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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
ŚNIADECKICH 8
00-950 WARSZAWA, POLAND

DEPARTMENT OF MATHEMATICS
TECHNION
32000 HAIFA, ISRAEL

Received December 17, 1993
Revised version September 27, 1994 (3210)

STUDIA MATHEMATICA 113 (3) (1995)

Property (wM^*) and the unconditional metric compact approximation property

by

ÅSVALD LIMA (Kristiansand)

Abstract. The main objective of this paper is to give a simple proof for a larger class of spaces of the following theorem of Kalton and Werner.

THEOREM. *Let X be a separable or reflexive Banach space. Then $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$ if and only if*

(a) *X has property (M^*), and*
(b) *X has the metric compact approximation property.*

Our main tool is a new property (wM^*) which we show to be closely related to the unconditional metric approximation property.

1. Introduction. We shall give characterizations of Banach spaces X such that $\mathcal{K}(X)$, the space of compact linear operators on X, is an M-ideal in $\mathcal{L}(X)$, the space of bounded linear operators on X. We shall give a new argument for the known fact that such spaces have the metric compact approximation property.

A closed subspace M of a Banach space X is called an *M-ideal* if there exists a projection P on X^* such that $\ker P = M^\perp$ and

$$\|x^*\| = \|x^* - Px^*\| + \|Px^*\| \quad \text{for all } x^* \in X^*.$$

Such a projection is called an *L-projection*. M-ideals were first defined and studied by Alfsen and Effros in [1] in 1972.

Many authors have tried to characterize those Banach spaces X such that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$. Finally, Kalton succeeded in [15] by introducing property (M^*) and showing that it plays a key role. But he had to assume that X satisfies a very strong form of the metric compact approximation

1991 *Mathematics Subject Classification*: Primary 46B20.

Research supported by the Norwegian Research Council. Part of the research was performed while the author was a guest at the Fachbereich Mathematik of Freie Universität Berlin. The author wants to thank Dirk Werner and his colleagues for their warm hospitality.

property, in particular, that the approximating compact operators (T_n) on X satisfy $\|I-2T_n\| \to 1$. Finally, Kalton and Werner [16] showed that this property follows from property (M^*) and the metric compact approximation property. Our aim is to show that the condition $\|I-2T_n\| \to 1$ is very closely related to a weak form of property (M^*) which we call property (wM^*).

Let us fix some notation. X, Y denote Banach spaces. The dual space of X is denoted by X^*. If M is a subspace of X, then its annihilator in X^* is written $M^\perp$. $B(x,r)$ or $B_X(x,r)$ denotes the closed ball in X with center x and radius r. We write $B_X(0,1) = B_X = X_1$.

We denote the space of compact linear operators from X to Y by $\mathcal{K}(X,Y)$. If $X=Y$, then we write $\mathcal{K}(X)$. Similarly, $\mathcal{L}(X,Y)$ and $\mathcal{F}(X,Y)$ denote the spaces of bounded and finite rank operators. T^* denotes the adjoint operator when $T \in \mathcal{L}(X,Y)$.

For a set C, $\operatorname{conv}(C)$ denotes its convex hull, $\overline{C}$ denotes its closure and $\operatorname{ext} C$ denotes its set of extreme points.

2. Property (M^*) and M-ideals of compact operators. In [15], Kalton characterized those Banach spaces X such that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$ using property (M^*). We shall give a new proof of the implication $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X) \Rightarrow X$ has property (M^*). Our proof uses only the fact that $\mathcal{K}(X)$ has the two-ball property in $\mathcal{L}(X)$. M-ideals were characterized by an intersection property of 3 balls by Alfsen and Effros in [1]. In [17], Lima introduced semi-M-ideals, and he gave a characterization of semi-M-ideals using intersections of 2 balls.

A closed subspace M of a Banach space X is called a *semi-M-ideal* if for all $x \in B_X$, all $m_1, m_2 \in B_M$ and all $\varepsilon > 0$, there is $m \in M$ satisfying

$$\|x + m_i - m\| \le 1 + \varepsilon \quad \text{for } i = 1,2.$$

DEFINITION 2.1. We say that a Banach space X has property (M^*) if whenever $u^*, v^* \in X^*$ with $\|u^*\| = \|v^*\|$ and (x^*_α) is a bounded weak* null net in X^*, then

$$\limsup_\alpha \|u^* + x^*_\alpha\| = \limsup_\alpha \|v^* + x^*_\alpha\|.$$

Property (M) is defined similarly using bounded weakly null nets in X.

Properties (M^*) and (M) were defined by Kalton in [15] using sequences. Thus there are two versions of these properties, defined by nets or by sequences. It is shown in [22] that if X is separable, then properties (M^*) defined using sequences or nets are equivalent. Moreover, if X^* is separable, then properties (M) defined using sequences or nets are equivalent. As the next result shows, property (M^*) is closely connected to M-ideals.

THEOREM 2.2. *If $\mathcal{K}(X)$ is a semi-M-ideal in $\mathcal{L}(X)$, then X has property (M^*).*

Proof. Let (x^*_α) be a bounded weak* null net in X^*. Define $\phi : X^* \to \mathbb{R}$ by

$$\phi(x^*) = \limsup_\alpha \|x^* + x^*_\alpha\|.$$

It is easily seen that ϕ is a convex norm continuous function. In fact, $|\phi(u^*) - \phi(v^*)| \le \|u^* - v^*\|$. It suffices to show the following claim:

CLAIM. *If $\|u^*\| = \|y^*\| = 1$ and y^* is weak* strongly exposed in B_{X^*}, then $\phi(y^*) \le \phi(u^*)$.*

Assume the claim is proved. Let $\|u^*\| = 1$ and let $\varepsilon > 0$. Since $B_{X^*} = \overline{\operatorname{conv}}^{\|\,\|}(w^*\text{-str.exp.}\, B_{X^*})$ (see [18]), we can find $x^* = \sum_{i=1}^m \lambda_i y_i^* \in \operatorname{conv}(w^*\text{-str.exp.}\, B_{X^*})$ such that $\|u^* - x^*\| \le \varepsilon$. Here $\lambda_i > 0$ and $\sum_{i=1}^m \lambda_i = 1$. Let $y^* \in B_{X^*}$ be weak* strongly exposed. From the claim, it follows that

$$\begin{aligned}\phi(u^*) - \varepsilon \le \phi(x^*) \le \sum_{i=1}^m \lambda_i \phi(y_i^*) &= \sum_{i=1}^m \lambda_i \phi(y^*) \\ &= \phi(y^*) \le \phi(u^*).\end{aligned}$$

Thus $\phi(u^*) = \phi(y^*)$.

In order to prove the claim, we use intersections of balls. Let y^* and u^* be as in the claim. Assume $y \in B_X$ strongly exposes y^*. Let $\varepsilon > 0$ and let $\varepsilon > \delta > 0$ be such that $\operatorname{diam} S(y,\delta) < \varepsilon$, where

$$S(y,\delta) = \{x^* \in B_{X^*} : x^*(y) \ge 1 - \delta\}.$$

Let $u \in B_X$ be such that $u^*(u) > 1 - \delta/5$. Define $S = y^* \otimes u \in B_{\mathcal{K}(X)}$. Since $\mathcal{K}(X)$ has the two-ball intersection property, there exists $U \in \mathcal{K}(X)$ such that

$$\max_\pm \|I \pm S - U\| < 1 + \delta/5.$$

Let $\psi = y \otimes u^* \in B_{\mathcal{L}(X)^*}$. Then

$$\begin{aligned}1 + \delta/5 &> \max_\pm |\psi(I \pm S - U)| \\ &= \max_\pm |y^*(y)u^*(u) \pm (u^* - U^*u^*)(y)| \\ &\ge 1 - \delta/5 + |(u^* - U^*u^*)(y)|\end{aligned}$$

so $|(u^* - U^*u^*)(y)| \le 2\delta/5$. Thus

$$(u^*(u)y^* \pm (u^* - U^*u^*))(y) \ge 1 - 3\delta/5 \ge (1-\delta)(1+\delta/5)$$

and

$$\frac{u^*(u)y^* \pm (u^* - U^*u^*)}{1 + \delta/5} \in S(y,\delta).$$

Hence

$$\begin{aligned}2\|u^*-U^*u^*\| &= \|(u^*(u)y^*+(u^*-U^*u^*))-(u^*(u)y^*-(u^*-U^*u^*))\|\\ &\le \varepsilon(1+\delta/5)<2\varepsilon\end{aligned}$$

and $\|u^*-U^*u^*\|<\varepsilon$. Since (x_α^*) is bounded weak* null, $\|U^*x_\alpha^*\|\to 0$ and $\|S^*x_\alpha^*\|\to 0$. We also have $\|S^*u^*-y^*\|=|1-u^*(u)|<\delta<\varepsilon$. Thus

$$\begin{aligned}\phi(y^*)-\varepsilon\le\phi(S^*u^*) &= \limsup_\alpha \|S^*(u^*+x_\alpha^*)+(I-U^*)x_\alpha^*\|\\ &\le \|S+I-U\|\phi(u^*)+\|u^*-U^*u^*\|\le(1+\varepsilon)\phi(u^*)+\varepsilon.\end{aligned}$$

Let $\varepsilon\to 0$, and the claim follows. ∎

3. The metric compact approximation property and extension operators. In this section we focus on the connection between the metric compact approximation property and the existence of certain norm one projections. The results are closely related to those of [19].

THEOREM 3.1. *Let X be a Banach space. Consider the following statements*:

(a) *X has the metric compact approximation property.*

(b) *For every Banach space Y, there exists a norm one projection P on $\mathcal{L}(Y,X)^*$ with $\ker P=\mathcal{K}(Y,X)^\perp$.*

(c) *There exists a norm one projection P on $\mathcal{L}(X)^*$ with $\ker P=\mathcal{K}(X)^\perp$.*

(d) *There exists a linear norm preserving extension operator $\Phi:\mathcal{K}(X)^*\to\operatorname{span}(\mathcal{K}(X),I)^*$.*

Then (a)⇒(b)⇒(c)⇒(d). *All statements above are equivalent if every functional on $\mathcal{K}(X)$ has a unique norm preserving extension to* $\operatorname{span}(\mathcal{K}(X), I)$. *Similar statements hold for the metric approximation property.*

Proof. (a)⇒(b) follows by a procedure due to J. Johnson [13]. Let (K_α) be a net in $B_{\mathcal{K}(X)}$ which converges to the identity in the strong operator topology. Using the weak* compactness of $B_{\mathcal{K}(X)^{**}}$, by passing to a subnet, we may assume that $\lim_\alpha\chi(K_\alpha)$ exists for all $\chi\in\mathcal{K}(X)^*$. Define the projection P by

$$P(\phi)(T)=\lim_\alpha\phi(K_\alpha T),$$

where $T\in\mathcal{L}(Y,X)$ and $\phi\in\mathcal{L}(Y,X)^*$. Note that $\psi:S\to\phi(ST)$ is in $\mathcal{K}(X)^*$. This P has the right properties.

(b)⇒(c) is obvious.

(c)⇒(d) is straightforward. For $\phi\in\mathcal{K}(X)^*$, define $\Phi(\phi)=P(\widehat{\phi})$ where $\widehat{\phi}$ is any bounded extension of ϕ to $\mathcal{L}(X)$.

(d)⇒(a). Let Φ be the extension operator in (d), and let $(x_n)\subseteq X$ and $(x_n^*)\subseteq X^*$ be such that $\sum_{n=1}^\infty\|x_n\|\|x_n^*\|<\infty$ and $|\sum_{n=1}^\infty x_n^*(Tx_n)|\le\|T\|$ for all $T\in\mathcal{K}(X)$. Define $\phi_n=x_n\otimes x_n^*\in\mathcal{K}(X)^*$ and $\phi=\sum_{n=1}^\infty\phi_n\in\mathcal{K}(X)^*$. Then $\|\phi\|\le 1$. By uniqueness of norm preserving extensions of functionals, $\Phi(\phi_n)=\widehat{\phi}_n=x_n\otimes x_n^*$ for all n. Since Φ is linear and has norm one, we get

$$\Phi(\phi)=\sum_{n=1}^\infty\Phi(\phi_n)=\sum_{n=1}^\infty\widehat{\phi}_n$$

and

$$\Big|\sum_{n=1}^\infty x_n^*(x_n)\Big|=\Big|\sum_{n=1}^\infty\widehat{\phi}_n(I)\Big|=|\Phi(\phi)(I)|\le\|\Phi(\phi)\|\le 1.$$

By Proposition 1.e.14 of [21], X has the metric compact approximation property. ∎

COROLLARY 3.2. *Assume $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$. Then X has the metric compact approximation property.*

The corollary follows from Theorem 3.1 using the L-projection on $\mathcal{L}(X)^*$. Note that every functional on an M-ideal has a unique norm preserving extension. (See Proposition I.1.12 of [11], or [18].)

Corollary 3.2 was first proved by Harmand and Lima [10]. Their argument used intersection properties of balls. Observe that if $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$, then by Proposition 4.1, $B_{X^*}=\overline{\operatorname{conv}}^{\|\ \|}(w^*\text{-str.exp.}\,B_{X^*})$. Let $u^*\in B_{X^*}$ be strongly exposed by $u\in B_X$ and define $U=u^*\otimes u\in\mathcal{K}(X)$. Now let

$$T\in\mathcal{K}(X)\cap B(0,1+\varepsilon)\cap B(I\pm U,1+\delta).$$

Then $\|T\|\le 1+\varepsilon$ (this gives the metric in the metric compact approximation property) and $\|U\pm(I-T)\|\le 1+\delta$. Evaluate at $\phi=u\otimes u^*$ to get $|1-T^*u^*(u)|\le 1+\delta$. If $\delta>0$ is properly chosen, we get $\|u^*-T^*u^*\|\le 1+\varepsilon$.

Using many weak* strongly exposed points, we can approximate I at all these points simultaneously. This shows that we can find a net (T_α) such that $T_\alpha^*x^*\to x^*$ in norm for all $x^*\in X^*$.

If we replace the ball $B(0,1+\varepsilon)$ above by $B\big(\frac12 I,\frac12+\varepsilon\big)$, then we see that $\limsup_\alpha\|I-2T_\alpha\|=1$.

Another approach uses the L-projection P on $\mathcal{L}(X)^*$ with $\ker P=\mathcal{K}(X)^\perp$. Then $P^*I\in B_{\mathcal{K}(X)^{**}}$. If $(S_\beta)\subset B_{\mathcal{K}(X)}$ and $S_\beta\to P^*I$ in the weak* topology, then these (S_β) have the same properties as (T_α) above.

Grothendieck has shown that reflexive and separable dual spaces with the approximation property have the metric approximation property [9, p. 181], [21, p. 39] or [6, p. 246]. In [6] it is shown that if a Banach space has the Radon–Nikodym property and is norm one complemented in its bidual, then it has the approximation property if and only if it has the metric approximation property. The next result is less general, but it shows the use of extension operators.

PROPOSITION 3.3. *Let X be an Asplund space. The following statements are equivalent:*

(a) *X^* has the metric approximation property.*
(b) *X^* has the bounded approximation property.*

Proof. We only have to prove (b)$\Rightarrow$(a).

Let (A_i) be a net of finite rank operators on X^* such that $\sup_i \|A_i\|<\infty$ and $\lim_i \|A_i x^* - x^*\| = 0$ for all $x^* \in X^*$. By using the "principle of local reflexivity", we may assume all A_i are conjugate operators, i.e. $A_i = S_i^*$ where $S_i \in \mathcal{F}(X)$. Taking a subnet if necessary, we may assume that $\lim_i \phi(S_i)$ exists for all $\phi \in \mathcal{F}(X)^*$. This follows from the compactness of $B_{\mathcal{F}(X)^{**}}$. Thus we can define an extension operator by the J. Johnson procedure:

$$\Phi(\phi)(T) = \lim_i \phi(S_i T).$$

If $\phi = x^{**} \otimes x^* \in X^{**} \otimes X^*$, then for $T \in \mathcal{L}(\mathcal{X})$,

$$\Phi(\phi)(T) = \lim_i T^{**}x^{**}(S_i^* x^*) = T^{**}x^{**}(x^*) = \phi(T).$$

Thus $\Phi(\phi) = \phi$ and $\|\Phi\| \le \sup_i \|A_i\| < \infty$.

Let $\phi \in \mathcal{F}(X)^*$ and let $\varepsilon > 0$. By a result of Feder and Saphar [7], since X is an Asplund space, there exist $(x_n^{**}) \subset X^{**}$ and $(x_n^*) \subset X^*$ such that $\phi = \sum_{n=1}^\infty x_n^{**} \otimes x_n^*$ and $\sum_{n=1}^\infty \|x_n^{**}\|\|x_n^*\| \le \|\phi\| + \varepsilon$. Since Φ is bounded and linear, we get

$$\Phi(\phi) = \sum_{n=1}^\infty \Phi(x_n^{**} \otimes x_n^*) = \sum_{n=1}^\infty x_n^{**} \otimes x_n^*.$$

Thus $\|\Phi(\phi)\| \le \|\phi\| + \varepsilon$. Hence $\|\Phi\| = 1$.

Assume next that $(x_n^*) \subset X^*$ and $(x_n^{**}) \subset X^{**}$ are such that

$$\sum_{n=1}^\infty \|x_n^*\|\|x_n^{**}\| < \infty \quad \text{and} \quad \Big|\sum_{n=1}^\infty x_n^{**}(Tx_n^*)\Big| \le \|T\|$$

for all $T \in \mathcal{F}(X^*)$. Define $\phi_n = x_n^{**} \otimes x_n^* \in \mathcal{F}(X)^*$ and $\phi = \sum_{n=1}^\infty \phi_n \in \mathcal{F}(X)^*$. Then $\|\phi\| \le 1$ since $|\sum_{n=1}^\infty x_n^{**}(T^* x_n^*)| \le \|T\|$ for all $T \in \mathcal{F}(X)$. By definition, $\Phi(\phi_n) = \phi_n$ for all n. Since Φ is linear and has norm one, we get

$$\Phi(\phi) = \sum_{n=1}^\infty \Phi(\phi_n) = \sum_{n=1}^\infty \phi_n$$

and

$$\Big|\sum_{n=1}^\infty x_n^{**}(x_n^*)\Big| = \Big|\sum_{n=1}^\infty \phi_n(I)\Big| = |\Phi(\phi)(I)| \le \|\Phi(\phi)\| \le 1.$$

By Proposition 1.e.14 of [21], X^* has the metric approximation property. ■

In Theorem 3.1, we can show that (d)$\Rightarrow$(a) when sufficiently many functionals have a unique norm preserving extension. This was used in [19]. Lemma 11 of [19] can be improved, as the next result shows. Lemma 3.4 below will be used in the proof of Theorem 4.3.

Lemma 11 of [19] is also extended by Theorem 3.7 of [20]. Note that by Proposition 3.3 of [20], in some cases the existence of denting points follows from unique extension of functionals.

LEMMA 3.4. *Let X and Y be Banach spaces, let $x \in X$ and $y^* \in Y^*$ and let $\phi = x \otimes y^* \in \mathcal{F}(X,Y)^*$.*

(a) *If y^* is a weak* denting point in B_{Y^*}, then ϕ has a unique norm preserving extension to $\mathcal{L}(X,Y)^*$.*

(b) *If x is a denting point in B_X, then ϕ has a unique norm preserving extension to $\mathcal{L}(X,Y)^*$.*

Proof. (a) We can assume $\|y^*\| = 1$ and $\|x\| = 1$. Let $\varepsilon > 0$ and let $T \in \mathcal{L}(X,Y)$. By Lemma 4.5 of D. Werner [24], we can choose $\delta \in (0,\varepsilon]$ sufficiently small and $y \in Y$ such that

$$y^*(y) = 1, \quad \|y\| \le 1 + \varepsilon\delta,$$
$$(z^*(y) > 1 - \delta \text{ and } \|z^*\| \le 1) \Rightarrow \|y^* - z^*\| \le \varepsilon.$$

Choose $x^* \in X^*$ such that $x^*(x) = 1$ and $\|x^*\| = 1 + \varepsilon\delta$. Define $S, U \in \mathcal{F}(X,Y)$ by $S = x^* \otimes y$ and $U = T^*y^* \otimes y$. Let $\widehat{\phi}$ be the natural extension of ϕ to $\mathcal{L}(X,Y)$ and let ψ be any norm preserving extension of ϕ to $\mathcal{L}(X,Y)$. We now show that $\widehat{\phi}(T) = \psi(T)$ so that ϕ has a unique extension to $\mathcal{L}(X,Y)$.

Note that $\|\phi\| \le 1$ and

$$\psi(U) = \phi(U) = y^*(y)(T^*y^*)(x) = \widehat{\phi}(T).$$

Since the unit ball of $\mathcal{L}(X,Y)^*$ is the weak*-closed convex hull of $B_X \otimes B_{Y^*}$, we can find m, $\lambda_i > 0$, $x_i \in B_X$ and $y_i^* \in B_{Y^*}$ such that $\sum_{i=1}^m \lambda_i = 1$ and

$$|\psi(U) - \eta(U)| < \varepsilon, \quad |\psi(T) - \eta(T)| < \varepsilon, \quad |\psi(S) - \eta(S)| < \delta^2,$$

where $\eta = \sum_{i=1}^m \lambda_i x_i \otimes y_i^*$. But then

$$\begin{aligned}|\widehat{\phi}(T) - \psi(T)| &= |\psi(U) - \psi(T)| \\ &\le |\psi(U) - \eta(U)| + |\psi(T) - \eta(T)| + |\eta(U) - \eta(T)| \\ &\le 2\varepsilon + |\eta(U) - \eta(T)|.\end{aligned}$$

We have

$$\eta(U) = \sum_{i=1}^m \lambda_i T^*y^*(x_i) y_i^*(y), \qquad \eta(T) = \sum_{i=1}^m \lambda_i T^*y_i^*(x_i),$$

and $\psi(S)=\phi(S)=y^*(y)x^*(x)=1$. Thus we get

$$1-\delta^2<\eta(S)=\sum_{i=1}^m \lambda_i y_i^*(y)x^*(x_i).$$

We can assume all $x^*(x_i)\ge 0$. Let $J=\{i: y_i^*(y)\le 1-\delta\}$. Then

$$1-\delta^2<\sum_{i\in J}\lambda_i(1-\delta)(1+\delta\varepsilon)+\sum_{i\notin J}\lambda_i(1+\delta\varepsilon)^2$$

so that $\sum_{i\in J}\lambda_i<\delta+2\varepsilon$. For $i\notin J$ we have $y_i^*(y)>1-\delta$. Hence $\|y^*-y_i^*\|\le\varepsilon$. Thus we get

$$\begin{aligned}|\eta(U)-\eta(T)|&\le\sum_{i=1}^m\lambda_i|y_i^*(y)T^*y^*(x_i)-T^*y_i^*(x_i)|\\&\le(2+\varepsilon\delta)\|T\|\sum_{i\in J}\lambda_i\\&\quad+\sum_{i\notin J}\lambda_i|(y_i^*(Tx_i)-y^*(Tx_i))-y^*(Tx_i)(y_i^*(y)-1)|\\&\le 3\|T\|(\delta+2\varepsilon)+\sum_{i\notin J}\lambda_i(\|T\|\varepsilon+\|T\|\delta)\le\|T\|(7\varepsilon+4\delta).\end{aligned}$$

Thus $\widehat{\phi}(T)=\psi(T)$ and ϕ has a unique norm preserving extension to $\mathcal{L}(X,Y)$.

(b) follows by the same reasoning. If x is a denting point in B_X, then we choose $x^*\in X^*$ such that

$$x^*(x)=1,\quad \|x^*\|\le 1+\varepsilon\delta,$$
$$(x^*(z)>1-\delta \text{ and } \|z\|\le 1)\Rightarrow\|x-z\|\le\varepsilon.$$

We may assume $\|y^*\|=1$. Choose $S=x^*\otimes y$, where $y\in Y$ with $y^*(y)=1$ and $\|y\|<1+\varepsilon\delta$, and choose $U=x^*\otimes Tx$. Then proceed as for (a). ∎

4. Property (wM^*) and the unconditional metric compact approximation property. Casazza and Kalton [3] introduced the notion of a u-ideal. If X is a subspace of Y, then we say that X is a *u-ideal* in Y if there is a projection Q on Y^* with $\ker Q=X^\perp$ and $\|I-2Q\|=1$.

An important step in the proof of Theorem 4.5 below is to show that $\mathcal{K}(X)$ is a u-ideal in $\mathcal{L}(X)$. We shall say that X has *property* (wM^*) if whenever (x_α^*) is a bounded net converging weak* to x^* in X^*, then

$$\limsup_\alpha\|x_\alpha^*\|=\limsup_\alpha\|2x^*-x_\alpha^*\|.$$

Clearly if X has property (M^*), then X has property (wM^*). In the proof of Theorem 4.5, we use property (wM^*). First we prove some preliminary results.

Let π be the projection on X^{***} with $\ker\pi=X^\perp$ and $\operatorname{im}\pi=X^*$. Following an argument by Kalton [15], we now show that if X has property (wM^*), then $\|I-2\pi\|=1$.

PROPOSITION 4.1. *We have* (a)⇒(b)⇒(c), *where*

(a) *X has property* (wM^*).
(b) $\|I-2\pi\|=1$.
(c) *X^* has the RNP and* $B_{X^*}=\overline{\mathrm{conv}}^{\|\ \|}(w^*\text{-str.exp.}\,B_{X^*})$.

Proof. (a)⇒(b). Let $y^*\in X^*$ and let $\widehat{y}\in X^\perp$. Then $(I-2\pi)(y^*+\widehat{y})=\widehat{y}-y^*$. Let $(x_\alpha^*)\subset X^*$ be such that $x_\alpha^*\to\widehat{y}$ weak* in X^{***} and $\|y^*+x_\alpha^*\|\le\|y^*+\widehat{y}\|$. Then (x_α^*) is weak* null in X^* and by (wM^*) and the weak* lower semicontinuity of the norm,

$$\begin{aligned}\|y^*+\widehat{y}\|&\ge\limsup_\alpha\|y^*+x_\alpha^*\|\\&=\limsup_\alpha\|2y^*-(y^*+x_\alpha^*)\|\ge\|y^*-\widehat{y}\|\end{aligned}$$

so $\|I-2\pi\|=1$.

(b)⇒(c). We can show that X^* has the RNP by an argument similar to that used in Lemma 2.6 of [20]. In fact, to prove that X^* has the RNP, we only need that $\|I-\lambda\pi\|\le a$ where $1\le a<\lambda\le 2$. From Theorem 5.12 of [23], it follows that $B_{X^*}=\overline{\mathrm{conv}}^{w^*}(w^*\text{-str.exp.}\,B_{X^*})$. We also have $B_{X^*}=\overline{\mathrm{conv}}^{\|\ \|}(\text{str.exp.}\,B_{X^*})$. Let $C=\overline{\mathrm{conv}}^{\|\ \|}(w^*\text{-str.exp.}\,B_{X^*})$. Let $x^*\in B_{X^*}$ be strongly exposed by $x^{**}\in B_{X^{**}}$, and let $(x_\alpha^*)\subset C$ be such that $x_\alpha^*\to x^*$ weak*. We may assume that $x_\alpha^*\to\widehat{x}\in X^{***}$ weak*. We are going to show that x^* has a unique norm preserving extension to X^{**}, from which we get $x^*=\widehat{x}$. Thus $1=\lim_\alpha x^{**}(x_\alpha^*)$, and $\|x^*-x_\alpha^*\|\to 0$. Hence $x^*\in C$ and $C=B_{X^*}$.

It remains to show that x^* has a unique norm preserving extension to X^{**}. Assume $\widehat{y}\in X^\perp$ and that $1=\|x^*+\widehat{y}\|$. Then since $\|I-2\pi\|=1$, we get $\|x^*\pm\widehat{y}\|=1$ and $x^{**}(\widehat{y})=0$. Let $(y_d^*)\subset X^*$ be such that $y_d^*\to\widehat{y}$ weak* and $\|x^*+y_d^*\|\le 1$. Then $x^{**}(y_d^*)\to 0$ and $1\ge x^{**}(x^*+y_d^*)\to 1$. Thus $\|y_d^*\|\to 0$, so $\widehat{y}=0$. ∎

THEOREM 4.2. *Consider the following statements*:

(a) *There is a net (T_α) of compact operators on X such that*

$$\|I-2T_\alpha\|\to 1,$$
$$\|x-T_\alpha x\|\to 0\quad \textit{for all } x\in X,$$
$$\|x^*-T_\alpha^*x^*\|\to 0\quad \textit{for all } x^*\in X^*.$$

(b) *There is a net (T_α) of compact operators on X such that*

$$\|I - 2T_\alpha\| \to 1,$$
$$x^{**}(T_\alpha^* x^*) \to 1 \quad \textit{if } \|x^{**}\| = \|x^*\| = 1 = x^{**}(x^*).$$

(c) *X has property (wM^*) and the metric compact approximation property.*

Then (a)⇔(b)⇒(c). *If X is separable or reflexive, then* (c)⇒(a).

Proof. (a)⇒(b) is clear.

To prove (b)⇒(a), we first have to show that X is an Asplund space. Assume that it is not. Then there exists a separable subspace Z of X with a nonseparable dual. Let $\varepsilon > 0$ and $c = 1 - \varepsilon^2/4$. There are (compare the proof of Lemma 2.6 in [20]) sequences (z_n) in Z_1 and (f_n) in Z_1^* such that

(1) $$f_m(z_n) \ge c \quad \text{for } m \ge n,$$
(2) $$|f_m(z_n)| \le \varepsilon \quad \text{for } m < n.$$

Denote still by f_m a norm preserving extension to X, and let f be a weak* limit point of (f_m) in X_1^*. Let z^{**} be a weak* limit point of (z_n) in X_1^{**}. Then

(1′) $$z^{**}(f) \ge c,$$
(2′) $$|z^{**}(f_m)| \le \varepsilon \quad \forall m.$$

Therefore
$$0 \le 1 - z^{**}(f) \le 1 - c = \varepsilon^2/4.$$

By the Bishop–Phelps–Bollobás theorem [2, p. 7] there exist $g \in X^*$ and $x^{**} \in X^{**}$, both of norm 1, satisfying $x^{**}(g) = 1$, $\|f - g\| \le \varepsilon$ and $\|z^{**} - x^{**}\| \le \varepsilon$. Using (b), pick $T = T_\alpha$ in the unit ball of $\mathcal{K}(X)$ such that $|1 - (x^{**} \otimes g)(T)| \le \varepsilon$ and $\|I - 2T\| \le 1 + \varepsilon$. Hence

$$\begin{aligned} z^{**}(T^* f) &= x^{**}(T^* g) + (z^{**} - x^{**})(T^* g) + (T^{**} z^{**})(f - g) \\ &\ge 1 - \varepsilon - \varepsilon - \varepsilon \;=\; 1 - 3\varepsilon. \end{aligned}$$

Since $T^{**}z^{**} \in X$ by compactness of T and since f is a weak* limit point of the f_m, we may assume that $|(f_m - f)(T^{**}z^{**})| \le \varepsilon$ for large m. Thus

$$1 - 3\varepsilon \le |z^{**}(T^* f)| \le |z^{**}(T^* f_m)| + |(f_m - f)(T^{**} z^{**})| \le \varepsilon + |z^{**}(T^* f_m)|,$$

so $1 - 4\varepsilon \le |(z^{**} \otimes f_m)(T)|$ for large m. But by (2′) and (b),

$$\begin{aligned} 1 + \varepsilon \ge \|I - 2T\| &\ge |(z^{**} \otimes f_m)(I - 2T)| = |z^{**}(f_m) - 2z^{**}(T^* f_m)| \\ &\ge 2|z^{**}(T^* f_m)| - |z^{**}(f_m)| \ge 2|z^{**}(T^* f_m)| - \varepsilon \ge 2 - 9\varepsilon. \end{aligned}$$

Choosing ε small enough gives a contradiction. Thus X is Asplund.

Now we can argue as in [10, Lemma 5.1].

(b)⇒(c). From the proof of (b)⇒(a) it follows that X is an Asplund space, and that we may assume $\|x - T_\alpha x\| \to 0$ for all $x \in X$ and $\|x^* - T_\alpha^* x^*\| \to 0$ for all $x^* \in X^*$.

Assume (x_d^*) is a bounded net in X^* converging weak* to x^*. Let $\varepsilon > 0$ and choose a compact operator T such that $\|I - 2T\| \le 1 + \varepsilon$ and $\|x^* - T^* x^*\| \le \varepsilon$. Since compact operators are weak*-to-norm continuous on bounded sets in X^*, we see that $T^* x_d^* \to T^* x^*$ in norm. Thus

$$(I - 2T^*)x_d^* = (x_d^* - 2x^*) + 2(x^* - T^* x^*) + 2(T^* x^* - T^* x_d^*)$$

so that

$$\limsup_d \|2x^* - x_d^*\| \le 2\varepsilon + \limsup_d \|(I - 2T^*)x_d^*\| \le 2\varepsilon + (1 + \varepsilon)\limsup_d \|x_d^*\|.$$

Hence $\limsup_d \|2x^* - x_d^*\| \le \limsup_d \|x_d^*\|$. Since

$$(I - 2T^*)(2x^* - x_d^*) = -x_d^* + 2(x^* - T^* x^*) + 2(T^* x_d^* - T^* x^*)$$

we get the converse inequality, and X has property (wM^*).

(c)⇒(a). We assume that X is separable. Let (K_n) be a sequence in $B_{\mathcal{K}(X)}$ such that $K_n x \to x$ for all $x \in X$. Assume $u^* \in B_{X^*}$ is strongly exposed by $u \in B_X$. Then $K_n^* u^*(u) \to u^*(u) = 1$, and hence $K_n^* u^* \to u^*$. Since X has property (wM^*), it follows from Proposition 4.1 that $B_{X^*} = \overline{\text{conv}}^{\|\,\|}(w^*\text{-str.exp.}\, B_{X^*})$. Hence (K_n) is shrinking, i.e. $K_n^* x^* \to x^*$ for all $x^* \in X^*$.

Next, we show that we may assume $\lim_n \|I - 2K_n\| = 1$. Let $\phi \in \mathcal{K}(X)^*$. We show that $\lim_n \phi(K_n)$ exists. Following an idea of K. John [12], we choose subsequences (K_{n_k}) and (K_{m_k}) such that

$$\lim_k \phi(K_{n_k}) = \limsup_n \phi(K_n) \quad \text{and} \quad \lim_k \phi(K_{m_k}) = \liminf_n \phi(K_n).$$

Let $T_k = K_{n_k} - K_{m_k} \in \mathcal{K}(X)$. Then

$$\lim_k x^{**}(T_k^* x^*) = 0$$

for all $x^* \in X^*$ and all $x^{**} \in X^{**}$. By a result of Kalton [14], $T_k \to 0$ weakly. Thus $\lim_n \phi(K_n)$ exists for all $\phi \in \mathcal{K}(X)^*$.

Let $\phi \in \overline{\text{ext}}^{w^*} B_{\text{span}(\mathcal{K}(X),I)^*}$. Since $\|T\| = \sup\{x^*(Tx) : \|x\| \le 1, \|x^*\| \le 1\}$ for all $T \in \mathcal{L}(X)$, from Milman's converse to the Krein–Milman theorem we deduce that there exist $\phi_\alpha = x_\alpha \otimes x_\alpha^* \in B_X \otimes B_{X^*}$ such that $\phi_\alpha \to \phi$ weak* in $\text{span}(\mathcal{K}(X), I)^*$. Clearly, we may assume that $x_\alpha^* \to x^*$ weak*.

Let $\varepsilon > 0$ and let N be such that $\|K_n^* x^* - x^*\| \le \varepsilon$ for all $n \ge N$. From property (wM^*), it follows that

$$1 = \lim_\alpha \|x_\alpha^*\| = \lim_\alpha \|x^* + (x_\alpha^* - x^*)\| = \lim_\alpha \|x_\alpha^* - 2x^*\|.$$

Thus for each $n \ge N$, since $K_n^* x_\alpha^* \to K_n^* x^*$ in norm,

$$|\phi(I-2K_n)|$$
$$= |\lim_\alpha (x_\alpha \otimes x^*_\alpha)(I-2K_n)|$$
$$= \lim_\alpha |(x^*_\alpha - 2x^*)(x_\alpha) + 2(x^* - K^*_n x^*)(x_\alpha) + 2(K^*_n x^* - K^*_n x^*_\alpha)(x_\alpha)|$$
$$\le \limsup_\alpha \|x^*_\alpha - 2x^*\| + 2\|x^* - K^*_n x^*\| \le 1 + 2\varepsilon.$$

Hence, $\lim_n |\phi(I-2K_n)| \le 1$.

Let $C = B_{\mathrm{span}(\mathcal{K}(X),I)^*}$ and let $\phi \in C$. By the Choquet integral representation theorem, there exists a regular Borel probability measure μ concentrated on $\operatorname{ext} C$ and representing ϕ. Thus by Lebesgue's bounded convergence theorem we get

$$\lim_n |\phi(I-2K_n)| = \lim_n \Big| \int (I-2K_n)\, d\mu \Big| = \Big| \int \lim_n (I-2K_n)\, d\mu \Big| \le 1.$$

From the Hahn–Banach theorem, it follows that

$$B(0,1+\delta) \cap \operatorname{conv}\{I-2K_n : n \ge 1\} \ne \emptyset$$

for all $\delta>0$. Hence we can find a sequence (S_n) with $S_n \in \operatorname{conv}(K_n, K_{n+1}, \ldots)$ such that $\lim_n \|I-2S_n\| = 1$.

Next, assume that X is reflexive. Let $(T_d) \subset B_{\mathcal{K}(X)}$ be a net such that $T_d x \to x$ for all $x \in X$. Since $B_{\mathcal{K}(X)^{**}}$ is compact, we may assume that $\lim_d \phi(T_d)$ exists for all $\phi \in \mathcal{K}(X)^*$. As above, $T^*_d x^* \to x^*$ for all $x^* \in X^*$. Moreover, the argument above shows that

$$\lim_d |\phi(I-2T_d)| \le 1 \quad \text{for all } \phi \in \overline{\operatorname{ext}}^{w^*} B_{\mathrm{span}(\mathcal{K}(X),I)^*}.$$

Since X^* has the RNP and contains no copy of l_1, we see [4] that $\mathcal{K}(X)$ contains no copy of l_1. Thus $\mathrm{span}(\mathcal{K}(X),I)$ contains no copy of l_1. By a result of Haydon [5, p. 215], $B_{\mathrm{span}(\mathcal{K}(X),I)^*}$ is the norm closed convex hull of its extreme points. Alternatively, we can use the fact that $X \hat{\otimes}_\pi X^* = \mathcal{K}(X)^*$ has the Radon–Nikodym property [6, p. 249], so that $\mathcal{K}(X)$ is an Asplund space. Thus

$$\lim_d |\phi(I-2T_d)| \le 1 \quad \text{for all } \phi \in B_{\mathrm{span}(\mathcal{K}(X),I)^*}.$$

Now use the Hahn–Banach theorem as above. ■

The next result is a generalization of Theorem 3.9 of [3] and Theorem 8.3 of [8].

THEOREM 4.3. *Consider the following statements*:

(a) *There is a net (T_α) of compact operators on X such that*

$$\|I-2T_\alpha\| \to 1, \quad \|x - T_\alpha x\| \to 0 \quad \textit{for all } x \in X.$$

(b) *$\mathcal{K}(X)$ is a u-ideal in $\mathcal{L}(X)$.*

(c) *$\mathcal{K}(X)$ is a u-ideal in* $\mathrm{span}(\mathcal{K}(X),I)$.

Then (a)$\Rightarrow$(b)$\Rightarrow$(c). *If X has the RNP or $B_{X^*} = \overline{\mathrm{conv}}^{\|\ \|}(w^*\text{-str.exp.}\, B_{X^*})$, then* (c)$\Rightarrow$(a).

Following [3], we shall say that X has the *unconditional metric compact approximation property* if (a) in Theorem 4.3 is satisfied.

Proof of Theorem 4.3. (a)$\Rightarrow$(b). Define the projection P on $\mathcal{L}(X)^*$ by the J. Johnson procedure

$$P(\phi)(T) = \lim_\alpha \phi(T_\alpha T).$$

(See [12] or [19].) This P satisfies $\ker P = \mathcal{K}(X)^\perp$ and $\|I-2P\| = 1$. See also the proof of Theorem 8.2 in [8].

(b)$\Rightarrow$(c). This follows from Proposition 3.6 of [8].

(c)$\Rightarrow$(a). Let P be the u-projection on $\mathrm{span}(\mathcal{K}(X),I)^*$. Define

$$\mathcal{T} : \mathrm{span}(\mathcal{K}(X),I) \to \mathcal{K}(X)^{**} \quad \text{by} \quad \mathcal{T}(T)(\phi_{|\mathcal{K}(X)}) = (P\phi)(T),$$

where $T \in \mathrm{span}(\mathcal{K}(X),I)$ and $\phi \in \mathrm{span}(\mathcal{K}(X),I)^*$. By Lemma 2.2 of [8], there is a net $(T_d) \subset \mathcal{K}(X)$ such that $\lim_d T_d = \mathcal{T}(I)$ in the weak* topology and $\limsup_d \|I-2T_d\| \le 1$. Let $x \in B_X$ and $x^* \in B_{X^*}$. Suppose x is strongly exposed or that x^* is weak* strongly exposed. Let $\phi = x \otimes x^* \in \mathrm{span}(\mathcal{K}(X),I)^*$. By Lemma 3.4, ϕ has a unique norm preserving extension from $\mathcal{K}(X)$. Thus $P(\phi) = \phi$ so that $T^*_d x^*(x) \to \mathcal{T}(I)(\phi) = \phi(I) = x^*(x)$. If x is strongly exposed, then $T_d x \to x$ and this holds for all $x \in X$ if X has the RNP. If x^* is weak* strongly exposed and $B_{X^*} = \overline{\mathrm{conv}}^{\|\ \|}(w^*\text{-str.exp.}\, B_{X^*})$, then as in the proof of (c)$\Rightarrow$(a) in Theorem 4.2, $T^*_d x^* \to x^*$ for this x^*, and hence for all $x^* \in X^*$. The convex combination argument that we used in (a)$\Rightarrow$(b) in Theorem 4.2 shows that we may assume $T_d x \to x$ for all $x \in X$. ■

From Theorems 4.2 and 4.3 we get the following result.

COROLLARY 4.4. *Let X be a reflexive Banach space. Then the following statements are equivalent*:

(a) *$\mathcal{K}(X)$ is a u-ideal in $\mathcal{L}(X)$.*

(b) *$\mathcal{K}(X)$ is a u-ideal in* $\mathrm{span}(\mathcal{K}(X),I)$.

(c) *X has property (wM^*) and the metric compact approximation property.*

(d) *There is a net (T_α) of compact operators on X such that*

$$\|I-2T_\alpha\| \to 1,$$
$$\|x - T_\alpha x\| \to 0 \quad \textit{for all } x \in X,$$
$$\|x^* - T^*_\alpha x^*\| \to 0 \quad \textit{for all } x^* \in X^*.$$

(e) *There is a net* (T_α) *of compact operators on* X *such that*

$$\|I - 2T_\alpha\| \to 1,$$

$$x^{**}(T_\alpha^* x^*) \to 1 \quad \text{if } \|x^{**}\| = \|x^*\| = 1 = x^{**}(x^*).$$

(f) *There is a net* (T_α) *of compact operators on* X *such that*

$$\|I - 2T_\alpha\| \to 1, \quad \|x - T_\alpha x\| \to 0 \quad \text{for all } x \in X.$$

Proof. (f)⇒(d) follows from the proof of Theorem 4.2. ■

We shall give a new proof of the following theorem of Kalton and Werner [16].

THEOREM 4.5. *Let* X *be a separable or reflexive Banach space. Then* $\mathcal{K}(X)$ *is an M-ideal in* $\mathcal{L}(X)$ *if and only if*

(a) X *has property* (M^*), *and*
(b) X *has the metric compact approximation property.*

That (a) and (b) are necessary was proved above. That they are also sufficient is proved in Theorem 3.7 of [16]. The proof we give here is shorter and simpler. We give the details only for separable spaces.

Proof. Let (K_n) be a sequence in $B_{\mathcal{K}(X)}$ such that $K_n x \to x$ for all $x \in X$. As shown in the proof of Theorem 4.2, $K_n^* x^* \to x^*$ in norm for all $x^* \in X^*$. Also, in the proof of (c)⇒(a) in Theorem 4.2 we showed that $\lim_n \phi(K_n)$ exists for all $\phi \in \mathcal{K}(X)^*$. Moreover, we can find a sequence $(S_n) \subset \operatorname{conv}(K_n)$ such that $\lim_n \|I - 2S_n\| = 1$.

We can define a projection P on $\mathcal{L}(X)^*$ by

$$P(\phi)(T) = \lim_n \phi(S_n T).$$

It is easily checked that $\|I - 2P\| = 1$, $\ker P = \mathcal{K}(X)^\perp$ and $\operatorname{im} P$ is isometric to $\mathcal{K}(X)^*$. It remains to show that P is an L-projection. This can be done as in Kalton [15] or as in [11, p. 299]. ■

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DEPARTMENT OF MATHEMATICS
AGDER COLLEGE
65, TORDENSKJOLDSGATE
N-4604 KRISTIANSAND, NORWAY
E-mail: AASVALDL@ADH.NO

Received May 30, 1994
Revised version September 21, 1994 (3283)