}$. One obtains by induction disjoint convex blocks $u_k = \sum_{j=m_k}^{n_k} c_j f_j$, where $l_1 \le m_1 < n_1 < m_2 < \ldots$ and $S_1 u_k \to S_1^{**} f$ in norm as $k \to \infty$. Notice that $\|u_k\| = 1$ for all k by (2.4). We may assume that $\|S_1 u_k - S_1^{**} f\| < \varepsilon/(2n)$ whenever $k \in \mathbb{N}$. Consequently,

$$\|(I - P_{l_1}) S_1 u_k\| \le \|I - P_{l_1}\| \cdot \|S_1 u_k - S_1^{**} f\| + \|(I - P_{l_1}) S_1^{**} f\| < \varepsilon/n$$

for all k.

Observe that $u_k \xrightarrow{w^*} f$ in J^{**} as $k \to \infty$, since (u_k) converges coordinatewise to f in the shrinking basis (e_k). Choose an integer $l_2 \ge l_1$ so that $\|(I - P_{l_2}) S_2^{**} f\| < \varepsilon/(2n)$. Apply the preceding argument to $(S_2 u_k)$ and recover as above disjoint convex blocks $v_k = \sum_{j=r_k}^{s_k} d_j u_j$ of (u_k) that are supported after l_2 with respect to (f_k), so that $\|S_2 v_k - S_2^{**} f\| < \varepsilon/(2n)$ for all k. We deduce as before that $\|(I - P_{l_2}) S_2 v_k\| < \varepsilon/n$. Note further that (v_k) are disjoint convex blocks of (f_k) and

$$\|(I - P_{l_2}) S_1 v_k\| \le \|(I - P_{l_1}) S_1 v_k\| \le \sum_{j=r_k}^{s_k} d_j \|(I - P_{l_1}) S_1 u_j\| < \varepsilon/n$$

for all k.

These observations allow us to repeat the above procedure in order to find eventually an integer l and disjoint convex blocks $z_k = \sum_{j=p_k}^{q_k} c_j f_j$ so that $\|(I - P_l) S_j z_k\| < \varepsilon/n$ for any $j = 1, \ldots, r$ and $k \in \mathbb{N}$. These estimates clearly imply that $\|(I - P_l) S_{j|[z_1, \ldots, z_n]}\| < \varepsilon$. This completes the proof of Proposition 2.8. ■

Proof of Lemma 2.9. By approximation there is no loss of generality in assuming that $\sum_{k=1}^{\infty} b_k f_k$ is finitely supported, $b_k = 0$ for $k \ge m$ and some $m \in \mathbb{N}$. According to (2.4) there are integers $1 = m_1 < m_2 < \ldots < m_l = m$ so that

$$\Big\|\sum_{k=1}^{m-1} b_k f_k\Big\|^2 = \sum_{r=1}^{l-1} \Big|\sum_{k=m_r}^{m_{r+1}-1} b_k\Big|^2. \tag{2.8}$$

Set $d_i = c_i b_k$ if $n_k \le i < n_{k+1}$ for some $k = 1, \ldots, l-1$, and $d_i = 0$ otherwise. Thus $\sum b_k z_k = \sum d_i f_i$, where $\sum_{k=m_r}^{m_{r+1}-1} b_k = \sum_{i=n_{m_r}}^{n_{m_{r+1}}-1} d_i$. Hence the right-hand side of (2.8) is a lower bound for $\|\sum b_k z_k\|$ so that $\|\sum_{k=1}^m b_k z_k\| \ge \|\sum_{k=1}^m b_k f_k\|$.

In order to prove the reverse inequality let l and $m_1, \ldots m_l$ be integers satisfying $1 = m_1 < m_2 < \ldots < m_l = n_m$. Put

$$N((m_r)) = \sum_{r=1}^{l-1} \Big|\sum_{i=m_r}^{m_{r+1}-1} d_i\Big|^2.$$

for each (m_r). Assume now that (m_r) is chosen so that $\|\sum_{k=1}^m b_k z_k\| = N((m_r))$. We verify below that (m_r) can be transformed to a sequence (m'_r) where each $m'_r \in \{n_k : 1 \le k \le m\}$, in such a way that $N((m_r)) \le N((m'_r))$. Clearly the convexity of the blocks and (2.4) together imply that $N((m'_r)) \le \|\sum b_k f_k\|^2$. This proves the lemma once (m'_r) is found.

The alteration proceeds as follows. Consider a fixed m_r and assume that $n_k < m_r < n_{k+1}$ for some k. Set

$$u = \sum_{i=m_{r-1}}^{m_r - 1} d_i \quad \text{and} \quad v = \sum_{i=m_r}^{m_{r+1}-1} d_i.$$

If $uv \ge 0$, then $(u+v)^2 \ge u^2 + v^2$ and $N(m_1, \ldots, m_{r-1}, m_{r+1}, \ldots m_l) \ge N((m_j))$. Simply discard m_r in this case.

In the case $uv < 0$ we proceed differently. We may suppose by symmetry that $u < 0$ and $v > 0$. There are two possibilities.

Case 1. Suppose that $b_k \ge 0$. We have $m_{r-1} < n_k$, since otherwise $u \ge 0$. Hence we get

$$\sum_{i=m_{r-1}}^{n_k - 1} d_i \le u < 0 \quad \text{and} \quad \sum_{i=n_k}^{m_{r+1}-1} d_i \ge v > 0$$

(here the fact that $c_j \ge 0$ for all j is used). This yields that $N(m_1, \ldots, m_{r-1}, n_k, m_{r+1}, \ldots, m_l) \ge N((m_j))$. Replace m_r by n_k.

Case 2. Suppose that $b_k < 0$. This implies that $m_{r+1} > n_{k+1}$. Deduce as above that $N(m_1, \ldots, m_{r-1}, n_{k+1}, m_{r+1}, \ldots, m_l) \ge N((m_j))$. Replace m_r by n_{k+1}.

By repeating the above procedure a finite number of times one arrives at the desired sequence (m'_r). This completes the proof of Lemma 2.9 and thus of Theorem 2.6. ∎

We consider as an application weak analogues of the Fredholm operators. Let E be a Banach space and set

$$\Phi_w(E) = \{S \in L(E) : S + W(E) \text{ is invertible in } L(E)/W(E)\},$$
$$\Phi_i(E) = \{S \in L(E) : R(S) \text{ is a bijection}\},$$

so that $\Phi_w(E) \subset \Phi_i(E)$. Yang [Y2, p. 522] states without citing examples that these concepts appear to be different. Theorem 2.6 gives rise to such examples. We refer to [T1] for additional motivation.

COROLLARY 2.10. *Let J be the complex James space. Then $\Phi_w(\ell^2(J)) \subsetneq \Phi_i(\ell^2(J))$.*

Proof. The proof of Theorem 2.6 carries through with some modifications in the case of complex scalars and (2.5) is replaced by the inequalities $c\|R(S)\|_r \le \|S\|_w \le \|R(S)\|_r$ for some $c > 0$ and all $S \in L(\ell^2(J))$. Here $\|(a_{ij})\|_r = \|(|a_{ij}|)\|$ for complex matrices (a_{ij}). The following additional facts are used.

• (2.7) admits as a complex counterpart $\|\sum_{k=1}^{\infty} b_k f_k\| \le \|\sum_{k=1}^{\infty} b_k z_k\| \le \sqrt{2}\|\sum_{k=1}^{\infty} b_k f_k\|$ for convex blocks (z_k) of (f_k) (apply (2.7) separately to the real and complex parts).

• The complex spaces $\ell_1^n(\mathbb{C})$ embed with uniform constant in the complex linear span $[f_1, \ldots, f_m]$ for m large enough. Indeed, it suffices to check that $\ell_\infty^r(\mathbb{C})$ embeds uniformly in the complex James space, and this is easily deduced from the fact that $\ell_\infty^r(\mathbb{R})$ embeds $(1+\delta)$-isomorphically in the real James space [GJ, Thm. 4] for all $\delta > 0$ and $r \in \mathbb{N}$.

It follows that $S \in \Phi_w(\ell^2(J))$ if and only if $R(S)$ is an isomorphism and its inverse $R(S)^{-1}$ is a regular operator. Ando (see [S, Ex. 1]) gave an example of a regular operator U on ℓ^2 so that its spectrum $\sigma(U) \subsetneq \sigma_r(U)$. Here $\sigma_r(U)$ denotes the spectrum of U in $\mathrm{Reg}(\ell^2)$. Lift U to an operator $\widehat{U} \in L(\ell^2(J))$ so that $R(\widehat{U}) = U$. Then $\sigma(\widehat{U} + W(\ell^2(J))) \subsetneq \sigma(R(\widehat{U}))$, which yields the claim. ∎

PROBLEM. The Yosida–Hewitt decomposition theorem implies that $(\ell^1)^{**} = \ell^1 \oplus c_0^{\perp}$ coincides with $(\ell^1)^{**} = ba(2^{\mathbb{N}}) = ca(2^{\mathbb{N}}) \oplus M_s$, where $M_s = \{\mu \in ba(2^{\mathbb{N}}) : \mu \text{ is purely finitely additive}\}$. Find conditions on $U \in L(M_s)$ so that U identifies with $R(S)$ for some $S \in L(\ell^1)$.

Buoni and Klein [BK] introduced a sequential representation of the quotient space $L(E,F)/W(E,F)$ (see [AT] for some further properties). Let E be a Banach space, $\ell^\infty(E) = \{(x_k) : (x_k) \text{ is bounded in } E\}$ equipped with the supremum norm and $w(E)$ its closed subspace $\{(x_k) \in \ell^\infty(E) :$

$\{x_k : k \in \mathbb{N}\}$ is relatively weakly compact in $E\}$. Set $Q(E) = \ell^\infty(E)/w(E)$ and consider $Q(S) \in L(Q(E), Q(F))$ for $S \in L(E,F)$, where

$$Q(S)((x_k) + w(E)) = (Sx_k) + w(F), \quad (x_k) \in \ell^\infty(E).$$

We have $Q(S) = 0$ if and only if $S \in W(E,F)$, $Q(\mathrm{Id}_E) = \mathrm{Id}_{Q(E)}$ and $Q(ST) = Q(S)Q(T)$ whenever ST is defined. Moreover, $\|Q(S)\| \le \omega(S)$, $S \in L(E,F)$, and equality holds if E is a separable Banach space [AT, Lemma 9]. Thus $S + W(E,F) \to Q(S)$ displays the same metric behaviour as $(L(E,F)/W(E,F), \omega)$ for separable E. [AT, Thm. 1] and [T2, 1.2] characterize the cases where the maps $S + W(E,F) \to Q(S)$ and $S + W(E,F) \to Q(S^*)$ have closed range within the class of separable Banach spaces.

$Q(E)$ is more difficult to handle than E^{**}/E. However, in Example 1.5 the map $Q : L(\ell^1, E^*)/W(\ell^1, E^*) \to L(Q(\ell^1), Q(E^*))$ has closed range in view of (1.5), but $\mathrm{Im}(R)$ fails to be closed. Hence Q and R have different properties in general. On the other hand, the proof of Theorem 2.2 implies that $Q(\mathcal{W}(E)) \cap I(Q(E)) = \{0\}$ if E is among the spaces mentioned in the theorem.

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DEPARTAMENTO DE MATEMÁTICAS
FACULTAD DE CIENCIAS
UNIVERSIDAD DE CANTABRIA
E-39071 SANTANDER, SPAIN

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HELSINKI
P.O.BOX 4 (HALLITUSKATU 15)
FIN-00014 UNIVERSITY OF HELSINKI, FINLAND

Received July 8, 1994 (3313)

STUDIA MATHEMATICA 113 (3) (1995)

Adjoint characterisations of unbounded weakly compact, weakly completely continuous and unconditionally converging operators

by

T. ALVAREZ (Oviedo), R. W. CROSS (Cape Town) and A. I. GOUVEIA (Cape Town)

Abstract. Characterisations are obtained for the following classes of unbounded linear operators between normed spaces: weakly compact, weakly completely continuous, and unconditionally converging operators. Examples of closed unbounded operators belonging to these classes are exhibited. A sufficient condition is obtained for the weak compactness of T'' to imply that of T.

1. Introduction and preliminaries. In this paper we shall be considering a linear operator $T : X \supset D(T) \to Y$ where X and Y are normed spaces.

Let us first recall some facts about bounded operators. Let T be bounded and everywhere defined and let X and Y be Banach spaces. Then T is *weakly compact* if it transforms bounded sequences into sequences having a weakly convergent subsequence; T is *weakly completely continuous* if it transforms weak Cauchy sequences into weakly convergent sequences; and T is *unconditionally converging* if it transforms weakly unconditionally convergent series into unconditionally convergent series. In order to characterize these classes of operators we introduce, for a given normed space E, the following subsets of E'':

$$K(E) = \{e'' \in E'' : \text{there exists a sequence } (e_n) \text{ in } E \text{ such that } e'' = \sigma(E'', E')\text{-lim } Je_n\},$$
$$N(E) = \{e'' \in E'' : \text{there exists a weakly unconditionally Cauchy series } \textstyle\sum e_i \text{ in } E \text{ such that } e'' = \sigma(E'', E')\text{-lim } \sum_{i=1}^{n} Je_i\}$$

1991 *Mathematics Subject Classification*: Primary 47B07.

This paper is an amalgamation of the two preprints: (1) R. W. Cross and A. I. Gouveia, *Unbounded weakly compact operators*, and (2) T. Alvarez and R. W. Cross, *Unbounded weakly compact, weakly completely continuous and unconditionally converging operators and their adjoints*.