with (6.8) leads to

$$(6.11)\qquad (2x_1-1)_+ + \sum_{j=1}^{n-1}(2x_{j+1}-2x_j)_+ < x_n,$$

where $t_+ = t$ if $t > 0$ and $t_+ = 0$ if $t \le 0$. It remains to observe that (6.11) is satisfied by (6.1′), (6.1″).

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The Bourgain algebra of the disk algebra $A(\mathbb{D})$ and the algebra QA

by

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Abstract. It is shown that the Bourgain algebra $A(\mathbb{D})_b$ of the disk algebra $A(\mathbb{D})$ with respect to $H^\infty(\mathbb{D})$ is the algebra generated by the Blaschke products having only a finite number of singularities. It is also proved that, with respect to $H^\infty(\mathbb{D})$, the algebra QA of bounded analytic functions of vanishing mean oscillation is invariant under the Bourgain map as is $A(\mathbb{D})_b$.

Introduction. Let $A \subseteq B$ be two commutative Banach algebras. The *Bourgain algebra $(A,B)_b$ of A with respect to B* is the set of all $f \in B$ for which $\operatorname{dist}(ff_n, A) := \inf_{g\in A}\|ff_n + g\|_B \to 0$ whenever f_n is a sequence in A converging weakly to zero (i.e., such that $\varphi(f_n) \to 0$ for every bounded linear functional φ on A). In a recent paper [4] Cima, Stroethoff and Yale characterized the Bourgain algebra $(A(\mathbb{D}), L^\infty(\mathbb{D}))_b$ of the disk algebra with respect to the algebra $L^\infty(\mathbb{D})$ of Lebesgue measurable, essentially bounded functions on the unit disk $\mathbb{D}$. They showed that

$$(A(\mathbb{D}), L^\infty(\mathbb{D}))_b = (H^\infty(\mathbb{D}) \cap W(\mathbb{D})) + UC(\mathbb{D}) + V,$$

where

$$W(\mathbb{D}) = \{f \in L^\infty(\mathbb{D}) : \text{for every } \delta > 0 \text{ the set } \{\zeta \in T : \omega(f,\zeta) \ge \delta\} \text{ is finite}\}$$

is the set of all functions in $L^\infty(\mathbb{D})$ whose essential oscillations

$$\omega(f,\zeta_n) = \lim_{\delta\to 0} \operatorname{ess\,sup}\{|f(z)-f(w)| : z,w \in \mathbb{D},\ |z-\zeta_n| < \delta,\ |w-\zeta_n| < \delta\}$$

converge to zero whenever $\zeta_n \in T$ is a sequence of different points of the unit circle T. Moreover,

$$V = \{f \in L^\infty(\mathbb{D}) : \|f\chi_{\mathbb{D}\setminus r\mathbb{D}}\|_\infty \to 0 \text{ as } r \to 1\}\ (^1)$$

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(1) χ_E is the characteristic function of a subset $E \subseteq \mathbb{D}$.

is an ideal of functions in $L^\infty(\mathbb{D})$ which vanish in an appropriate sense near the boundary of $\mathbb{D}$ and $UC(\mathbb{D})$ is the set of uniformly continuous functions on $\mathbb{D}$. Restricting to $H^\infty(\mathbb{D})$, we see that the Bourgain algebra $A(\mathbb{D})_b := (A(\mathbb{D}), H^\infty(\mathbb{D}))_b$ of $A(\mathbb{D})$ with respect to $H^\infty(\mathbb{D})$ has the form

$$A(\mathbb{D})_b = H^\infty(\mathbb{D}) \cap W(\mathbb{D}).$$

In another paper [3] Cima, Stroethoff and Yale show that $A(\mathbb{D})_b$ contains every Blaschke product whose zeros cluster only at a finite number of points but, on the other hand, no Blaschke product with infinitely many boundary singularities is contained in $A(\mathbb{D})_b$. At the AMS Summer Meeting on Function Spaces in Orono (Maine, U.S.A., 1991), the following question was asked: Does $A(\mathbb{D})_b$ coincide with the algebra generated by the Blaschke products which have only a finite number of singularities?

It is the aim of the first section of this paper to give a positive answer to this question. Moreover, we shall obtain a characterization of $A(\mathbb{D})_b$ in terms of the size of the cluster sets of functions in H^∞. Finally, we show that $A(\mathbb{D})_b$ itself is invariant under the Bourgain map. The latter result has also been proved by Izuchi [14] in the case $A = A(T)$ and $B = L^\infty(T)$. For related results see also [10].

In the second section of the paper we show that the algebra QA of bounded analytic functions of vanishing mean oscillation is invariant under the Bourgain map, i.e.,

$$(QA, H^\infty(\mathbb{D}))_b = QA.$$

This result should be compared with a theorem of Izuchi, Stroethoff and Yale [15] which tells us that

$$(QC, L^\infty(T))_b = QC,$$

where QC is the algebra of quasicontinuous functions on the unit circle.

Recall that QA can be viewed as the set of all functions in QC whose Poisson integral is analytic in $\mathbb{D}$ (see [20], [8]).

Some results of this paper have also been proved by Pamela Gorkin and Donald E. Marshall. We are deeply indebted to Donald Marshall for allowing us to include his elegant proof of Lemma 1.1 in this paper. Our original proof will be briefly sketched later.

1. The Bourgain algebra of $A(\mathbb{D})$. Let $L^\infty = L^\infty(T)$ be the space of Lebesgue measurable, essentially bounded functions on the unit circle T. For a closed subset $F \subseteq T$, let L_F^∞ be the space of all functions f in L^∞ for which there exists a function q continuous on $T \setminus F$ such that q coincides with f almost everywhere. If $f \in L^\infty$, then $\|f\|_\infty$ will denote its essential sup norm.

For a function $f \in H^\infty$, let f^* denote its *boundary function*, i.e.,

$$f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) \quad \text{a.e.}$$

Note that f^* is defined almost everywhere and that $f^* \in L^\infty$. When it causes no confusion, we will identify f^* with f. As usual, the *cluster set* $\mathrm{Cl}(f,\zeta)$ of a function $f \in H^\infty$ at a point $\zeta \in T$ is defined to be the set of all points $w \in \mathbb{C}$ for which there exists a sequence z_n in $\mathbb{D}$ converging to ζ such that $f(z_n) \to w$.

LEMMA 1.1. *Let $f \in H^\infty$ and let $\varepsilon > 0$. Suppose there exists a finite subset $F = \{\zeta_1, \ldots, \zeta_N\}$ of T such that* $\operatorname{diam} \mathrm{Cl}(f,\zeta) < \varepsilon$ *for all $\zeta \in T \setminus F$. Then there exists $q_\varepsilon \in C(T \setminus F)$ such that*

$$\|q_\varepsilon - f^*\|_\infty < 3\varepsilon.$$

Proof (D. E. Marshall). For every $\lambda \in T \setminus F$ choose $a_\lambda \in \mathbb{C}$ and open disks D_λ centered at λ with $D_\lambda \cap F = \emptyset$ and such that $|f(z) - a_\lambda| < \varepsilon$ for all $z \in D_\lambda \cap \mathbb{D}$. Let $I_\lambda = \frac{1}{2}D_\lambda \cap T$, where $\frac{1}{2}D_\lambda$ is the disk centered at λ with half the radius of D_λ. Choose a countable subcollection $\{I_{\lambda_k}\}_{k=1}^\infty$ of $\{I_\lambda : \lambda \in T \setminus F\}$ which still covers $T \setminus F$. By deleting unnecessary ones, we may assume without loss of generality that no I_{λ_j} is contained in $\bigcup_{k\neq j} I_{\lambda_k}$. Define a function $f_\varepsilon : T \to \mathbb{C}$ by $f_\varepsilon(\lambda_k) = a_{\lambda_k}$, $f_\varepsilon(\zeta_j) = 0$ $(j = 1, \ldots, N)$ and extend f_ε to be linear between two adjacent λ_j's. (Note that $\zeta_1, \ldots, \zeta_N$ are the only cluster points of the λ_k and that $\bigcup_{j=1}^\infty I_{\lambda_j} = T \setminus F$.) More precisely, if there is no λ_j within the arc (λ_k, λ_l), let

$$f_\varepsilon(\zeta) = \left(\frac{\zeta - \lambda_l}{\lambda_k - \lambda_l}\right) a_{\lambda_k} + \left(\frac{\zeta - \lambda_k}{\lambda_l - \lambda_k}\right) a_{\lambda_l} \quad \text{for } \zeta \in (\lambda_k, \lambda_l).$$

Obviously, f_ε is continuous on $T \setminus F$. Because the radius of D_{λ_k} is twice that of the arc I_{λ_k}, at least one of D_{λ_k} or D_{λ_l}, say D_{λ_k}, contains the whole arc (λ_k, λ_l). Let $z \in D_{\lambda_k} \cap D_{\lambda_l} \cap \mathbb{D}$. Then

$$|a_{\lambda_k} - a_{\lambda_l}| \le |a_{\lambda_k} - f(z)| + |f(z) - a_{\lambda_l}| < 2\varepsilon.$$

Hence for $\zeta \in (\lambda_k, \lambda_l)$ and $r\zeta \in D_{\lambda_k}$ $(0 < r < 1)$ we have

$$|f(r\zeta) - f_\varepsilon(\zeta)| \le |f(r\zeta) - a_{\lambda_k}| + |a_{\lambda_k} - f_\varepsilon(\zeta)| \le 3\varepsilon.$$

Therefore $|f^*(\zeta) - f_\varepsilon(\zeta)| \le 3\varepsilon$ for a.e. $\zeta \in T$. This implies that $\|f^* - f_\varepsilon\|_\infty \le 3\varepsilon$. ■

Remark. The assertion of the lemma also follows from the proof of a general result of Helmer and Pym [12] on the approximation by bounded functions having only a finite number of discontinuities. We have just to define a suitable boundary function of $f \in H^\infty$ by putting

$$f^*(\zeta) = a_\zeta,$$

where a_ζ is any (fixed) point of the radial cluster set of f at ζ (note that Zermelo's axiom of choice is needed here), and to realize that for any $\zeta \in T \setminus F$ the oscillation

$$\operatorname{osc}(f^*, \zeta) := \limsup_{\delta \to 0} \{|f^*(\lambda) - f^*(\beta)| : \lambda, \beta \in T,\ |\lambda - \zeta| < \delta,\ |\beta - \zeta| < \delta\}$$

of f^* at ζ is less than ε whenever f satisfies the assumptions of Lemma 1.1.

Let $\mathcal{A}$ denote the set

$$\mathcal{A} = \{f \in H^\infty : \text{for every } \varepsilon > 0 \text{ the set } \{\zeta \in T : \operatorname{diam} \mathrm{Cl}(f, \zeta) \ge \varepsilon\} \text{ is finite}\}$$

endowed with the sup norm $\|\cdot\|$, where, as usual, $\operatorname{diam} E = \sup\{|a - b| : a, b \in E\}$ is the diameter of a bounded subset E of $\mathbb{C}$.

PROPOSITION 1.2. *$\mathcal{A}$ is a closed subalgebra of H^∞.*

Proof. Since

$$\operatorname{diam} \mathrm{Cl}(f + g, \zeta) \le \operatorname{diam} \mathrm{Cl}(f, \zeta) + \operatorname{diam} \mathrm{Cl}(g, \zeta)$$

and

$$\operatorname{diam} \mathrm{Cl}(f \cdot g, \zeta) \le \|g\| \operatorname{diam} \mathrm{Cl}(f, \zeta) + \|f\| \operatorname{diam} \mathrm{Cl}(g, \zeta),$$

it is easy to see that $\mathcal{A}$ is an algebra.

To prove that $\mathcal{A}$ is closed, let $f_n \in \mathcal{A}$ and $f \in H^\infty$ be such that $\|f_n - f\| < \varepsilon$. Then $\operatorname{diam} \mathrm{Cl}(f, \zeta) < 2\varepsilon + \operatorname{diam} \mathrm{Cl}(f_n, \zeta)$. Hence

$$\{\zeta \in T : \operatorname{diam} \mathrm{Cl}(f, \zeta) \ge 3\varepsilon\} \subseteq \{\zeta \in T : \operatorname{diam} \mathrm{Cl}(f_n, \zeta) > \varepsilon\},$$

from which we conclude that $f \in \mathcal{A}$. ∎

THEOREM 1.3. *$\mathcal{A}$ is the smallest closed subalgebra of H^∞ containing the set of all Blaschke products with a finite number of singularities.*

Proof. In what follows let

$$\mathcal{B} = \operatorname{clos} \Big\{ \sum_{j=1}^{n} \lambda_j B_j : \lambda_j \in \mathbb{C},\ B_j \text{ a BP},\ \operatorname{Sing} B_j \text{ finite},\ n \in \mathbb{N} \Big\}$$

be the uniform algebra generated by the set of Blaschke products which have a finite number of singularities. We have to show that $\mathcal{A} = \mathcal{B}$.

Obviously, $\mathcal{B} \subseteq \mathcal{A}$. (Note that $d := \operatorname{diam} \mathrm{Cl}(B, \zeta) = 2$ whenever $\zeta \in \operatorname{Sing} B$ and $d = 0$ otherwise.)

Now let $f \in \mathcal{A}$ and let $\varepsilon > 0$. By definition, there exists a finite set $F = \{\zeta_1, \ldots, \zeta_N\} \subseteq T$ such that $\operatorname{diam} \mathrm{Cl}(f, \zeta) < \varepsilon$ for all $\zeta \in T \setminus F$. Choose a function $q_\varepsilon \in C(T \setminus F)$, according to Lemma 1.1, such that

$$\|q_\varepsilon - f^*\|_\infty < 3\varepsilon. \tag{1}$$

Obviously, $q_\varepsilon \in L_F^\infty$. Because $H^\infty + L_F^\infty$ is a closed subalgebra of L^∞ (see [7] and [8, p. 399]), a theorem of Davie, Gamelin and Garnett yields that

$$\operatorname{dist}(q_\varepsilon, H^\infty) = \operatorname{dist}(q_\varepsilon, H^\infty \cap L_F^\infty)$$

(see also [8, pp. 386, 399]). Hence, by (1), there exists $h_\varepsilon \in H^\infty \cap L_F^\infty$ with $\|q_\varepsilon - h_\varepsilon\|_\infty < 3\varepsilon$.

Therefore $\|f^* - h_\varepsilon\|_\infty < 6\varepsilon$ from which we conclude that

$$f^* \in \overline{\bigcup_{\substack{F \subseteq T \\ F \text{ finite}}} (H^\infty \cap L_F^\infty)}.$$

By [1] or [7] we have

$$H^\infty \cap L_F^\infty = \operatorname{clos} \Big\{ \sum_{j=1}^{n} \lambda_j B_j : \lambda_j \in \mathbb{C},\ B_j \text{ a BP},\ \operatorname{Sing} B_j \subseteq F,\ n \in \mathbb{N} \Big\}.$$

Hence $f^* \in \mathcal{B}$. ∎

Remark. Theorem 1.3 has also been proven by P. Gorkin and D. Marshall.

We are now in a position to prove the result on Bourgain algebras.

THEOREM 1.4. *The Bourgain algebra $A(\mathbb{D})_b := (A(\mathbb{D}), H^\infty(\mathbb{D}))_b$ of the disk algebra with respect to $H^\infty(\mathbb{D})$ is generated by the set of Blaschke products which have only a finite number of singularities, i.e.,*

$$A(\mathbb{D})_b = \mathcal{A} = \mathcal{B}.$$

Proof. 1) Let B be a Blaschke product such that $\operatorname{Sing} B$ is finite. Then by exactly the same arguments as in [3], $B \in A(\mathbb{D})_b$. Since $A(\mathbb{D})_b$ is a closed algebra, we obtain

$$\mathcal{B} \subseteq A(\mathbb{D})_b. \tag{1}$$

2) Assume that $f \notin \mathcal{A}$. Then there exist $\varepsilon > 0$ and $\zeta_n \in T$ such that $\operatorname{diam} \mathrm{Cl}(f, \zeta_n) \ge \varepsilon$ for all n. Without loss of generality, let $\zeta_n \to \zeta \in T$. Define

$$f_n(z) = \left(\frac{z + \zeta_n}{2} \right)^{k_n}.$$

Then $f_n(\zeta_n) = 1$ and $f_n \to 0$ weakly in $A(\mathbb{D})$ for k_n sufficiently large (e.g., $k_n = [1/|a_n| + 1]^2$, $a_n = \log|(\zeta + \zeta_n)/2|$). Let $g_n \in A(\mathbb{D})$. Then

$$\begin{aligned} 2\|f f_n - g_n\| &\ge \operatorname{diam} \mathrm{Cl}(f f_n - g_n, \zeta_n) \\ &= \operatorname{diam} \mathrm{Cl}(f f_n, \zeta_n) = \operatorname{diam} \mathrm{Cl}(f, \zeta_n) \ge \varepsilon. \end{aligned}$$

Thus $f \notin A(\mathbb{D})_b$. Consequently, we have

$$A(\mathbb{D})_b \subseteq \mathcal{A}. \tag{2}$$

By Theorem 1.3, we can conclude from (1) and (2) that $A(\mathbb{D})_b = \mathcal{A} = \mathcal{B}$. ■

In order to determine the "bi-Bourgain" algebra $A(\mathbb{D})_{bb}$ of $A(\mathbb{D})$, we shall first prove the following lemma.

LEMMA 1.5. *Let $\zeta_j \in T$ be a sequence of different points converging to $\zeta \in T$. Then there exist functions $f_n \in \mathcal{A}$ converging weakly to zero such that $f_n(\zeta_{j_k}^{(n)}) = 1$ for some subsequence $(\zeta_{j_k}^{(n)})_k$ of $(\zeta_j)_j$ depending on n.*

Proof. Let G be the slit domain $G = \{z \in \mathbb{C} : |z| < 2\} \setminus [\zeta, 2\zeta]$ and let Φ be the Riemann mapping of G onto $\mathbb{D}$ with $\Phi(0) = 0$, $\Phi'(0) > 0$. Let $w_n = \Phi(\zeta_n)$. By taking a subsequence, we may assume without loss of generality that $(w_n)_n$ is an interpolating sequence for $H^\infty(\mathbb{D})$. Take Per Beurling functions $F_n \in H^\infty(\mathbb{D})$ such that $F_n(w_k) = \delta_{nk}$, $\sum_{n=1}^\infty |F_n(w)| \le M$, $w \in \mathbb{D}$ (see [8, p. 294]). Let $G_n = F_n \circ \Phi$. Then $G_n \in H^\infty(G)$, $G_n(\zeta_k) = \delta_{nk}$ and $\sum_{n=1}^\infty |G_n(z)| \le M$ for $z \in G$.

Let $\{I_n\}_{n=1}^\infty$ be any partition of $\mathbb{N}$ into infinite subsets. Let $f_n = \sum_{j\in I_n} G_j$. Then $f_n \in H^\infty(\mathbb{D})$ and $f_n \in C(\overline{\mathbb{D}} \setminus \{\zeta\})$. Hence $f_n \in \mathcal{A}$. Moreover, $\sum_n |f_n| = \sum_j |G_j| \le M$ in $\mathbb{D}$. By [2] (see also [11]), $f_n \to 0$ weakly in $H^\infty(\mathbb{D})$, hence in $\mathcal{A}$, and $f_n(\zeta_j) = 1$ whenever $j \in I_n$. Letting $(\zeta_{j_k}^{(n)})_k = (\zeta_j)_{j\in I_n}$ gives the assertion of the lemma. ■

THEOREM 1.6. *The Bourgain algebra $A(\mathbb{D})_b$ of the disk algebra with respect to $H^\infty(\mathbb{D})$ is invariant under the Bourgain map, i.e.,*

$$A(\mathbb{D})_{bb} = A(\mathbb{D})_b.$$

Proof. In view of Theorem 1.4, we have to show that $(\mathcal{A}, H^\infty)_b = \mathcal{A}$; that is, $\mathcal{A}_b = \mathcal{A}$. Since $\mathcal{A}$ is an algebra, we trivially have $\mathcal{A} \subseteq \mathcal{A}_b$. So let $f \in \mathcal{A}_b$. Assume that $f \notin \mathcal{A}$. Then there exists, by the definition of $\mathcal{A}$, $\varepsilon_0 > 0$ and a sequence $\zeta_n \in T$ such that $\operatorname{diam} C(f, \zeta_n) \ge \varepsilon_0$ for all n. Without loss of generality, let $\zeta_n \to \zeta \in T$.

Let f_n be the functions of Lemma 1.5. Because $f_n \to 0$ weakly and since $f \in \mathcal{A}_b$, there exists $g_n \in \mathcal{A}$ such that

$$\|ff_n + g_n\| \to 0. \tag{1}$$

Because $\mathcal{A} = \mathcal{B}$, we may assume without loss of generality that $\operatorname{Sing} g_n$ is finite. Fix n. For $k \ge k(n)$ sufficiently large we then have

$$\begin{aligned} 2\|ff_n + g_n\| &\ge \operatorname{diam} \mathrm{Cl}(ff_n + g_n, \zeta_{j_k}^{(n)}) = \operatorname{diam} \mathrm{Cl}(ff_n, \zeta_{j_k}^{(n)}) \\ &= \operatorname{diam} \mathrm{Cl}(f, \zeta_{j_k}^{(n)}) \ge \varepsilon_0. \end{aligned}$$

This is a contradiction to (1).

Hence $f \in \mathcal{A}$ and so $\mathcal{A} = \mathcal{A}_b$. ■

Remark. It has been communicated to us by D. Marshall that the result also follows from Izuchi [14].

We shall now prove that the corona theorem is true in $A(\mathbb{D})_b$. The idea is to reduce the problem to the investigation of a certain Douglas algebra. To this end, let D be the Douglas algebra generated by H^∞ and the complex conjugates of all Blaschke products with only a finite number of singularities, i.e.,

$$D = \operatorname{clos}\Big\{\sum_{j=1}^n h_j \overline{B}_j : B_j \text{ a BP, } \operatorname{Sing} B_j \text{ finite, } h_j \in H^\infty,\ n \in \mathbb{N}\Big\},$$

and $M(D)$ its spectrum. As usual, we shall identify a function in D with its Gelfand transform $\widehat{f} : M(D) \to \mathbb{C}$, defined by $\widehat{f}(m) = m(f)$ for $m \in M(D)$.

Along the same lines as in [16, p. 147] we first prove the following lemma.

LEMMA 1.7. *An inner function I is invertible in D (i.e., $\overline{I} \in D$) if and only if $\operatorname{Sing} I$ is finite.*

Proof. 1. Let I be an inner function such that $\operatorname{Sing} I$ is finite. By Frostman's theorem, there exists a sequence (a_n) in $\mathbb{D}$ converging to zero such that the $B_n = (I - a_n)/(1 - \overline{a}_n I)$ are Blaschke products for all $n \in \mathbb{N}$. Obviously, $\operatorname{Sing} B_n$ is finite. Hence $\overline{B}_n \in D$. Since $\overline{B}_n$ tends uniformly to $\overline{I}$, we conclude that $\overline{I} \in D$.

2. Suppose that I is an inner function invertible in D. Then there exist Blaschke products b_j $(j = 1, \ldots, n)$ with $\operatorname{Sing} b_j$ finite and $f_j \in H^\infty$ such that

$$\Big\|\overline{I} - \sum_{j=1}^n f_j \overline{b}_j\Big\|_\infty < \varepsilon.$$

Let $E = \bigcup_{j=1}^n \operatorname{Sing} b_j$. Then E is a finite set. Suppose that $\lambda \in T \setminus E$. Then

$$\Big\|\overline{I} - \sum_{j=1}^n f_j \overline{b}_j\Big\|_{M_\lambda(L^\infty)} < \varepsilon, \tag{1}$$

where $M_\lambda(L^\infty) = \{m \in M(L^\infty) : m(z - \lambda) = 0\}$ is the fiber over the point λ of the spectrum $M(L^\infty)$ of L^∞.

Since b_j is analytic at each of those λ's, it follows that $b_j|_{M_\lambda(L^\infty)}$ is constant. Suppose $\overline{b}_j|_{M_\lambda(L^\infty)} \equiv a_j$, where $|a_j| = 1$ $(a_j = a_j(\lambda))$. Thus we obtain from (1) the estimate

$$\Big\|1 - I\Big(\sum_{j=1}^n f_j a_j\Big)\Big\|_{M_\lambda(L^\infty)} < \varepsilon. \tag{2}$$

(Note that I is unimodular on $M_\lambda(L^\infty)$.) Because $0 \in \mathrm{Cl}(I, \lambda)$ whenever $\lambda \in \operatorname{Sing} I$, we conclude from (2) and [13, §10] that I has no singularity at λ. ■

For a Douglas algebra $\mathcal{D}$, let $C\mathcal{D}$ denote the C^*-algebra generated by the set of all invertible inner functions in $\mathcal{D}$. These algebras were studied in [1] (see also [8]). We recall that $\mathcal{B}$ is the smallest closed subalgebra of H^∞ containing the set of all Blaschke products with a finite number of singularities.

THEOREM 1.8. *Let D be the Douglas algebra generated by H^∞ and the complex conjugates of all Blaschke products with a finite number of singularities. Then*

$$H^\infty \cap CD = A(\mathbb{D})_b.$$

In particular, the corona theorem holds in $A(\mathbb{D})_b$.

Proof. By [1] the closed unit ball of $H^\infty \cap CD$ is the convex hull of the Blaschke products in $H^\infty \cap CD$. Moreover, by Lemma 1.7, any Blaschke product in CD has at most a finite number of singularities. Hence $H^\infty \cap CD = \mathcal{B}$. Theorem 1.4 now implies that $A(\mathbb{D})_b = \mathcal{B} = H^\infty \cap CD$.

Since by a result of Chang–Marshall [1] the corona theorem is true for any algebra of the form $H^\infty \cap C\mathcal{D}$, we conclude that it also holds in $A(\mathbb{D})_b$. ■

2. The Bourgain algebra of QA. Let QA be the algebra of bounded analytic functions in $\mathbb{D}$ with vanishing mean oscillation on the boundary (see [8], [18]). Associated with QA is the algebra QC of quasicontinuous functions on the unit circle (see [18]). Let $C = C(T)$ be the space of continuous functions on T. It is well known that $QC = (H^\infty + C(T)) \cap \overline{(H^\infty + C(T))}$ and that $QA = QC \cap H^\infty$. From the work of Wolff [20] we know that the algebra QA behaves in many cases like the disk algebra. It is therefore quite surprising that, in contrast to $A(\mathbb{D})$, the Bourgain algebra of QA with respect to H^∞ is not larger than QA itself.

We first present some auxiliary results.

For a point $\zeta \in M(QC)$ let

$$E_\zeta = \{m \in M(L^\infty) : f(m) = f(\zeta) \text{ for every } f \in QC\}$$

be the QC-level set of ζ in $M(L^\infty)$. Obviously, the level sets form a partition of $M(L^\infty)$ into closed sets. By Shilov's decomposition theorem [9, §44], a function $f \in L^\infty$ belongs to QC if and only if f is constant on every QC-level set in $M(L^\infty)$.

Finally, we recall that a Blaschke product b is said to be *thin* if its zeros z_n in $\mathbb{D}$ satisfy

$$\prod_{n\neq k}\left|\frac{z_n - z_k}{1-\overline{z}_k z_n}\right| \to 1 \quad \text{as } k\to\infty.$$

LEMMA 2.1. *Let $f \in H^\infty \setminus QC$. Then there exist uncountably many QC-level sets on which f is not constant.*

Proof. By our assumptions we see that $\overline{f} \notin H^\infty + C$. Thus $H^\infty[\overline{f}] \neq H^\infty + C$, where $H^\infty[\overline{f}]$ is the Douglas algebra generated by H^∞ and $\overline{f}$. Hence by the Chang–Marshall theorem (see [8, §9]), there exists an interpolating Blaschke product b with $\overline{b} \in H^\infty[\overline{f}]$. Let b_1 be a subfactor of b such that the zero sequence of b_1 is a thin sequence. Now $\overline{b}_1 \in H^\infty[\overline{f}]$, and it is easy to see that if $\overline{f}|E_\zeta$ is constant, then $\overline{b}_1|E_\zeta \in H^\infty|E_\zeta$. By Clancey and Gosselin's theorem [6], $\overline{b}_1|E_\zeta$ is also constant.

Let $\{z_n\}$ be the zero sequence of b_1 in D. By [19] and [20] this is an interpolating sequence for QA. Hence by [13, p. 205] the closure $S = \overline{\{z_n\}}^{M(QA)}$ of $\{z_n\}$ in $M(QA)$ is homeomorphic to the Stone–Čech compactification of the integers. This implies that S is uncountable. Because $M(QC) = M(QA) \setminus \mathbb{D}$, the QC-level sets corresponding to points $\zeta \in S \setminus \mathbb{D}$ are pairwise disjoint. Now b_1 is not constant on any QC-level set where it has a zero, and we have just seen that there are uncountably many such sets. Hence, by the considerations above, f is not constant on uncountably many QC-level sets. ■

THEOREM 2.2. *The algebra QA is invariant under the Bourgain map, i.e.,*

$$QA_b := (QA, H^\infty)_b = QA.$$

Proof. For a function $f \in L^\infty$ and a point $\zeta \in M(QC)$ let

$$\omega(f,\zeta) = \sup\{|f(x) - f(y)| : x, y \in E_\zeta\}.$$

Now let $f \in QA_b$. Assume that $f \notin QA$; thus $f \in H^\infty \setminus QC$. By Lemma 2.1, f is not constant on uncountably many QC-level sets in $M(L^\infty)$. Hence there exists $\varepsilon_0 > 0$ such that $\omega(f,\zeta_n) \geq \varepsilon_0$ for infinitely many $\zeta_n \in M(QC)$ $(n = 1, 2, \ldots)$.

Without loss of generality let $\zeta_k \notin \operatorname{clos}\{\zeta_n : n \neq k\}$. Since QC is a C^*-algebra, we have $QC \simeq C(M(QC))$. Thus there exists $f_n \in QC$, $f_n(\zeta_n) = 1$, f_n invertible in QC, such that $f_n \to 0$ weakly in QC. (Note that the latter is equivalent to pointwise and bounded convergence on $M(QC)$.)

Since QA is strongly logmodular on $M(QC)$ [17], there exists $g_n \in QA$ satisfying $\log|g_n| = \log|f_n|$. Hence $|g_n(\zeta_n)| = 1$ and $g_n \to 0$ weakly in QA. The latter holds because $|g_n| = |f_n|$ on $M(QC)$ and $f_n \to 0$ on the Shilov boundary $M(QC)$ of QA.

Let $h_n \in QA$. Then

$$\begin{aligned}2\|fg_n + h_n\| &\geq \max_{m,\widetilde{m}\in E_{\zeta_n}} |(fg_n + h_n)(m) - (fg_n + h_n)(\widetilde{m})|\\ &= \max_{m,\widetilde{m}\in E_{\zeta_n}} |f(m) - f(\widetilde{m})| = \omega(f,\zeta_n) \geq \varepsilon_0.\end{aligned}$$

Thus $f \notin QA_b$. ■

Remark. Lemma 2.1 also yields another proof of the fact that $(QC, L^\infty(T))_b = QC$ (see [15]).

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