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Singularities and normal forms of generic 2-distributions on 3-manifolds

by

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Abstract. We give a complete classification of germs of generic 2-distributions on 3-manifolds. By a 2-distribution we mean either a module generated by two vector fields (at singular points its dimension decreases) or a Pfaff equation, i.e. a module generated by a differential 1-form (at singular points the dimension of its kernel increases).

1. Introduction. The aim of this paper is to give a description of generic singularities of 2-distributions on 3-manifolds. In particular, we give a complete classification of such singularities and a list of local normal forms.

By a *smooth 2-distribution* on a differentiable 3-manifold M we mean an object which is more general than the usually considered *field of planes* $\{\Delta_p \subset T_pM\}_{p\in M}$, $\dim \Delta_p = 2$, depending smoothly on p (a distribution of constant dimension 2). We allow the dimension to vary and, in generic cases considered further, the dimension can drop by one or increase by one outside a dense open subset of M. More precisely, our 2-distribution Q on M will be described by either of the following two objects:

(1) *a module* (X,Y) *of vector fields* (over the ring of smooth functions) generated by smooth vector fields X and Y,

(2) *a Pfaffian equation* $\omega = 0$, $\omega \in \Lambda^1(M)$, which will be represented by the module (ω) of differential 1-forms generated by a smooth 1-form ω.

The latter description means that we actually have in mind the field of kernels of the differential 1-form ω.

We shall use the term "2-distribution" in this general sense, i.e. by a 2-distribution we understand either (1) or (2).

A generic 2-distribution at a generic point is equivalent to the Pfaff–Darboux normal form. Such points are called nonsingular.

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First classification results on singularities of distributions were obtained by Jean Martinet [M] who identified singularities of codimension one. Further, singularities were studied in [JP] (using the language of vector fields), in [P], and in [Z1], [Z2] (using the language of Pfaffian equations). In all those papers the main emphasis was put on obtaining general results concerning most regular singularities in dimension n.

In this paper we are able to complete the study of generic singularities of 2-distributions in dimension 3 and give a complete classification and a list of normal forms corresponding to all typical singularities. In particular, we describe completely the most complicated generic singularity of 2-modules of vector fields (the normal form (3.6) in Theorem 3.3, and case (e) in Theorem 8.1). It was this singularity which was missing in the complete classification of generic 2-distributions on 3-manifolds.

Below we informally describe the contents of the paper. We assume all objects considered in this paper to be smooth (of class C^∞).

For a 2-distribution Q we denote by $Q(p)$ the subspace $\operatorname{span}(X(p), Y(p)) \subset T_pM$ or, if Q is a Pfaffian equation, the subspace $\operatorname{Ker}\omega|_p \subset T_pM$. The number $\dim Q(p)$ is called the dimension of Q at p. For a 2-generated module of vector fields it may be either 2, 1, or 0, for Pfaffian equations (ω) it is either 2 (if $\omega|_p \neq 0$) or 3 (if $\omega|_p = 0$).

Definition 1.1. Points at which the dimension of a 2-distribution is not 2 are called *degenerate points* of this distribution. The set of degenerate points is denoted by D.

Definition 1.2. A point $p \in M$ is called a *singular point* of a 2-distribution Q if either it is degenerate, or $\dim Q(p) = 2$ but the field of planes which Q defines in a neighbourhood of p is not a contact structure. The latter means that $\omega \wedge d\omega|_p = 0$, or, equivalently, $\dim\operatorname{span}(X(p), Y(p), [X,Y](p)) < 3$. The set of singular points is denoted by S.

The Darboux theorem says that the germs of 2-distributions at nonsingular points are all equivalent to the germ at zero

$$(dz + x\,dy) \tag{1.1}$$

or, equivalently, to the germ at zero

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right). \tag{1.1'}$$

The normal forms (1.1) and (1.1′) are called the standard contact structure on $\mathbb{R}^3$ or the Pfaff–Darboux normal form.

In Section 2 the natural local equivalence of fields of planes, 2-generated modules and Pfaffian equations is defined. We give complete lists of local normal forms of generic fields of planes, Pfaffian equations and modules of vector fields in Section 3 (Theorems 3.1–3.3). In this section results about stability and finite determinacy are also formulated (Theorems 3.4 and 3.5).

In Sections 4–5 we explain the word "generic" in the formulation of the theorems of Section 3. The results of Section 3 hold true for 2-distributions satisfying two genericity conditions. The first of them, condition (4.G), is given in Section 4. This condition guarantees that S is a smooth surface, the set D of degenerate points of the 2-module is a smooth curve in S, and a few more geometric facts (Propositions 4.2–4.4). One of them is the so-called "typical nongenericity" (this term is taken from [JP]): $Q(p) \subset T_pS$ for any degenerate point of the 2-module Q.

The second genericity condition, condition (5.G), is given in Section 5. In order to state it we introduce a vector field Z on the surface S which is invariantly assigned to any 2-distribution, up to multiplication by a nonvanishing function. Condition (5.G) concerns the linear approximation of Z at the singular points of Z. The vector field Z is connected with the following natural distribution on S: $d = \{d_p = Q(p) \cap T_pS\}_{p\in S}$. Namely, d is a smooth field of lines near nonsingular points of Z (and in this case d_p is generated by $Z(p)$), and d is singular at the points where Z is.

Singular points of Z are called irregular, and nonsingular points of Z are called regular points of the 2-distribution (Section 6). Nondegenerate points might be both regular and irregular, the same is true for degenerate points of 2-modules; all degenerate points of Pfaffian equations are irregular. We also give another, equivalent definition of regular and irregular points in geometric terms (Proposition 6.1). Under genericity condition (5.G) irregular points are isolated (in particular, degenerate points of a Pfaffian equation are isolated).

In Section 7 we define types of irregular points in terms of the eigenvalues of the linear approximation of the vector field Z. Nondegenerate irregular points are divided into two types (hyperbolic and elliptic points), and degenerate irregular points are divided into three types (node, saddle and focus). The topological behaviour of the distribution d defined above, near its singular point, is determined by the type of this point. As already noticed in [JP], near a hyperbolic irregular point, d is topologically a saddle; near an elliptic irregular point it is a focus (the latter fact is nontrivial). The geometry of singularities is "richest" near irregular degenerate points of the 2-module Q (which are isolated points of the curve D) of node or saddle type. If p is such a point then there are four invariant directions in T_pS: $Q(p)$, T_pD and the eigenspaces of the linear approximation of the vector field Z. All these directions are different. A numerical module in the classification of 4-tuples of straight lines in $\mathbb{R}^2$ corresponds to the parameter in the normal form (3.6).

In Section 8 we give the main classification result of this paper (Theorem 8.1); its formulation contains all typical singularity classes and the correspondence between a singularity class and a normal form. Theorem 8.1 and the results of Sections 4–7 imply the results of Section 3. The normal forms (3.1)–(3.3) and (3.5) were obtained in [M] ((3.1)), [JP] ((3.5)), and [Z1, Z2] ((3.3), (3.4)), but the genericity conditions in these articles were different. Ours are more effective (they are more explicit and formulated in the same terms for both Pfaffian equations and 2-modules of vector fields for all possible degenerations). We show that the genericity conditions (4.G) and (5.G) imply those of [M], [JP] and [Z2]. After proving this fact the reducibility to the normal forms (3.1)–(3.3) and (3.5) follows from the results of these papers.

Local normal forms near degenerate points of Pfaffian equations were obtained in [L] for the even-dimensional case (in this case degenerate points correspond to singularities of first order partial differential equations), and in [Z1, Z2] for the odd-dimensional (in particular, 3-dimensional) case. The normal form (3.4) given in this paper is similar to those obtained in [Z1, Z2], but the advantage of the normalization (3.4) is that it is a 1-parameter family of germs (in [Z1, Z2] a germ reduces to one of two 1-parameter families), and any two different real values of the parameter correspond to nonequivalent germs. Nevertheless, the reduction to (3.4) is proved in almost the same manner as in [Z2].

The principally new result of this paper is the normal form (3.6) for germs of 2-modules at irregular degenerate points (the most difficult case). Though this degeneration was also considered in [JP], the authors of [JP] only managed to normalize the 2-jet of a germ (under more complicated genericity conditions). We prove (Section 9) that the normalization (3.6) holds using the homotopy method. This reduces the proof to solvability of a singular system of partial differential equations. We succeed in reducing this system to a single equation and prove its solvability in formal series. We use the results of [B] (on the relation between formal and smooth solvability of certain partial differential equations) in order to complete the proof.

2. Equivalence. We consider the following natural local equivalence of fields of planes, 2-generated modules of vector fields and Pfaffian equations defined on a manifold M.

We call two germs at $p \in M$ of modules (X, Y) and $(\widetilde{X}, \widetilde{Y})$ *equivalent* if there exists a germ of a diffeomorphism $\Psi : (M,p) \to (M,p)$ and a germ at p of a 2×2 matrix $H = H(x) = \{h_{ij}(x)\}$, $x \in M$, $H(p)$ nonsingular, such that

$$\Psi_* X = h_{11}\widetilde{X} + h_{12}\widetilde{Y}, \qquad \Psi_* Y = h_{21}\widetilde{X} + h_{22}\widetilde{Y}.$$

Using the fact that the ring of germs of smooth functions is local it can be easily proved that this definition is independent of the choice of the generators of the modules (cf. also [JP], Appendix). Similarly, two germs at $p \in M$ of Pfaffian equations (ω), $(\widetilde{\omega})$ are said to be *equivalent* if there exists a germ of a diffeomorphism $\Psi : (M,p) \to (M,p)$ and a function f, $f(p) \neq 0$, such that

$$\Psi^*\omega = f\widetilde{\omega}.$$

Finally, two germs at $p \in M$ of fields of planes Δ and $\widetilde{\Delta}$ are *equivalent* if the corresponding germs of modules (or of Pfaffian equations) are equivalent. Equivalence at different points $p, q \in M$ can be defined analogously.

R e m a r k. A field of planes can be represented, at least locally, by either a 2-generated module of vector fields or a Pfaffian equation. On the other hand, a 2-distribution Q defines a germ at $p \in M$ of a field of planes if and only if $\dim Q(p) = 2$, for all p. In this case the passage from any of the two descriptions to a field of planes is given by $Q \to \{Q(p)\}_{p \in M}$, and the correspondence between a 2-generated module V of vector fields and a Pfaffian equation $P = (\omega)$ is established by the duality between 1-forms and vector fields, namely: $V \to P = V^\perp = \{\omega \mid \omega(Z) \equiv 0,\ Z \in V\}$, and $P \to V = P^\perp = \{Z \mid \omega(Z) \equiv 0\}$. If the distribution is not of constant dimension, then the above correspondence $V \leftrightarrow P$ still makes sense except that the number of generators may be different from two and one, respectively. For example, if $P = (\omega)$ and $\omega = x\,dx + y\,dy + z\,dz$, then the corresponding V is 3-generated, $V = P^\perp = (x\partial/\partial y - y\partial/\partial x, y\partial/\partial z - z\partial/\partial y, z\partial/\partial x - x\partial/\partial z)$.

In the space of all 2-generated modules of vector fields on M and in the space of all Pfaffian equations on M we can introduce the Whitney topology (see e.g. [AVG] or [GG]) using the generators of the corresponding modules. Then we can use usual notions and results of singularity theory, in particular the Thom transversality theorem. In the formulations of our results we will say that a property holds for *generic* objects if it holds for objects in a sufficiently large set, namely, a countable intersection of sets open and dense in the Whitney topology.

3. Normal forms, stability, and finite determinacy. We begin formulating our results with listing normal forms for generic fields of planes, modules of vector fields, and Pfaffian equations. This means that the classification results of this section hold for 2-distributions which form a countable intersection of open dense sets. This set of distributions is described in Sections 4–5.

THEOREM 3.1. *The germ of a generic field of planes on M at any point $p \in M$ is equivalent to one and only one of the germs* (1.1), (3.1), (3.2), *and* (3.3) (*or, equivalently, the germs* (1.1′), (3.1′), (3.2′), (3.3′)), *where*

$$(dz + x^2dy), \tag{3.1}$$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x^2\frac{\partial}{\partial z}\right), \tag{3.1′}$$

$$(dz + (xz + x^2y + bx^3y^2)dy), \tag{3.2}$$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - (xz + x^2y + bx^3y^2)\frac{\partial}{\partial z}\right), \tag{3.2′}$$

$$(dz + (xz + x^3/3 + xy^2 + bx^3y^2)dy), \tag{3.3}$$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - (xz + x^3/3 + xy^2 + bx^3y^2)\frac{\partial}{\partial z}\right). \tag{3.3′}$$

Here x, y, z are coordinates on $\mathbb{R}^3$, the germs are taken at $0 \in \mathbb{R}^3$, and $b \in \mathbb{R}$ is a parameter (an invariant distinguishing nonequivalent germs).

THEOREM 3.2. *The germ of a generic Pfaffian equation (ω) on M at any point $p \in M$ is equivalent to one and only one of the germs* (1.1), (3.1), (3.2), (3.3) (*if $\omega|_p \neq 0$*) *or* (3.4) (*if $\omega|_p = 0$*), *where the germ* (3.4) *is defined by*

$$(zdz - \theta xdx + (\theta x + y)dy). \tag{3.4}$$

Here, as before, we consider the germs at $0 \in \mathbb{R}^3$. The real parameter θ is an invariant distinguishing nonequivalent germs.

THEOREM 3.3. *The germ of a generic module (X, Y) on M at any point $p \in M$ is equivalent to one and only one of the germs* (1.1′), (3.1′), (3.2′), (3.3′) (*if* $\dim\operatorname{span}(X(p), Y(p)) = 2$), *and* (3.5), (3.6) (*if* $\dim\operatorname{span}(X(p), Y(p)) < 2$), *where we define* (3.5) *and* (3.6) *by*

$$\left(\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right), \tag{3.5}$$

$$\left(\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} + (z + \lambda x^2 + y^2)\frac{\partial}{\partial z}\right). \tag{3.6}$$

The real parameter λ is an invariant distinguishing nonequivalent germs.

We postpone the explanation of the genericity conditions, the geometry of the above singularities and the characterization of the equivalence classes of (3.1)–(3.6) to Sections 4–8. The main result describing the geometry of the above normal forms is Theorem 8.1. Here we only mention that for a generic module or Pfaffian equation the normal forms (1.1) and (1.1′) hold on an open set in M (which is the complement of the smooth surface S), the normal forms (3.1) and (3.1′) hold at generic points of the surface S, the normal form (3.5) holds at generic points of a smooth curve $D \subset S$, and the other normal forms hold at isolated points (see Fig. 1).

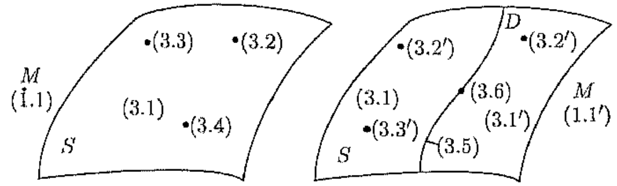


Fig. 1. The set of points at which a generic 2-distribution is equivalent to one of the normal forms (1.1) and (3.1)–(3.6)

To describe structural stability and finite determinacy of the singularities (3.1)–(3.6) we recall the usual definitions. Let Q be a 2-distribution on M, and let $p \in M$. The germ of Q at p is called *stable* if for any close element $\widetilde{Q}$ there exists a point $\widetilde{p}$ close to p such that the germ of $\widetilde{Q}$ at $\widetilde{p}$ is equivalent to the germ of Q at p.

A germ μ at p is *k-determined* if any germ $\widetilde{\mu}$ at p satisfying $j^k_p\mu = j^k_p\widetilde{\mu}$ is equivalent to μ. Here j^k_p denotes the kth jet at p.

THEOREM 3.4. *Let Q be a generic 2-distribution on a 3-manifold M. There exists a set $\Sigma \subset M$ consisting of isolated points such that the germ of Q at any point $p \in M \backslash \Sigma$ is stable and equivalent to one and only one of the germs* (1.1), (3.1), (3.5). *The other germs* (3.2), (3.3), (3.4) *and* (3.6) *are not stable.*

THEOREM 3.5. *Let Q be a generic 2-distribution on a 3-manifold M. Then the germ of Q at any point $p \in M$ is 5-determined. Moreover, the germs* (1.1), (3.4) *and* (3.5) *are 1-determined,* (3.1) *and* (3.6) *are 2-determined, and* (3.2), (3.3) *are 5-determined.*

Stability and finite determinacy of differential 1-forms (not Pfaffian equations) are more restrictive properties and hold for Darboux and Martinet normal forms only (see [GT] and [Z3]).

In the following four sections we describe the geometry of our singularities, including the genericity conditions, and we establish preparatory facts needed for proving the main classification result (Theorem 8.1). The theorems stated in this section are consequences of this result.

4. The first genericity condition. Basic geometry of singularities. To explain the word "generic" in the formulations of Theorems 3.1–3.5 we have to formulate genericity conditions under which the classification results of these theorems hold true. In this section we give the first genericity condition (4.G) (the other is given in Section 5). We also describe the basic geometry of singularities (Propositions 4.2–4.4) of a 2-distribution satisfying (4.G) and we present preliminary normal forms.

To formulate the first genericity condition we take a nondegenerate volume form Ω on M (local if M is not orientable) and any generators X, Y or ω of a distribution Q (here, as in the previous sections, Q is either a 2-generated module (X, Y) of vector fields, or a Pfaffian equation (ω)). We introduce a function H on M by

$$H = \Omega(X, Y, [X, Y]) \quad \text{if } Q = (X, Y), \tag{4.1}$$

$$H\Omega = \omega \wedge d\omega \quad \text{if } Q = (\omega). \tag{4.2}$$

Notation. The set of all singular points of Q is denoted by S or $S(Q)$, and the set of all degenerate points by D or $D(Q)$.

Clearly, S is the 0-level set of H:

$$S = \{p \in M \mid H(p) = 0\}. \tag{4.3}$$

The first genericity condition says that the 1-jet of H is nonzero at any point of M:

$$j_p^1 H \neq 0, \quad p \in M \quad (\text{equivalently, } H(p) = 0 \Rightarrow dH|_p \neq 0,\ p \in M). \tag{4.G}$$

Note that the function H depends on the choice of Ω and the generators, but the condition (4.G) is invariant (changing the volume form or the generators leads to multiplication of H by a nonvanishing function).

PROPOSITION 4.1. *In the set of all 2-distributions the subset of those distributions satisfying* (4.G) *is open and dense in the Whitney topology.*

Proof. Condition (4.G) can be checked in terms of the 1-jet of a 2-distribution at p. The 1-jets violating (4.G) form a codimension 4 submanifold in the space of all 1-jets, and the proposition follows from the Thom transversality theorem. ∎

The following three theorems give basic information about the local geometry of singularities of a 2-distribution satisfying (4.G).

PROPOSITION 4.2. *Let Q be a 2-distribution satisfying* (4.G). *Then S is a smooth surface* (*unless $S = \emptyset$*). *If Q is a 2-module then the set D of degenerate points of Q is a smooth curve in S* (*unless $D = \emptyset$*); *the dimension of $Q(p)$ is equal to 1 at any point $p \in D$.*

Proof. The set S is a smooth surface by (4.G) and (4.3). In the case where Q is a 2-module (X, Y) and $X(p) = Y(p) = 0$ the 1-jet of H (defined by (4.1)) at p is zero. Therefore $\dim Q(p) = 1$ at any degenerate point p, and the germ of Q at $p \in D$ is reducible to a normal form

$$X = \partial/\partial x, \quad Y = A\partial/\partial y + B\partial/\partial z, \tag{4.4}$$

where A and B are some function germs vanishing at p. Condition (4.G) implies

$$j_p^1\left(A\frac{\partial B}{\partial x} - B\frac{\partial A}{\partial x}\right) \neq 0.$$

Since $A(p) = B(p) = 0$ this means that either

$$\frac{\partial A}{\partial y}(p)\frac{\partial B}{\partial x}(p) - \frac{\partial B}{\partial y}(p)\frac{\partial A}{\partial x}(p) \neq 0$$

or

$$\frac{\partial A}{\partial z}(p)\frac{\partial B}{\partial x}(p) - \frac{\partial B}{\partial z}(p)\frac{\partial A}{\partial x}(p) \neq 0,$$

therefore $D = \{(x, y, z) \mid A = B = 0\}$ is a smooth curve. ∎

PROPOSITION 4.3. *Let $Q = (X, Y)$ be a 2-module of vector fields satisfying* (4.G), *and let p be a degenerate point of Q. Then*

(a) $Q(p) \subset T_pS$;

(b) $Q(p) + T_pD = T_pS$;

(c) $\dim \operatorname{span}(X(p), Y(p), [X, Y](p)) = 2$;

(d) *the germ of Q at p is reducible* (*equivalent*) *to a normal form at* $p = 0$,

$$(X, Y) = (\partial/\partial x,\ x\partial/\partial y + C\partial/\partial z), \qquad C(p) = \frac{\partial C}{\partial x}(p) = 0. \tag{4.5}$$

Proof. (a) As in the proof of Proposition 4.2 we can reduce the germ at p to the normal form (4.4). The surface S is given by the equation $H = A\frac{\partial B}{\partial x} - B\frac{\partial A}{\partial x} = 0$. Since p is a degenerate point of Q, it follows that $A(p) = B(p) = 0$, $Q(p) = (\partial/\partial x)$, $\frac{\partial H}{\partial x}(p) = 0$, and so we get (a).

(c) This statement is a direct consequence of (4.G).

(d) We can use the normal form (4.4), where $A(p) = B(p) = 0$. By (c), either $\frac{\partial A}{\partial x} \neq 0$ or $\frac{\partial B}{\partial x} \neq 0$. We assume, without loss of generality, that $\frac{\partial A}{\partial x} \neq 0$. Introducing a new coordinate $\widetilde{x} = A$ and multiplying the first generator of Q by a nonzero function we obtain the normal form (4.5) with some function germ C vanishing at p. The condition $\frac{\partial C}{\partial x}(p) = 0$ can be met on replacing z by $\widetilde{z} = z - \frac{\partial C}{\partial x}(p)y$.

(b) We use the normal form (4.5). The surface S and the curve D are given by the equations

$$S = \left\{(x, y, z) \,\middle|\, C - x\frac{\partial C}{\partial x} = 0\right\}, \tag{4.6}$$

$$D = \{(x, y, z) \in S \mid x = 0\}, \tag{4.7}$$

and (b) follows. ∎

PROPOSITION 4.4. *Let (ω) be a Pfaffian equation satisfying* (4.G), *and let p be a degenerate point. Then the space $\operatorname{Ker} d\omega|_p \subset T_pM$ is invariant* *(does not depend on the generator ω) and 1-dimensional (i.e. $d\omega|_p \neq 0$); the germ of (ω) at p is equivalent to the germ at $0 \in \mathbb{R}^3$ of the form*

$$(\omega) = (x\,dy + dR(x,y,z)), \qquad j_0^1 R = 0. \tag{4.8}$$

Proof. The invariance of $\operatorname{Ker} d\omega|_p$ follows from the relation $d(f\omega)|_p = f(p)d\omega|_p$ which is true for every point at which ω vanishes. The nondegeneracy of $d\omega$ follows from (4.G): if $\omega|_p = 0$ and $d\omega|_p = 0$ then $j_p^1(\omega \wedge d\omega) = 0$. By the Darboux theorem for closed 2-forms the germ at p of the 2-form $d\omega$ is equivalent to $dx \wedge dy$, therefore the germ of ω at p is equivalent to the normal form (4.8). ∎

5. Dynamical systems on the surface of singular points. The second genericity condition. In this section we introduce a 1-distribution $d = \{d_p\}_{p\in S}$ (with singularities) on the surface S of singular points, which is canonically defined by any 2-distribution Q on M satisfying the first genericity condition. There exists a smooth vector field Z on S such that $\operatorname{span}(Z)|_p \subset d_p$ at every $p \in S$ and equality holds except at isolated points. The module of vector fields generated by Z is also invariantly assigned to Q. We shall formulate the second genericity condition in terms of Z.

We define a distribution on S as the intersection of Q with S:

$$d = \{d_p = Q(p) \cap T_pS\}_{p\in S}.$$

This distribution is smooth at nondegenerate singular points $p \in S$ such that the 2-dimensional space $Q(p)$ is transversal to T_pS in T_pM (then, near p, d is a nonsingular field of lines); it is singular at the other nondegenerate points (the dimension of d_p increases from 1 to 2 if $Q(p)$ and T_pS coincide). The dimension of d_p also increases from 1 to 2 when passing from a generic point of a Pfaffian equation to a degenerate one. On the other hand, it follows from Proposition 4.3(a) that if Q is a 2-module of vector fields, then the dimension of d_p at a degenerate point p is equal to 1 (i.e. it is the same as at a generic point). We will show in this section that for generic 2-modules of vector fields the distribution d is smooth near a generic degenerate point.

There exists another, dynamical invariant which is "stronger" than the distribution d. It turns out that there exists a smooth vector field Z on S which defines the distribution d at every point where d is of dimension 1 and which is singular at the other points.

To construct Z for the case $Q = (\omega)$ we take a volume form Ω_S on S (the construction is local if S is not orientable), and define Z by

$$Z \lrcorner \Omega_S = \omega|_S. \tag{5.1}$$

The construction of Z for the case where $Q = (X,Y)$ can be done in two steps. First, we pass from (X,Y) to a differential 1-form ω using a volume form Ω on M: $\omega = \Omega(X,Y,\cdot)$, i.e.

$$\omega(Z') = \Omega(X,Y,Z')$$

(for any vector field Z' on M), and define a vector field Z_1 on S by

$$Z_1 \lrcorner \Omega_S = \omega|_S. \tag{5.2}$$

Note that Z_1 vanishes at every degenerate point of Q (since so does ω). Using the fact that the set D of all degenerate points of Q is a smooth curve (see Proposition 4.2) we take a function f on S defining D, i.e. $D = \{f = 0\}$, and such that $df|_p \neq 0$ for any $p \in D$, and obtain the vector field Z on dividing Z_1 by f: $fZ = Z_1$. It is clear that Z is a smooth vector field.

Thus, we have defined a smooth vector field Z on S both for Pfaffian equations and for 2-modules of vector fields. Note that Z is not an invariant of Q and it is defined up to multiplication by a nonzero function. However, the 1-generated module (Z) is invariant, i.e. (Z) does not depend on the choice of the volume forms Ω and Ω_S and of the choice of the generator(s) of Q. The same is true for the vector field Z_1 and the module (Z_1).

In the following section we will use the vector field Z in order to define singularity types of distributions. In fact, our definitions will depend on the module (Z) only, and so they will be invariantly given.

PROPOSITION 5.1. *Under the genericity condition* (4.G) *the vector field Z defines the distribution d, i.e.* $\operatorname{span}(Z(p)) = d_p$ *at every $p \in S$ such that $Z(p) \neq 0$.*

Proof. In the case of a Pfaffian equation (ω) we have $\omega|_S(Z) = 0$, therefore $\operatorname{span}(Z(p)) \subset \operatorname{Ker}\omega|_p$. If p is a nonsingular point of Z then p is a nondegenerate point of (ω), and $\operatorname{Ker}\omega|_p$ is transversal to T_pS, i.e. $\dim d_p = 1$ and the theorem follows.

In the case of 2-modules (X,Y) and a nondegenerate point p the kernel at p of the 1-form $\omega = \Omega(X,Y,\cdot)$ coincides with $\operatorname{span}(X(p),Y(p))$, and $(Z(p)) = (Z_1(p))$, i.e. the proof reduces to the case of Pfaffian equations.

It remains to prove the proposition for a 2-module (X,Y) and a degenerate point p. In this case we use the normal form (4.5). Let $j_p^1C = ay + bz$. The surface S is defined by the equation $H = C - x\frac{\partial C}{\partial x} = 0$. This equation can be written in the form $ay + bz + \varphi(x,y,z) = 0$, where $j_p^1\varphi = 0$. Taking $\Omega = dx \wedge dy \wedge dz$, we get $\omega = \Omega(X,Y,\cdot) = x\,dz - C\,dy$. If $a \neq 0$ then (x,z) is a local coordinate system on S near p, and $j_p^1(\omega|_S) = x\,dz$. We take $\Omega|_S = dx \wedge dz$. Then $j_p^1 Z_1 = x\partial/\partial x$. The curve D is $\{x = 0\}$, and so $Z(p) = \partial/\partial x$. Therefore $Z(p) \in d_p$, and the conclusion follows from Propositions 4.2 and 4.3.

In the case where $a = 0$ the surface S is given by the equation $z = \psi(x,y)$, where $j_p^1\psi = 0$. Choosing (x,y) as local coordinates on S and taking into account that $C(p) = 0 = \frac{\partial C}{\partial x}$ we get in this case $j^1_p(\omega|_S) = 0$, and consequently $Z(p) = 0$. ∎

Proposition 5.1 shows that the distribution d, corresponding to 2-modules, remains smooth near those degenerate points at which the field Z does not vanish.

As a byproduct of the above proof we obtain the following lemma.

LEMMA 5.1. *Let $Q = (X, Y)$ be a module satisfying* (4.G), *and let p be a degenerate point of Q. Assume that X and Y have the normal form* (4.5) *in coordinates (x, y, z) near p. Then $Z(p) = 0$ if and only if $\frac{\partial C}{\partial y}(p) = 0$.*

Now we will formulate the second genericity condition for a 2-distribution Q:

(5.G) Let $p \in S$ be a singular point of the vector field Z. Then the linear approximation of Z at p is nondegenerate. If p is a degenerate point of Q then, additionally, the linear approximation is nonresonant.

By a *nonresonant* linear operator we mean here a linear operator whose eigenvalues λ_1, λ_2 have the following property:

$$m_1\lambda_1 + m_2\lambda_2 \neq 0 \quad \text{for any integers } m_1 \geq -1,\ m_2 \geq -1.$$

The following statement shows that (5.G) is a genericity condition.

PROPOSITION 5.2. *In the set of all Pfaffian equations (respectively, 2-modules of vector fields) the subset of those satisfying conditions* (4.G) *and* (5.G) *is a countable intersection of sets which are open and dense in the Whitney topology. Additionally,* (5.G) *is a condition on the 3-jet at p if p is a nondegenerate singular point, it concerns the 1-jet at p if p is a degenerate singular point of a Pfaffian equation, and it concerns the 2-jet at p if p is a degenerate point of a 2-module.*

Proof. Let us introduce the following singularity classes of germs at $0 \in \mathbb{R}^3$ of 2-distributions satisfying (4.G):

- the singularity class P of germs (ω) such that 0 is a singular point, $\omega|_0 \neq 0$ and $Z(0) = 0$;
- the singularity class W of germs (ω) such that $\omega|_0 = 0$ (and consequently $Z(0) = 0$);
- the singularity class E of germs (X, Y) such that the origin is a degenerate point and $Z(0) = 0$.

Denote by P^* the subclass of P consisting of germs for which the linear approximation of Z at $0 \in \mathbb{R}^3$ is degenerate. Let W_r and E_r be the subclasses of W and E respectively consisting of germs for which either the linear approximation of Z at $0 \in \mathbb{R}^3$ is degenerate or the ratio of the eigenvalues of Z at the origin is equal to r (r is a fixed real number).

Proposition 5.2 is a corollary of the transversality theorem and the following statement.

PROPOSITION 5.3. *The codimension of each of the singularity classes P, W and E is equal to* 3 *(in the space of all germs); the codimension of each of the singularity classes P^*, W_r and E_r in P, W and E, respectively, is* 1. *The condition $(\omega) \in P$ (resp. $(\omega) \in P^*$) concerns the 2-jet (resp. the 3-jet) of ω at* 0. *The condition $(\omega) \in W$ (resp. $(\omega) \in W_r$) concerns the 0-jet (resp. the 1-jet) of ω at* 0. *The condition $(X, Y) \in E$ (resp. $(X, Y) \in E_r$) concerns the 1-jets (resp. the 2-jets) of X and Y at* 0.

Proof. First consider the singularity class P. Any germ at a nondegenerate point of a Pfaffian equation is equivalent to a germ of the form

$$(\omega) = (dy + C(x, y, z)dz), \qquad C(0) = 0 \tag{5.3}$$

(this follows e.g. from the fact that its kernel contains a nonzero vector field which can be taken in the form $\partial/\partial x$). Then the condition "0 is a singular point" means that $\frac{\partial C}{\partial x}(0) = 0$ and the singular surface S is given by the equation $\frac{\partial C}{\partial x} = 0$. The condition $Z(0) = 0$ is equivalent to $\operatorname{Ker}\omega|_p \subset T_pS$. For the germ (5.3) this means that

$$\frac{\partial^2 C}{\partial x^2}(0) = \frac{\partial^2 C}{\partial x \partial z}(0) = 0. \tag{5.4}$$

Therefore the singularity class P is distinguished by a condition on the 2-jet of a germ, and it has codimension 3 in the space of all germs.

Under the degeneration (5.4) condition (4.G) for $p = 0 \in \mathbb{R}^3$ implies

$$\frac{\partial^2 C}{\partial x \partial y}(0) \neq 0. \tag{5.5}$$

It is shown in [Z2, p. 69] that the 3-jet of the germ (5.3) satisfying (5.4) and (5.5) is reducible to a normal form

$$(\omega) = (dy + (xy + f(x, z))dz),$$

where f is a homogeneous polynomial of degree 3, and the matrix of the linear approximation at the origin of the vector field Z has the form

$$\begin{pmatrix} -\frac{\partial^3 f}{\partial x^2 \partial z} & -\frac{\partial^3 f}{\partial x \partial z^2} \\ \frac{\partial^3 f}{\partial x^3} & \frac{\partial^3 f}{\partial x^2 \partial z} \end{pmatrix}(0) \tag{5.6}$$

(up to a nonzero numerical factor). Therefore the condition $(\omega) \in P^*$ concerns the 3-jet of ω, and the singularity class P^* has codimension 1 in P.

Consider now the singularity class W. It is clear that it is distinguished by a condition on the 0-jet of a germ and has codimension 3. By Proposition 4.4 any germ $(\omega) \in W$ is reducible to the normal form (4.8). Denote by W^1 the subclass of W consisting of germs for which $\operatorname{Ker} d\omega|_0 \subset T_0S$. For the germ

(4.8) this degeneration means that $\frac{\partial^2 R}{\partial z^2}(0) = 0$, therefore the codimension of W^1 in W equals 1. Let $W^0 = W \setminus W^1$. Any germ of W^0 is equivalent to a germ

$$(\omega) = (\pm z dz + x dy + dT(x,y)), \quad j_0^1 T = 0 \tag{5.7}$$

(the reduction of (4.8) to (5.7) follows from the Morse lemma with parameters, we introduce a new coordinate $\widetilde{z}$ instead of z and use the condition $\frac{\partial^2 R}{\partial z^2}(0) \neq 0$). Now $S = \{z = 0\}$ and it is easy to compute the matrix of the linear approximation of Z at the origin: it has the form

$$\begin{pmatrix} 1 + \frac{\partial^2 T}{\partial x \partial y}(0) & \frac{\partial^2 T}{\partial y^2}(0) \\ -\frac{\partial^2 T}{\partial x^2}(0) & -\frac{\partial^2 T}{\partial x \partial y}(0) \end{pmatrix} \tag{5.8}$$

(up to a nonzero numerical factor). Therefore the singularity class W_r is distinguished by a condition on the 1-jet of a germ (the 2-jet of R depends on the 1-jet of ω), and the codimension of $W_r \cap W^0$ in W^0 equals 1. Consequently, the codimension of W_r in W is also 1.

Consider now the last case—the singularity class E. It is a subclass of the singularity class G consisting of germs for which the origin is a degenerate point and condition (4.G) holds true for $p = 0$. We have already proved that G is distinguished by a condition on the 0-jet of a germ and the codimension of G in the space of all germs is 2 (see the proof of Proposition 4.2). By Proposition 4.3 any germ $(X, Y) \in G$ is equivalent to a germ

$$(X,Y) = (\partial/\partial x, x\partial/\partial y + C\partial/\partial z), \quad C(0) = \frac{\partial C}{\partial x}(0) = 0. \tag{5.9}$$

By Lemma 5.1, $((X,Y)) \in E$ if and only if $\frac{\partial C}{\partial y}(0) = 0$. Therefore the singularity class E is distinguished by a condition on the 1-jet of a germ and has codimension 3 in the space of all germs.

Assume now that $(X,Y) \in E$. Then $\frac{\partial C}{\partial x}(0) = \frac{\partial C}{\partial y}(0) = 0$ and by condition (4.G), $\frac{\partial C}{\partial z}(0) \neq 0$. In this case (5.9) can be easily reduced to a normal form

$$(X,Y) = (\partial/\partial x, x\partial/\partial y + (z + f(x,y,z))\partial/\partial z), \quad j_0^1 f = 0. \tag{5.10}$$

Let $f(x,y,z) = f(x,y,0) + z f_1(x,y,z)$. Dividing Y by $1 + f_1$ and introducing a new coordinate $\widetilde{x} = x/(1+f_1)$ instead of x we reduce (5.10) (after a simple change of the generators) to a normal form

$$(X,Y) = (\partial/\partial x, x\partial/\partial y + (z + g(x,y) + q(x,y,z))\partial/\partial z),$$
$$j_0^1 g = 0, \ j_0^2 q = 0.$$

Let $j_0^2 g = ax^2 + by^2 + cxy$. The change of the coordinate z to $\widetilde{z} = z - (c/2)y^2$ reduces c to 0, and we obtain a normal form

$$(X,Y) = (\partial/\partial x, x\partial/\partial y + (z + \alpha x^2 + \beta y^2 + h(x,y,z))\partial/\partial z), \ j_0^2 h = 0. \tag{5.11}$$

Now it is easy to compute the linear approximation of the vector field Z: its matrix is

$$\begin{pmatrix} \alpha & \beta \\ \alpha & 0 \end{pmatrix}. \tag{5.12}$$

Therefore the singularity class E_r is distinguished by a condition on the 2-jet of a germ and it has codimension 1 in E. The proof of Proposition 5.3 is complete. ∎

6. Regular and irregular points. In this section we assume that our 2-distribution satisfies the genericity conditions (4.G) and (5.G). We define regular and irregular points in terms of the vector field Z introduced in the preceding section. In fact, these definitions depend on the module (Z) only, and so they are invariant. We also give equivalent definitions of these types of points in geometric terms.

DEFINITION 6.1. A singular point $p \in S$ of a 2-distribution Q is called *regular* if $Z(p) \neq 0$ and *irregular* if $Z(p) = 0$.

It follows from the definition of Z that *all degenerate points of a Pfaffian equation are irregular*. For 2-modules this is not true: degenerate points may be both regular and irregular.

EXAMPLE 6.1. For the distribution (3.5) we have

$$S = \{y = 0\}, \quad \omega = dx \wedge dy \wedge dz(X, Y, \cdot) = xdz - ydy,$$
$$\omega|_S = xdz, \quad (Z_1) = (x\,\partial/\partial x), \quad D = \{x = y = 0\}, \quad (Z) = (\partial/\partial x),$$

therefore every degenerate point $(0,0,z)$ is regular.

EXAMPLE 6.2. For the distribution (3.6) we have

$$S = \{z - \lambda x^2 + y^2 = 0\},$$
$$\omega = dx \wedge dy \wedge dz(X, Y, \cdot) = xdz - (z + \lambda x^2 + y^2)dy.$$

One can take the coordinate system (x, y) on S. In this coordinate system we have

$$\omega|_S = xd(\lambda x^2 - y^2) - 2\lambda x^2 dy,$$
$$(Z_1) = ((xy + \lambda x^2)\partial/\partial x + \lambda x^2 \partial/\partial y),$$

and $D = \{x = 0\} \cap S$. Therefore, we obtain

$$(Z) = ((y + \lambda x)\partial/\partial x + \lambda x \partial/\partial y),$$

and so the origin is an irregular degenerate point (the other degenerate points are regular).

The nondegenerate points may be regular or irregular for both Pfaffian equations and 2-modules.

EXAMPLE 6.3. For the field of planes (3.1) we have

$$S = \{x = 0\}, \quad \omega|_S = dz, \quad (Z) = (Z_1) = (\partial/\partial y),$$

therefore all singular points are regular nondegenerate.

EXAMPLE 6.4. For the field of planes (3.2) we have

$$S = \{z + 2xy + 3bx^2y^2 = 0\}, \quad \omega|_S = -d(2xy + 3bx^2y) - (x^2y + 2bx^3y^2)dy.$$

After choosing the coordinate system (x, y) on S we obtain

$$(Z) = (Z_1) = ((2x + 3bx^2 + x^2y + 2bx^3y^2)\partial/\partial x - (2y + 6bxy)\partial/\partial y),$$

therefore the origin is a nondegenerate irregular point.

The following theorem gives geometric characterizations of regular and irregular points. Recall that a degenerate point of a Pfaffian equation is always irregular.

PROPOSITION 6.1. *For any 2-distribution satisfying* (4.G) *and* (5.G) *the following holds.*

1) *A nondegenerate point p of a 2-distribution Q is irregular if and only if $Q(p) \subset T_pS$.*

2) *A degenerate point $p \in S$ of a 2-module $Q = (X, Y)$ is irregular if and only if*

$$\text{span}(X(p), Y(p), [X, Y](p)) \subset T_pS. \tag{6.1}$$

Remark. For 2-modules of vector fields the two statements of Proposition 6.1 can be replaced by one: a singular point is irregular if and only if (6.1) holds.

Proof of Proposition 6.1. The first statement follows from the definition (5.1) of the vector field Z. To prove the second one we use the normal form (4.5) (Proposition 4.3). By Lemma 5.1, p is an irregular point if and only if $\frac{\partial C}{\partial y}(p) = 0$. This condition is equivalent to (6.1) since $X(p) = \partial/\partial x$, $Y(p) = 0$, $[X, Y](p) = \partial/\partial y$ and $T_pS = \text{Ker}\left(\frac{\partial C}{\partial y}(p)dy + \frac{\partial C}{\partial z}(p)dz\right)$. ∎

PROPOSITION 6.2. 1) *For any Pfaffian equation satisfying* (4.G) *and* (5.G) *degenerate points and irregular nondegenerate points are isolated points in S.*

2) *For any 2-module of vector fields satisfying* (4.G) *and* (5.G) *irregular degenerate points are isolated points of D, and irregular nondegenerate points are isolated points of $S\backslash D$.*

Proof. Every irregular point of a 2-distribution is a singular point of Z, and by condition (5.G) singular points of Z are isolated. ∎

If p is a degenerate point of a Pfaffian equation then the direction $\text{Ker}\, d\omega|_p$ is invariant (see Proposition 4.3). Under the genericity condition (5.G) it has the following important property.

PROPOSITION 6.3. *Let (ω) be a Pfaffian equation satisfying* (4.G) *and* (5.G), *and let $p \in S$ be a degenerate point. Then $\text{Ker}\, d\omega|_p \subset T_pM$ is transversal to T_pS.*

Proof. By Proposition 4.3 we may assume that

$$(\omega) = (xdy + dR(x, y, z)), \quad j^1_pR = 0. \tag{6.2}$$

Then $d\omega = dx \wedge dy$ and $\text{Ker}\, d\omega|_p = \partial/\partial z$. On the other hand, $S = \{\frac{\partial R}{\partial z} = 0\}$. If $\text{Ker}\, d\omega|_p$ is not transversal to T_pS then $\frac{\partial^2 R}{\partial z^2}(p) = 0$, and so we can take the coordinate system (z, ψ) on S, where ψ is either x or y. Then it is easy to show that the 1-jet at p of the restriction of ω to S has the form $gd\psi$, where g is some linear function. It follows that the vector field Z has a degenerate linear approximation at p. The contradiction with (5.G) shows that the transversality holds. ∎

7. Types of irregular points. In this section we study the eigenvalues of the linear approximation of the vector field Z at irregular points. We divide irregular nondegenerate points into hyperbolic and elliptic ones, and irregular degenerate points into node, saddle and focus points. We continue the study of the geometry of singularities.

PROPOSITION 7.1 ([M]). *If p is a nondegenerate irregular point then the sum of the eigenvalues of Z at p is* 0.

Proof. It suffices to give the proof for a Pfaffian equation (ω), $\omega|_p \neq 0$. The relation $\omega \wedge d\omega(p) = 0$ means that $(d\omega|_p)|_{\text{Ker}\,\omega|_p} = 0$. If p is an irregular point then $\text{Ker}\,\omega|_p = T_pS$, therefore $(d\omega|_S)|_p = 0$. This implies that $\text{div}\, Z|_p = d(\Omega(Z, \cdot))|_p = (d\omega|_S)|_p = 0$ for any volume form Ω on S, hence the sum of the eigenvalues of Z at p is 0. ∎

DEFINITION 7.1. A nondegenerate irregular point is called *hyperbolic* (resp. *elliptic*) if the eigenvalues of Z at this point are real and nonzero (resp. purely imaginary, i.e. $\lambda_1 = -\lambda_2 = \alpha i$, $\alpha \neq 0$).

The parabolic case ($\lambda_1 = \lambda_2 = 0$) does not occur for generic 2-distributions: it is excluded by the genericity condition (5.G).

For degenerate irregular points there are no restrictions on the eigenvalues λ_1, λ_2 of Z. This follows from the formulas (5.8) (resp. (5.12)) for the matrix of the linear approximation of Z at degenerate points of a Pfaffian equation (resp. at irregular degenerate points of 2-modules) in suitable coordinates.

DEFINITION 7.2. A degenerate irregular point is called a *node* (resp. *saddle*, *focus*) if the eigenvalues λ_1, λ_2 at this point are real nonzero and of the same sign (resp. real nonzero and of different signs for a saddle, and $\lambda_{1,2} = a + ib$, $a \neq 0$, $b \neq 0$, for a focus).

From (5.G) it follows that the other possibilities for the eigenvalues at degenerate irregular points are excluded.

EXAMPLE 7.1. We have shown (see Example 6.2) that the matrix of the linear approximation at $0 \in \mathbb{R}^3$ of the field Z corresponding to the 2-module (3.6) is $\left(\begin{smallmatrix}\lambda & 1\\ \lambda & 0\end{smallmatrix}\right)$. Therefore the origin is an irregular degenerate point of (3.6) of saddle (resp. node, focus) type if $\lambda > 0$ (resp. $\lambda < -4$, $\lambda \in (-4, 0)$).

EXAMPLE 7.2. It is easy to compute the matrix of the linear approximation at $0 \in \mathbb{R}^3$ of the field Z corresponding to the Pfaffian equation (3.4): the result is $\left(\begin{smallmatrix}\theta & 1\\ \theta & 0\end{smallmatrix}\right)$. Therefore the origin is a degenerate point of (3.4) of saddle (resp. node, focus) type if $\theta > 0$ (resp. $\theta < -4$, $\theta \in (-4, 0)$).

The topological behaviour of the field d of lines (defined in Section 5) near an irregular degenerate point depends on the type of the point: d has the topological type of a node, saddle or focus in a neighbourhood of a degenerate point of the type with the same name. The topological behaviour of d distinguishes also irregular nondegenerate points. It follows from the definition that d is of saddle type near a hyperbolic nondegenerate singular point. The topological type of d near an elliptic nondegenerate singular point cannot be defined in terms of the linear approximation of Z (since at these points the eigenvalues are purely imaginary). Nevertheless, the following statement holds.

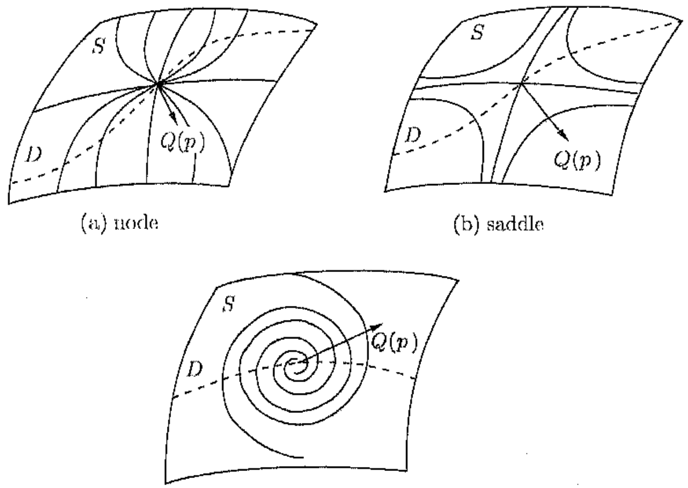
PROPOSITION 7.2. *Let Q be a generic (satisfying (4.G) and (5.G)) distribution, and let p be an elliptic nondegenerate irregular point of Q. Then d has the topological type of a focus in a neighbourhood of p.*

The proof can be found in [JP] and [Z2].

The local geometry of singularities is "richest" near an irregular degenerate point p of a 2-module Q of vector fields (Fig. 2). Assume that p is either a node or a saddle. Then under condition (5.G) the linear approximation of Z has two different eigenspaces: the lines $E_1(p)$ and $E_2(p)$ in T_pS. Thus, we have four invariant directions in T_pS: $Q(p)$, T_pD, $E_1(p)$ and $E_2(p)$.

PROPOSITION 7.3. *Let Q be a generic (satisfying (4.G) and (5.G)) 2-module, and let $p \in S$ be an irregular degenerate node or saddle. Then the directions $Q(p)$, T_pD, $E_1(p)$ and $E_2(p)$ are different.*

Proof. In Section 5 we have shown that the germ of Q at p is equivalent to the germ (5.11) at $0 \in \mathbb{R}^3$. The matrix of the linear approximation of Z at p has the form (5.12) (in the coordinate system (x, y) on S), $T_pS = \operatorname{Ker} dz$,



(a) node (b) saddle

(c) focus

Fig. 2. Local geometry of a generic 2-module (X, Y) near an irregular degenerate point p

$Q(p) = (\partial/\partial x)$, therefore to prove the theorem we have to show that neither the vector $(1,0)^t$ nor $(0,1)^t$ is an eigenvector of (5.12). This follows from (5.G) as $\alpha \neq 0$ and $\beta \neq 0$. ∎

8. Singularity classes. Main classification theorem. The following theorem may be considered as the main classification result of this paper. The classification results of Section 3 are corollaries of this theorem and of the results of Sections 4–7.

THEOREM 8.1. 1) *Any singular point of a generic (satisfying (4.G) and (5.G)) Pfaffian equation on a 3-manifold M is of one of the types listed below, and is equivalent to one of the normal forms (3.1)–(3.4) according to the following rule.*

(a) *A regular nondegenerate singular point is equivalent to the normal form (3.1).*

(b) *Irregular nondegenerate singular points are either*

(b1) *hyperbolic points, equivalent to the normal form (3.2), or*
(b2) *elliptic points, equivalent to the normal form (3.3).*

(c) *Irregular degenerate singular points are either*

(c1) *node points, with normal form (3.4), $\theta < -4$,*
(c2) *saddle points, with normal form (3.4), $\theta > 0$, or*
(c3) *focus points, with normal form (3.4), $\theta \in (-4, 0)$.*

2) *Any singular point of a generic (satisfying* (4.G) *and* (5.G)) *2-module of vector fields on M is of one of the types listed below, and each of the types is equivalent to one of the normal forms* (3.1′)–(3.3′), (3.5), *or* (3.6) *according to the following rule.*

(a′) *A regular nondegenerate singular point is equivalent to the normal form* (3.2′).

(b′) *Irregular nondegenerate singular points are either*

(b1′) *hyperbolic points, equivalent to the normal form* (3.2′), *or*
(b2′) *elliptic points, equivalent to the normal form* (3.3′).

(d) *A regular degenerate singular point is equivalent to the normal form* (3.5).

(e) *Irregular degenerate singular points are either*

(e1) *node points, with normal form* (3.6), $\lambda < -4$,
(e2) *saddle points, with normal form* (3.6), $\lambda > 0$, *or*
(e3) *focus points, with normal form* (3.6), $\lambda \in (-4, 0)$.

The classification results of Section 3 (Theorems 3.1–3.5) are corollaries of Theorem 8.1, Propositions 5.2, 5.3 and the fact that the parameter b in (3.2) is defined by the 5-jet (the latter is proved in [Z2]). The set Σ in the formulation of Theorem 3.4 consists of all irregular points of the 2-distribution Q (by Proposition 6.2, Σ consists of isolated points).

In this section we show how all the statements of Theorem 8.1 follow from the results of the preceding sections and from the preceding work, with one exception. Namely, the proof of the fact that the germ of a 2-module of vector fields at an irregular degenerate point is equivalent to the normal form (3.6) is given in Section 9.

The fact that under (4.G) and (5.G) every singular point has one of the listed types has already been proved in Sections 6 and 7.

The reducibility of a germ at a regular nondegenerate point to the normal form (3.1) follows from Proposition 6.1 and the results of [M] for Pfaffian equations where the reducibility to (3.1′) was proved under (4.G) and the transversality of $Q(p)$ and T_pS. The reducibility of a germ of a 2-module at a regular degenerate point to the normal form (3.5) follows (in the same way) from Proposition 6.1 and the results of [JP] (in [JP] condition (4.G) and the condition $\mathrm{span}(X(p), Y(p), [X, Y](p)) + T_pS = T_pM$ were used). Finally, the equivalence of the germ of a 2-distribution at a nondegenerate irregular hyperbolic (resp. elliptic) point to the normal form (3.2′) (resp. (3.3′)) follows from Proposition 6.1 and the results of [Z2, Ch. 4, Section 19] for Pfaffian equations.

Now we will prove the reducibility of the germ of a Pfaffian equation at a degenerate (and consequently irregular) point to the normal form (3.4).

By Proposition 4.3 we can use the normal form (4.8). It follows from Proposition 6.3 that $\frac{\partial^2 R}{\partial z^2}(0) \neq 0$, therefore the germ is equivalent to (5.7) (see the proof of Proposition 5.3). We can rewrite (5.7) in the form

$$(8.1) \qquad (\omega) = (\pm z dz + A(x,y)dx + B(x,y)dy),$$

where A and B are some function germs vanishing at p (see the proof of Proposition 5.3). Then $S = \{z = 0\}$ and, in the coordinates x, y on S we get

$$(8.2) \qquad Z = B(x,y)\partial/\partial x - A(x,y)\partial/\partial y.$$

Condition (5.G) makes it possible to reduce Z to a linear normal form $Z = (\theta x + y)\partial/\partial x + \theta x \partial/\partial y$, therefore (ω) reduces to

$$(3.4) \qquad (\pm z dz - \theta x dx + (\theta x + y)dy).$$

It can be directly shown that both normal forms with the plus and minus sign are equivalent, therefore we obtain the normal form (3.4). This also follows from the fact that the classification of Pfaffian equations of the form (8.1) is equivalent to the orbital classification of vector fields (8.2) (see [Z2], Ch. 3, Sect. 13). Example 7.2 explains how the saddle, node and focus cases are distinguished in terms of θ.

9. Reduction to the normal form (3.6). In this section we complete the proof of Theorem 8.1 by showing that the germ of a generic (satisfying (4.G) and (5.G)) 2-module $Q = (X, Y)$ at an irregular degenerate point is equivalent to the normal form (3.6). The proof is based on the homotopy method.

In proving Proposition 5.3 we have shown that Q is equivalent to the following germ at $0 \in \mathbb{R}^3$:

$$(9.1) \qquad (\partial/\partial x, x\partial/\partial y + (z + \alpha x^2 + \beta y^2 + h(x,y,z))\partial/\partial z), \qquad j_0^2 h = 0,$$

where $\alpha \neq 0$ and $\beta \neq 0$. Introducing a new coordinate $\tilde{z} = z/\beta$ we reduce (9.1) to

$$(9.2) \qquad (\widetilde{X}, \widetilde{Y}) = (\partial/\partial x, x\partial/\partial y + (z + \lambda x^2 + y^2 + f(x,y,z))\partial/\partial z), \qquad j_0^2 f = 0.$$

We have to prove that the germ (9.2) is equivalent to the germ

$$(9.3) \qquad (X, Y) = (\partial/\partial x, x\partial/\partial y + (z + \lambda x^2 + y^2)\partial/\partial z),$$

i.e. that we can reduce f to 0.

Proving this we shall use Pfaffian equations rather than modules of vector fields (to reduce computations). Let $\Omega = dx \wedge dy \wedge dz$ and

$$(9.4) \qquad \omega = \Omega(X, Y, \cdot) = x dz - (z + \lambda x^2 + y^2)dy,$$

$$(9.5) \qquad \widetilde{\omega} = \Omega(\widetilde{X}, \widetilde{Y}, \cdot) = x dz - (z + \lambda x^2 + y^2 + f(x,y,z))dy.$$

LEMMA 9.1. *The germs* (9.2) *and* (9.3) *are equivalent if and only if the germs of the Pfaffian equations* (ω) *and* ($\widetilde{\omega}$) *are equivalent.*

Proof. The proof can be easily reduced to the following property of the modules (9.2) and (9.3): if $\Omega(X,Y,U)=0$, then U belongs to (X,Y), and if $\Omega(\widetilde{X},\widetilde{Y},\widetilde{U})=0$, then $\widetilde{U}$ belongs to $(\widetilde{X},\widetilde{Y})$ (for arbitrary vector fields U and $\widetilde{U}$). We will check this property for the module (9.3) (for (9.2) the arguments are similar).

Let $U=U_1\partial/\partial x+U_2\partial/\partial y+U_3\partial/\partial z$. Then the equality $\Omega(X,Y,U)=0$ implies that $xU_3=(z+\lambda x^2+y^2)U_2$. The functions x and $z+\lambda x^2+y^2$ are differentially independent at 0, therefore there exists a function g such that $U_2=xg$ and $U_3=(z+\lambda x^2+y^2)g$, hence $U=U_1X+gY$. ∎

Remark. The above fact is not necessarily true for nongeneric germs (X,Y) and $(\widetilde{X},\widetilde{Y})$ (take for example a nonsingular vector field X and a singular vector field $\widetilde{X}$, and put $Y=X$, $\widetilde{Y}=\widetilde{X}$). On the other hand, this fact is true for a large set of $(n-1)$-generated modules of vector fields in $\mathbb{R}^n$ which can be effectively described ([MZ]).

By Lemma 9.1 it remains to prove the equivalence of the germs (ω) and ($\widetilde{\omega}$), where ω and $\widetilde{\omega}$ are given by (9.4) and (9.5) respectively.

In the proof of this equivalence we shall use the homotopy method. Let ω_t be the family of 1-forms

$$\omega_t=x\,dz-(z+\lambda x^2+y^2+tf)dy,\qquad t\in[0,1].$$

Consider the homology equation $L_{\xi_t}\omega_t+h_t\omega_t=f\,dy$, $\xi_t(0)=0$, i.e.

$$\xi_t\lrcorner\, d\omega_t+d(\xi_t\lrcorner\,\omega_t)+h_t\omega_t=f\,dy,\qquad \xi_t(0)=0,\tag{9.6}$$

with the unknown pair (h_t,ξ_t), where h_t is a family of germs of functions and ξ_t is a family of germs of vector fields ($L_{\xi_t}\omega_t$ is the Lie derivative of the 1-form ω_t along the vector field ξ_t).

PROPOSITION 9.1 (the homotopy method). *If the homology equation* (9.6) *is solvable, then the Pfaffian equations* (ω) *and* ($\widetilde{\omega}$) *are equivalent.*

Proof. Let (h_t,ξ_t) be a solution of the homology equation (9.6). Define a family of diffeomorphisms Φ_t by $d\Phi_t/dt=\xi_t(\Phi_t)$, $\Phi_0=\mathrm{id}$. From (9.6) it follows that

$$\frac{d}{dt}\Phi_t^*\omega_t=\Phi_t^*\Big(L_{\xi_t}\omega_t+\frac{d\omega_t}{dt}\Big)=\Phi_t^*(-h_t\omega_t).$$

Write $\Phi_t^*\omega_t=\widetilde{\omega}_t$, $h_t(\Phi_t)=\widetilde{h}_t$. Then $d\widetilde{\omega}_t/dt=-\widetilde{h}_t\widetilde{\omega}_t$, and so $\widetilde{\omega}_t=H_t\widetilde{\omega}_0=H_t\omega_0$, where $H_t=\exp(-\int_0^t\widetilde{h}_s ds)$. This means that $\Phi_1^*\omega_1=H_1\omega_0$, i.e. ω_1 and ω_0 are equivalent. ∎

Now, to show the reducibility of a germ at an irregular degenerate point to the normal form (3.6) it is enough to prove the solvability of the homology equation (9.6). We shall do this below by reducing the problem to solving a single singular partial differential equation.

Let

$$\xi_t=\phi_t\frac{\partial}{\partial x}+\psi_t\frac{\partial}{\partial y}+\eta_t\frac{\partial}{\partial z},\qquad u_t=\xi_t\lrcorner\,\omega_t.$$

Equation (9.6) can be rewritten as a system of equations for ϕ_t, ψ_t, η_t, h_t and u_t:

$$\begin{aligned}
&\frac{\partial u_t}{\partial x}-\eta_t+\Big(2\lambda x+t\frac{\partial f}{\partial x}\Big)\psi_t=0,\\
&\frac{\partial u_t}{\partial y}-\Big(2\lambda x+t\frac{\partial f}{\partial x}\Big)\phi_t-\Big(1+t\frac{\partial f}{\partial z}\Big)\eta_t-h_t(z+\lambda x^2+y^2+tf)=f,\\
&\frac{\partial u_t}{\partial z}+\phi_t+\Big(1+t\frac{\partial f}{\partial z}\Big)\psi_t+xh_t=0,\\
&u_t=x\eta_t-(z+\lambda x^2+y^2+tf)\psi_t.
\end{aligned}$$

After the elimination of η_t and ϕ_t we obtain the following equivalent system of two equations for u_t, ψ_t and h_t:

$$\begin{aligned}
&-\frac{\partial u_t}{\partial x}+\frac{\partial u_t}{\partial y}+2\lambda x\frac{\partial u_t}{\partial z}+V_t(u_t)-f=h_tc_t,\\
&u_t-x\frac{\partial u_t}{\partial x}=-\psi_tc_t,
\end{aligned}\tag{9.7}$$

where V_t is a vector field such that $j_0^1V_t=0$ (here and below we use the condition $j_0^2f\neq0$), and c_t is the function

$$c_t=z-\lambda x^2+y^2+t\Big(f-x\frac{\partial f}{\partial x}\Big).$$

Changing the coordinate z to $\widetilde{z}=c_t$ and leaving the other two coordinates unchanged we reduce (9.7) to a system with unknowns $\widetilde{u}_t(x,y,\widetilde{z})$, $\widetilde{h}_t(x,y,\widetilde{z})$ and $\widetilde{\psi}_t(x,y,\widetilde{z})$. This system has the form (we omit the tildes)

$$\begin{aligned}
&-\frac{\partial u_t}{\partial x}+\frac{\partial u_t}{\partial y}+(4\lambda x+2y)\frac{\partial u_t}{\partial z}+V_{1,t}(u_t)-f_{1,t}=h_tz,\\
&u_t-x\frac{\partial u_t}{\partial x}+x^2(2\lambda+f_{2,t})\frac{\partial u_t}{\partial z}=-\psi_tz,
\end{aligned}\tag{9.8}$$

where $V_{1,t}$ is a vector field, $f_{1,t}$ and $f_{2,t}$ are functions, and $j_0^1V_{1,t}=0$, $j_0^2f_{1,t}=0$, $f_{2,t}(0)=0$. Write the above equations as $L_1=h_tz$, $L_2=-\psi_tz$. Since h_t and ψ_t are arbitrary unknown functions which appear on the right-hand sides of the equations only, it follows that solving this system is equivalent to solving the system $L_1|_{z=0}=0$, $L_2|_{z=0}=0$, where these unknown functions do not appear any more. The only unknown left is u_t. However, on the surface $\{z=0\}$ its derivative $\partial u_t/\partial z$ can be considered as a separate unknown. We shall look for a solution of the form

$$u_t|_{z=0} = x\alpha_t \quad \text{and} \quad \frac{\partial u_t}{\partial z}\bigg|_{z=0} = \beta_t,$$

where α_t and β_t are unknown functions of x and y. Having α_t and β_t we can extend u_t outside the surface $\{z=0\}$ (not uniquely).

Our further considerations are restricted to the surface $\{z=0\}$. The system $L_1|_{z=0}=0$, $L_2|_{z=0}=0$ now takes the form

$$\text{(9.9)}\qquad \begin{aligned} &-\alpha_t - x\frac{\partial\alpha_t}{\partial x} + x\frac{\partial\alpha_t}{\partial y} + (4\lambda x+2y)\beta_t + W_t(\alpha_t) + A_t\alpha_t + B_t\beta_t - g_{1,t} = 0,\\ &-x^2\frac{\partial\alpha_t}{\partial x} + x^2(2\lambda+g_{2,t})\beta_t = 0,\end{aligned}$$

where W_t is a vector field on the surface $\{z=0\}$, A_t, B_t, $g_{1,t}$ and $g_{2,t}$ are functions of x and y, and $j_0^1A_t = j_0^1B_t = j_0^1W_t = 0$, $j_0^2g_{1,t}=0$, $g_{2,t}(0)=0$.

From the second equation we compute $\beta_t = (2\lambda+g_{2,t})^{-1}\partial\alpha_t/\partial x$ and plug it into the first. We obtain an equation for α_t of the form

$$\text{(9.10)}\qquad Y_t(\alpha_t) + C_t\alpha_t = d_t,$$

where Y_t is a vector field on $\mathbb{R}^2(x,y)$, and C_t and d_t are functions of x and y such that

$$j_0^1Y_t = Y = (\lambda x+y)\partial/\partial x + \lambda x\partial/\partial y,$$
$$C_t(0) = -\lambda, \qquad j_0^2 d_t = 0.$$

Thus, we have reduced our problem to solving the singular partial differential equation (9.10). It is easy to see that the condition $\xi_t(0)=0$ (see (9.6)) holds true if a solution α_t of (9.10) has zero 2-jet at the origin. Let us first prove that (9.10) has a formal solution (a formal series $\widehat{\alpha}_t$ with coefficients depending smoothly on t) with zero 2-jet. To show this it is enough to check that the equation

$$\text{(9.11)}\qquad Y(\alpha) - \lambda\alpha = d$$

has an r-homogeneous solution α for every r-homogeneous function d and any $r\ge 3$.

Recall that the 1-jet of the vector field Z corresponding to the module $Q=(X,Y)$ is defined by the 2-jets of X and Y. Therefore j_0^1Z can be computed using the normal form (9.3) (which coincides with (3.6)). Now note that $j_0^1Z = Y$ (see Example 6.2) up to multiplication by a nonzero factor. Therefore the eigenvalues λ_1 and λ_2 of Y satisfy the genericity condition (5.G). The solvability of (9.11) with an r-homogeneous polynomial d

is equivalent to the condition

$$\text{(9.12)}\qquad m_1\lambda_1 + m_2\lambda_2 - \lambda \ne 0$$

for all nonnegative integers m_1 and m_2 such that $m_1+m_2=r$. Since $\lambda_1+\lambda_2=\lambda$, (9.12) follows from (5.G).

Thus, we have proved that the equation (9.10) has a formal solution $\widehat{\alpha}_t$ such that $j_0^2\widehat{\alpha}_t = 0$. The vector field Y_t is hyperbolic at $0\in\mathbb{R}^2$ (i.e. its eigenvalues lie off the imaginary axis; this also follows from (5.G)). To solve (9.10) in smooth germs we shall use a result of G. R. Belitskiĭ [B] (an extension of a theorem by R. Roussarie [R]). One of the versions of his result says that if the differential equation $V(u)=h(x)$ in $\mathbb{R}^n$ with fixed vector field V hyperbolic at $0\in\mathbb{R}^n$ and any fixed function germ $h=h(x)$, $x\in\mathbb{R}^n$, has a formal solution $\widehat{u}=\widehat{u}(x)$ then it has a smooth solution $u=u(x)$ such that the formal series of u coincides with $\widehat{u}$. Using this result we can conclude that (9.10) has a smooth solution α_t with zero 2-jet at the origin. The proof of the reducibility of (X,Y) to the normal form (3.6) is complete.

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Property (wM^*) and the unconditional metric compact approximation property

by

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Abstract. The main objective of this paper is to give a simple proof for a larger class of spaces of the following theorem of Kalton and Werner.

THEOREM. *Let X be a separable or reflexive Banach space. Then $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$ if and only if*

(a) *X has property (M^*), and*
(b) *X has the metric compact approximation property.*

Our main tool is a new property (wM^*) which we show to be closely related to the unconditional metric approximation property.

1. Introduction. We shall give characterizations of Banach spaces X such that $\mathcal{K}(X)$, the space of compact linear operators on X, is an M-ideal in $\mathcal{L}(X)$, the space of bounded linear operators on X. We shall give a new argument for the known fact that such spaces have the metric compact approximation property.

A closed subspace M of a Banach space X is called an *M-ideal* if there exists a projection P on X^* such that $\ker P = M^\perp$ and

$$\|x^*\| = \|x^* - Px^*\| + \|Px^*\| \quad \text{for all } x^* \in X^*.$$

Such a projection is called an *L-projection*. M-ideals were first defined and studied by Alfsen and Effros in [1] in 1972.

Many authors have tried to characterize those Banach spaces X such that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$. Finally, Kalton succeeded in [15] by introducing property (M^*) and showing that it plays a key role. But he had to assume that X satisfies a very strong form of the metric compact approximation

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