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Denseness of the spaces Φ_V of Lizorkin type in the mixed $L^{\overline{p}}(\mathbb{R}^n)$-spaces

by

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Abstract. The spaces $\Phi_V(\mathbb{R}^n)$ are defined to consist of Schwartz test functions φ such that the Fourier transform $\widehat{\varphi}$ and all its derivatives vanish on a given closed set $V\subset\mathbb{R}^n$. Under the only assumption that $m(V)=0$ it is shown that Φ_V is dense in $C_0(\mathbb{R}^n)$ and in the space $L^{\overline{p}}(\mathbb{R}^n)$ with the mixed norm, for $1/\overline{p}$ in a certain pyramid. The result on the denseness for arbitrary $\overline{p}=(p_1,\ldots,p_n)$, $1<p_k<\infty$, $k=1,\ldots,n$, is proved for so-called quasibroken sets V.

1. Introduction. The space $\Phi=\Phi(\mathbb{R}^n)$ of Schwartz test functions, invariant relative to the Liouville fractional integrodifferentiation of functions of several variables or relative to the Riesz potential operator, is well known in the theory of function spaces and in operator theory ([5], [3], [6], see also [4]; [11], p. 18; [12], pp. 352, 359). It consists of those Schwartz functions whose Fourier transforms vanish with all their derivatives on the coordinate planes in the former case and at the origin in the latter case. Such spaces are convenient when treating Riesz potentials as distributions (see e.g. [8]).

Various problems of function theory and operator theory involve similar invariant spaces which correspond to convolution operators with Fourier transform vanishing on a given set $V\subset\mathbb{R}^n$ (e.g. for the hyperbolic Riesz potential operator, V is a conic surface, see [12], p. 410, while for the operator of Bochner–Riesz means, V is the sphere). The general case of the spaces Φ_V of Schwartz test functions with Fourier transform vanishing on an arbitrary closed set V was treated in [9], [10]. In particular, in [9] the convolutors in Φ_V were characterized and the space $\Psi_V=\widehat{\Phi}_V$ was applied to the problem of division by a regular distribution $f(x)$ which slowly decreases as x tends to the set of zeros of f. For a conic set V, a constructive characterization of Φ_V was given, not appealing to Fourier transforms. In [10] the denseness of Φ_V in $L^p(\mathbb{R}^n)$ was shown for any V with $m(V)=0$ if $2\le p<\infty$, and for

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special types of V if $1 < p < 2$. Here $m(V)$ denotes the Lebesgue measure of V.

In this paper the denseness of Φ_V is proved in the space $L^{\overline{p}}(\mathbb{R}^n)$ with $\overline{p} = (p_1, \ldots, p_n)$, for

$$1/\overline{p} := (1/p_1, \ldots, 1/p_n)$$

in a certain convex polyhedron. In the case of all admissible $p_k \in (1, \infty)$, $k = 1, \ldots, n$, the denseness is shown for so-called quasibroken sets V.

Moreover, we show that the given approximations by functions in Φ_V converge in all considered cases not only in $L^{\overline{p}}$-norm, but almost everywhere as well.

The paper is organized as follows. Section 2 contains the necessary preliminaries; we also formulate there the main results of this paper (Theorems A–D) and give, for the reader's convenience, the statements of some results from [9] as well. Sections 3–6 contain the proofs of Theorems A–D.

2. Preliminaries and statements of the results

2.1. *The spaces Ψ_V and Ψ'_V.* Let S be the space of Schwartz test functions. Let V be an arbitrary closed set in $\mathbb{R}^n$. We denote by Ψ_V the space of all functions in S which vanish on V together with all their derivatives:

$$\Psi_V = \{f \in S : (D^j f)(x) = 0,\ x \in V,\ |j| = 0, 1, 2, \ldots\}.$$

Obviously, Ψ_V is a closed subspace in S.

The space Ψ_V is non-trivial if $V \neq \mathbb{R}^n$, containing elements of S with support beyond V. If V is a surface defined by the equation $p(x) = 0$ with an infinitely differentiable function p such that $|D^k p(x)| \le c \exp((1-\varepsilon)|x|^2)$ for some $\varepsilon > 0$, every multiindex k and large $|x|$, then the function $\exp(-|x|^2 - [p(x)]^{-2})$, extended by zero to $x \in V$, is an example of a function in Ψ_V.

Let $\varrho(x, V) = \min_{y \in V} |x - y|$ and let

$$M_V(x) = \max\{\sqrt{1 + |x|^2},\ [\varrho(x, V)]^{-1}\}, \quad x \in \mathbb{R}^n \setminus V.$$

We equip Ψ_V with the countable set of norms

$$\|\psi\|_N = \max_{|j| \le N} \sup_{x \in \mathbb{R}^n \setminus V} [M_V(x)]^N |D^j \psi(x)|, \quad N = 1, 2, \ldots, \tag{2.1}$$

which turns Ψ_V into a linear topological space. It is complete in that topology (see [9], p. 679).

The space Ψ'_V of continuous linear functionals on Ψ_V can be identified with the quotient space

$$\Psi'_V = S'/\Psi^0_V, \quad \Psi^0_V = \{f \in S' : (f, \psi) = 0,\ \psi \in \Psi_V\} \tag{2.2}$$

of the Schwartz space S' by the subspace Ψ^0_V of functionals $f \in S'$ supported on V. Therefore, Ψ'_V can be considered as the space obtained from S' by identification of any two functionals which differ by a functional supported on V. For $f \in \Psi'_V$ and its "representative" $g \in S'$ we can formally write $f = g + \Psi^0_V$.

2.2. *The Fourier-dual space $\Phi_V = F^{-1}(\Psi_V)$.* Let

$$(Ff)(x) = \widehat{f}(x) = \int_{\mathbb{R}^n} e^{ixy} f(y)\, dy$$

be the Fourier transform of the function f. Let

$$\Phi_V = \{f \in S : \widehat{f} \in \Psi_V\}. \tag{2.3}$$

Obviously, $\Phi_V \subset S$ and Φ_V is closed in S. Similarly to (2.2),

$$\Phi'_V = S'/\Phi^0_V,$$

where $\Phi^0_V = \{f \in S' : (f, \varphi) = 0,\ \varphi \in \Phi_V\}$ consists of functionals which vanish on Φ_V.

It follows directly from (2.3) that $f \in S$ belongs to Φ_V if and only if it is orthogonal to all functions of the form $x^j e^{iax}$, $a \in V$:

$$\int_{\mathbb{R}^n} \varphi(x) x^j e^{iax}\, dx = 0$$

where $|j| = 0, 1, 2, \ldots$ and a is an arbitrary point of V.

A more effective characterization of Φ_V when V is a cone is given in Theorem 3 below.

2.3. The space $L^{\overline{p}}(\mathbb{R}^n)$, $\overline{p} = (p_1, \ldots, p_n)$, is defined by the mixed norm

$$\|f\|_{\overline{p}} = \|\ldots\|\,\|f\|^{(1)}_{p_1}\|^{(2)}_{p_2} \ldots\|^{(n)}_{p_n}, \tag{2.4}$$

where $\|f\|^{(k)}_{p_k}$ stands for the $L^{p_k}(\mathbb{R}^1)$-norm in the kth variable, $1 \le p_k \le \infty$ (see [1]). It is known ([1], p. 302) that the interpolating inequality

$$\|f\|_{\overline{p}} \le \|f\|^{1-\lambda}_{\overline{r}} \|f\|^{\lambda}_{\overline{q}} \tag{2.5}$$

holds for all $\overline{p}, \overline{r}$ and $\overline{q}$ with $p_k, r_k, q_k \in [1, \infty]$ such that $1/\overline{p} = (1-\lambda)/\overline{r} + \lambda/\overline{q}$, $0 \le \lambda \le 1$. We also recall that the Hausdorff–Young theorem is valid for $L^{\overline{p}}$-spaces:

$$\|\widehat{f}\|_{\overline{p}'} \le (2\pi)^s \|f\|_{\overline{p}}, \quad 1 \le p_n \le p_{n-1} \le \ldots \le p_1 \le 2, \tag{2.6}$$

with $\overline{p}' = (p'_1, \ldots, p'_n)$, $p'_k = p_k(p_k - 1)^{-1}$ and $s = \sum_{k=1}^n (p'_k)^{-1}$ (see [1], p. 322). It is also known that

$$\|f - f\chi_{G_m}\|_{\overline{p}} \to 0 \quad \text{as } m \to \infty, \tag{2.7}$$

$1 \le p_k < \infty$, where χ_{G_m} is the characteristic function of the set G_m, $G_m \subseteq G_{m+1}$, $\bigcup G_m = \mathbb{R}^n$ ([2], Ch. I, Sec. 1).

We denote by $C_0^\infty = C_0^\infty(\mathbb{R}^n)$ the space of infinitely differentiable functions in $\mathbb{R}^n$ with compact support. This space is dense in $L^{\overline{p}}(\mathbb{R}^n)$, $1 \le p_k < \infty$, $k = 1, \ldots, n$ ([2], Ch. I, Sec. 1).

2.4. *Some properties of Ψ_V and Φ_V*

THEOREM 1 ([9]). *A function f on $\mathbb{R}^n$ is a multiplier in Ψ_V if and only if $f \in C^\infty(\mathbb{R}^n \setminus V)$ and for every multiindex j there exist an integer $m_j \ge 0$ and a number $c_j > 0$ such that*

$$|D^j f(x)| \le c_j [M_V(x)]^{m_j}.$$

Theorem 1 can also be reformulated as follows: Multipliers in Ψ_V are precisely those functions $f(x)$ which are slowly growing as x tends to infinity and to the set V.

A function f defined beyond V is said to be *slowly vanishing as x tends to V* if there exist an integer $m > 0$ and a number $c > 0$ such that

$$|f(x)| \ge c[\varrho(x, V)]^m, \quad x \in \mathbb{R}^n \setminus V.$$

THEOREM 2 ([9]). *If f is a multiplier in Ψ_V, slowly vanishing as x tends to V, then $1/f$ is a multiplier in Ψ_V as well.*

THEOREM 3 ([9]). *Let V be a cone in $\mathbb{R}^n$ (i.e. $x \in V \Rightarrow \lambda x \in V$, $\forall \lambda \in \mathbb{R}^1$), $V \ne \{0\}$. Then $f \in \Phi_V$ if and only if*

$$\int\limits_{x\cdot a = \mathrm{const}} x^k f(x)\, ds_x = 0$$

for all $a \in V$ and $k = 0, 1, \ldots$, ds_x standing for the volume element at the point x in the hyperplane $x \cdot a = \mathrm{const}$.

We single out the case when $V = \bigcup Pa_j$ is the union of the hyperplanes $Pa_j = \{x \in \mathbb{R}^n : x \cdot a_j = 0\}$, $a_j = (a_1^j, \ldots, a_n^j)$, $j = 1, \ldots, l$, $1 \le l \le \infty$.

THEOREM 3′ ([9]). *Let $V = \bigcup Pa_j$. Then $f \in \Phi_V$ if and only if $f \in S$ and f is orthogonal to all polynomials along lines perpendicular to the hyperplanes Pa_j:*

$$\int\limits_{-\infty}^{\infty} \xi^m f(\xi a_j + h)\, d\xi = 0$$

for all $m = 0, 1, \ldots$, $j = 1, \ldots, l$ and $h \in \mathbb{R}^n$.

2.5. *Statements of the main results.* Let Π_0 and Π_1 be the following pyramids in $\mathbb{R}^n$:

$$\Pi_0 = \{x \in \mathbb{R}^n : 0 < x_n \le x_{n-1} \le \ldots \le x_1 \le 1/2\},$$
$$\Pi_1 = \{x \in \mathbb{R}^n : 1/2 \le x_1 \le x_2 \le \ldots \le x_n < 1\},$$

both contained in the unit cube $\{x : 0 < x_k < 1, k = 1, \ldots, n\}$ and symmetrical with respect to the center $(1/2, \ldots, 1/2)$ of this cube. Let $\mathrm{conv}(\Pi_0, \Pi_1)$ be their convex hull. It is characterized in Theorem D below.

THEOREM A. *Let $m(V) = 0$. Then Φ_V is dense in $L^{\overline{p}}(\mathbb{R}^n)$ if $1/\overline{p} \in \Pi_0$.*

CONJECTURE. If $m(V) = 0$, then Φ_V is dense in $L^{\overline{p}}(\mathbb{R}^n)$ if $1/\overline{p} \in \mathrm{conv}(\Pi_0, \Pi_1)$.

We call a set V *quasibroken* (see [10]) if, within every ball, it is contained in a union of a finite number of hyperplanes (of dimension $n-1$).

THEOREM B. *Let V be a quasibroken set. Then Φ_V is dense in $L^{\overline{p}}(\mathbb{R}^n)$ for all $\overline{p}$ with $1 < p_k < \infty$, $k = 1, \ldots, n$.*

THEOREM C. *Let $m(V) = 0$. For any $f \in L^{\overline{p}}(\mathbb{R}^n)$, $1 < p_k < \infty$, $k = 1, \ldots, n$, there exists a sequence $f_N \in \Phi_V$, $N = 1, 2, \ldots$, which converges to f almost everywhere. Moreover, if either V is quasibroken or $1/\overline{p} \in \Pi_0$, then f_N also converges to f in $L^{\overline{p}}$-norm.*

THEOREM D. *We have $1/\overline{p} \in \mathrm{conv}(\Pi_0, \Pi_1)$ if and only if $1 < p_k < \infty$, $k = 1, \ldots, n$, and*

$$\left|\frac{1}{p_n} - \frac{2}{p_{j_m}} + \frac{2}{p_{j_{m-1}}} - \ldots + \frac{2(-1)^m}{p_{j_1}} - \frac{(-1)^m}{2}\right| < \frac{1}{2} \tag{2.8}$$

for any choice of integers j_k, $k = 1, \ldots, m$, satisfying $1 \le j_1 < \ldots < j_m \le n-1$, for arbitrary $m = 1, \ldots, n-1$.

Sorting out all admissible j_k, $k = 1, \ldots, m \le n-1$, we see that the total number of inequalities in (2.8) is

$$C_{n-1}^1 + \ldots + C_{n-1}^{n-1} = 2^{n-1} - 1.$$

For $n = 2$ conditions (2.8) have a simple form:

$$\frac{2}{p_1} - 1 < \frac{1}{p_2} < \frac{2}{p_1}.$$

3. Proof of Theorem A

3.1. Our considerations essentially use the notion of the regularized distance $\Delta(x, V)$ from the point x to the set V ([13], Ch. VI), which can be constructed so that $\Delta(\cdot, V) \in C^\infty(\mathbb{R}^n \setminus V)$ and

$$C_1 \varrho(x, V) \le \Delta(x, V) \le C_2 \varrho(x, V), \tag{3.1}$$
$$|D^j \Delta(x, V)| \le B_j [\varrho(x, V)]^{1-|j|}, \quad |j| = 1, 2, \ldots \tag{3.2}$$

for all multiindices j, with the constants C_1, C_2 and B_j depending on the set V only. Moreover, the regularized distance may be chosen in such a way that

$$\Delta(-x, V) = \Delta(x, -V), \tag{3.3}$$

which will be essentially used below (to achieve (3.3) it is sufficient to take a radial function $\varphi(x)$ for the partition of unity used when constructing $\Delta(x, V)$, see [13]).

3.2. Our first step in proving Theorem A is to show the denseness of Ψ_V in $L^{\overline{p}}(\mathbb{R}^n)$. Let μ be a smooth step function, i.e. $\mu \in C^\infty(0,\infty)$, $\mu(r) \equiv 1$ if $r \geq 2$, $\mu(r) \equiv 0$ if $0 \leq r \leq 1$ and $0 \leq \mu(r) \leq 1$ for $1 \leq r \leq 2$. Let $\mu_N(x) = \mu[N\Delta(x,V)]$ and

$$\psi_N(x) = \mu_N(x)\psi(x)$$

so that

$$\psi \in S \Rightarrow \psi_N \in \Psi_V \tag{3.4}$$

(for the proof of (3.4), based on (3.1)–(3.2), see [9]).

LEMMA 1. *Let $m(V) = 0$ and $\psi \in L^{\overline{p}}(\mathbb{R}^n)$, $1 \leq p_k < \infty$. Then ψ_N converges to ψ in $L^{\overline{p}}(\mathbb{R}^n)$, so that Ψ_V (and $\Psi_V \cap C_0^\infty$) is dense in $L^{\overline{p}}(\mathbb{R}^n)$.*

Proof. Since C_0^∞ is dense in $L^{\overline{p}}$ with $1 \leq p_k < \infty$, in view of (3.4) it is sufficient to show that $\|\psi - \psi_N\|_{\overline{p}} \to 0$ as $N \to \infty$ for $\psi \in C_0^\infty$. We have $1 - \mu[N\Delta(x,V)] \equiv 0$ for $\varrho(x,V) \geq 2/(C_1N)$ by (3.1). So

$$\|\psi - \psi_N\|_{\overline{p}} \leq \|(1-\chi_{G_N})\psi\|_{\overline{p}}$$

where $G_N = \{x : \varrho(x,V) \geq 2/(C_1N)\}$. It remains then to refer to (2.7). The lemma is proved.

Let A_N^V be the operator defined via the Fourier transforms:

$$(F(A_N^V f))(x) = \mu_N(x)(Ff)(x), \quad f \in S, \tag{3.5}$$

so that $A_N^V f \in \Phi_V$ if $f \in S$. By the denseness of S in $L^{\overline{p}}(\mathbb{R}^n)$, to prove Theorem A it is sufficient to show the convergence

$$\|A_N^V f - f\|_{\overline{p}} \to 0, \quad f \in S. \tag{3.6}$$

By the Hausdorff–Young theorem (2.6) we immediately deduce from Lemma 1 that (3.6) holds for $p_n \geq p_{n-1} \geq \ldots \geq p_1 \geq 2$, i.e. for $1/\overline{p} \in \Pi_0$. This yields the assertion of Theorem A.

3.3. *Foundations of the Conjecture.* We need the following lemma.

LEMMA 2. *Let $m(V) = 0$. Then (3.6) holds for a given $\overline{p}$ if*

$$\|A_N^V f\|_{\overline{r}} \leq c, \quad f \in S, \tag{3.7}$$

with c not depending on N, for any $\overline{r}$ such that $1/\overline{p}$ is an interior point of the straight line segment with ends $1/\overline{r}$ and $(1/2,\ldots,1/2)$.

Proof. By (2.5) with $\overline{q} = (2,\ldots,2)$ we have

$$\|A_N^V f - f\|_{\overline{p}} \leq \|A_N^V f - f\|_{\overline{r}}^{1-\lambda} \|A_N^V f - f\|_2^{\lambda}, \quad f \in S,$$

for $\overline{p}$ and $\overline{r}$ as in the statement of the lemma. Then (3.7) yields (3.6), which proves the lemma.

Let now $1/\overline{p} \in \Pi_1$ so that $1/\overline{p}' \in \Pi_0$. We have

$$|(A_N^V f - f, \omega)| \leq \|f\|_{\overline{p}} \|(A_N^V)^*\omega - \omega\|_{\overline{p}'} \tag{3.8}$$

for $\omega \in L^{\overline{p}'}(\mathbb{R}^n)$, where $(A_N^V)^*$ is the operator conjugate to A_N^V. Taking (3.3) into account we see that

$$(A_N^V)^* = A_N^{-V}.$$

Since $m(-V) = m(V) = 0$ and $1/\overline{p}' \in \Pi_0$, in view of Theorem A we can hope that

$$\|A_N^{-V}\omega - \omega\|_{\overline{p}'} \to 0.$$

(Unfortunately, Theorem A only provides this for $\omega \in S$ or $\omega \in L^2 \cap L^{\overline{p}'}$.) Then (3.8) would yield the weak convergence of $A_N^V f$ to f in $L^{\overline{p}}(\mathbb{R}^n)$. Hence the sequence $\{A_N^V f\}_{N=1}^\infty$ would be bounded in $L^{\overline{p}}(\mathbb{R}^n)$ for every $1/\overline{p} \in \Pi_1$. Since Π_1 is convex, Lemma 2 provides then the convergence (3.6) for $1/\overline{p} \in \Pi_1$. Thus, Theorem A would be proved also for $1/\overline{p} \in \Pi_1$. Application of the inequality (2.5):

$$\|A_N^V f - f\|_{\overline{p}} \leq \|A_N^V f - f\|_{\overline{r}}^{1-\lambda} \|A_N^V f - f\|_{\overline{q}}^{\lambda}$$

with $1/\overline{r} \in \Pi_0$ and $1/\overline{q} \in \Pi_1$ would then yield (3.6) for $1/\overline{p} \in \mathrm{conv}(\Pi_0, \Pi_1)$.

4. Proof of Theorem B

4.1. *The case when V is a subspace in $\mathbb{R}^n$*

LEMMA 3. *Any space Φ_V with V a subspace in $\mathbb{R}^n$ is dense in $L^{\overline{p}}(\mathbb{R}^n)$ for $1 < p_k < \infty$, $k = 1,\ldots,n$.*

Proof. Without loss of generality we can take $V = \mathbb{R}^m = \{x \in \mathbb{R}^n : x_{m+1} = \ldots = x_n = 0\}$, $1 \leq m \leq n-1$. We have to justify the passage to the limit in (3.6). Let $x = (x', x'')$ with $x' = (x_1,\ldots,x_m)$. Now $\Delta(x,V) = \varrho(x,V) = |x''| = (x_{m+1}^2 + \ldots + x_n^2)^{1/2}$. Let

$$k(x'') = (2\pi)^{m-n} \int_{\mathbb{R}^{n-m}} [1 - \mu(|y|)] e^{-ix''y}\, dy$$

be the inverse Fourier transform of the function $1 - \mu(|y|) \in C_0^\infty(\mathbb{R}^{n-m})$. Considering $1 - \mu(N|x''|)$ in $\mathbb{R}^n$ as the direct product $1(x') \times [1 - \mu(N|x''|)]$ we see that the operator $I - A_N^V$ can now be interpreted as convolution with the generalized function $\delta(x') \times N^{m-n} k(x''/N)$, where $\delta(x')$ is the Dirac delta-function in $\mathbb{R}^m$. So

$$((I - A_N^V)f)(x) = \int_{\mathbb{R}^{n-m}} k(y) f(x', x'' - Ny)\, dy.$$

Then the desired convergence is derived from the following lemma.

LEMMA 4. *Let $k \in L^1(\mathbb{R}^{n-m})$ and let*

$$(K_N f)(x) = \int_{\mathbb{R}^{n-m}} k(y) f(x', x'' - Ny)\, dy.$$

Then $\|K_N f\|_{\overline{p}} \to 0$ as $N \to \infty$ for all $f \in L^{\overline{p}}(\mathbb{R}^n)$, $1 < p_k < \infty$, $k = 1, \ldots, n$.

Proof. By the generalized Minkowski inequality for mixed norm spaces ([2], Ch. 1, Sec. 1) we have the uniform estimate

$$\|K_N f\|_{\overline{p}} \le c\|f\|_{\overline{p}}, \quad c = \|k\|_{L^1(\mathbb{R}^{n-m})}. \tag{4.1}$$

So, by the Banach–Steinhaus theorem it suffices to prove that $\|K_N f\|_{\overline{p}} \to 0$ for $f \in C_0^\infty(\mathbb{R}^n)$. First, let $p_1 = \ldots = p_n = 2$. Then by the Parseval equality we have

$$(2\pi)^n \|K_N f\|_2^2 = \int_{\mathbb{R}^n} |\widehat{k}(Nx'')\widehat{f}(x)|^2\, dx \to 0 \tag{4.2}$$

by the Lebesgue dominated convergence theorem. Let now $\overline{p}$ be arbitrary, $1 < p_k < \infty$, $k = 1, \ldots, n$. Then there always exists $\overline{r}$ with $1 < r_k < \infty$, $k = 1, \ldots, n$, such that $1/\overline{p}$ is an interior point of the segment with ends $1/\overline{r}$ and $(1/2, \ldots, 1/2)$. The interpolating inequality (2.5) yields the estimate

$$\|K_N f\|_{\overline{p}} \le \|K_N f\|_{\overline{r}}^{1-\lambda} \|K_N f\|_2^{\lambda} \to 0$$

as $N \to \infty$, by (4.1) and (4.2). This proves Lemma 4 and, thereby, Lemma 3.

4.2. *The general case*

LEMMA 5. *Let V be a union of a finite number of shifted hyperplanes: $V = \bigcup_{j=1}^{l} V_j$, $V_j = H_j + x_0^j$, where $x_0^j \in \mathbb{R}^n$ and H_j are subspaces in $\mathbb{R}^n$, $j = 1, \ldots, l$. Then Φ_V is dense in $L^{\overline{p}}(\mathbb{R}^n)$ for $1 < p_k < \infty$, $k = 1, \ldots, n$.*

Proof. The approximating operator A_N^V in this case can be constructed as $\prod_{j=1}^{l} E_j^{-1} A_N^{H_j} E_j$, where $(E_j f)(x) = e^{ixx_0^j} f(x)$ and $A_N^{H_j}$ are the approximating operators constructed for the subspaces H_j (see the proof of Lemma 3). It is easily seen that

$$\|f - A_N^V f\|_{\overline{p}} \le \sum_{j=1}^{l} c_j \|f - A_N^{H_j} f\|_{\overline{p}} \to 0$$

as $N \to \infty$ by Lemma 3.

Proof of Theorem B. If V is bounded then, by definition, V is contained in a union of a finite number of hyperplanes and therefore Φ_V is dense in $L^{\overline{p}}(\mathbb{R}^n)$ by Lemma 5. Let V be unbounded. Every function $f \in L^{\overline{p}}(\mathbb{R}^n)$ can be approximated in $L^{\overline{p}}(\mathbb{R}^n)$ by functions $f_m \in S$ whose Fourier transforms $\widehat{f}_m$ have compact support (this is achieved by the identity approximation with a C^∞-kernel whose Fourier transform has a compact support). Let $V_* = V_*^m = V \cap \operatorname{supp} \widehat{f}_m$. Then V_* is bounded and contained in a union of a finite number of hyperplanes, by definition. Let $\widetilde{V}$ be this union and let $A_N^{\widetilde{V}}$ be the approximating operator constructed in the proof of Lemma 5. Then $\|A_N^{\widetilde{V}} f_m - f_m\|_{\overline{p}} \to 0$ as $N \to \infty$ by Lemma 5. Clearly, $A_N^{\widetilde{V}} f_m \in \Phi_{\widetilde{V}} \subset \Phi_{V^*}$. Moreover, $A_N^{\widetilde{V}} f_m \in \Phi_V$, because $F(A_N^{\widetilde{V}} f_m) \in \Psi_V$, which is the desired result.

5. Proof of Theorem C. Let A_N^V be the operator defined by (3.5).

LEMMA 6. *Let $m(V) = 0$ and $f \in S$. Then $A_N^V f$ uniformly converges to f.*

Proof. We have

$$\begin{aligned} |f(x) - A_N^V f(x)| &= |F^{-1}\{1 - \mu[N\Delta(x, V)]\} Ff| \\ &\le \|\{1 - \mu[N\Delta(x, V)]\}\widehat{f}(x)\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{5.1}$$

It was shown in the proof of Lemma 1 that $\|\{1 - \mu[N\Delta(x, V)]\}\psi\|_{\overline{p}} \to 0$ as $N \to \infty$ for all $\psi \in S$, including the case $p_1 = \ldots = p_n = 1$. So from (5.1) we obtain the assertion of Lemma 6.

To prove Theorem C we first note that every function $f \in L^{\overline{p}}(\mathbb{R}^n)$, for $1 \le p_k < \infty$, can be approximated by functions $f_m \in S$ both in $L^{\overline{p}}$-norm and almost everywhere simultaneously. The approximation in $L^{\overline{p}}$-norm is known ([2], Ch. 1, Sec. 1) and is achieved by means of the identity approximation with a "good" kernel and by multiplication by an expanding smooth step function. So, to guarantee the almost everywhere convergence we only have to choose the kernel of the identity approximation to be radial and monotone (see [13], Ch. III). Thus for any $\varepsilon > 0$ we can construct $g \in S$ such that both $\|f - g\|_{\overline{p}} < \varepsilon/2$ and $|f(x) - g(x)| < \varepsilon/2$ for almost all x. It now remains to apply Lemma 6 to g and take into account that the approximations $A_N^V g$ converge in $L^{\overline{p}}$-norm as well, as was shown in the proof of Theorem A or Theorem B.

6. Proof of Theorem D. We have to prove that $x \in \operatorname{conv}(\Pi_0, \Pi_1)$ if and only if $0 < x_k < 1$ and

$$\left| x_n - 2x_{j_m} + 2x_{j_{m-1}} - \ldots + 2 \cdot (-1)^m x_{j_1} - (-1)^m \frac{1}{2} \right| < \frac{1}{2} \tag{6.1}$$

for all $1 \le j_1 < \ldots < j_m \le n - 1$ and every $m = 1, \ldots, n - 1$. Clearly, $\operatorname{conv}(\Pi_0, \Pi_1)$ is the convex hull of $2n$ points

$$a^1 = (0, 0, \ldots, 0), \ a^2 = (\tfrac{1}{2}, 0, \ldots, 0), \ \ldots, \ a^n = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, 0)$$

and

$$a^{n+1} = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, 1), \ a^{n+2} = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, 1, 1), \ \ldots, \ a^{2n} = (1, 1, \ldots, 1).$$

So, as is well known, $\operatorname{conv}(\Pi_0, \Pi_1)$ is the set of points x representable as

$$x = \sum_{k=2}^{2n} \mu_k a^k \tag{6.2}$$

with $\mu_k \geq 0$ such that

$$\sum_{k=2}^{2n} \mu_k \leq 1. \tag{6.3}$$

We observe that (6.1) is equivalent to

$$0 < x_n - 2x_{j_1} + 2x_{j_2} - \ldots + 2x_{j_m} < 1 \tag{6.1'}$$

if m is even, and to

$$0 < 2x_{j_m} - 2x_{j_{m-1}} + \ldots + 2x_{j_1} - x_n < 1 \tag{6.1''}$$

if m is odd.

The "if" part. From (6.2) we obtain

$$2x_j = \sum_{k=j+1}^{2n-j} \mu_k + 2 \sum_{k=2n-j+1}^{2n} \mu_k$$

if $j = 1, \ldots, n-1$ and $x_n = \sum_{k=n+1}^{2n} \mu_k$. Hence direct calculations give

$$x_n - 2x_{j_1} + 2x_{j_2} - \ldots + 2x_{j_m} = \sum_{k=j_m+1}^{j_{m-1}} \mu_k + \ldots + \sum_{k=j_2+1}^{j_1} \mu_k + \sum_{k=n+1}^{2n-j_1} \mu_k + \sum_{k=2n-j_2+1}^{2n-j_3} \mu_k + \ldots + \sum_{k=2n-j_m+1}^{2n} \mu_k$$

if m is even, and

$$2x_{j_m} - 2x_{j_{m-1}} + \ldots + 2x_{j_1} - x_n = \sum_{k=j_m+1}^{j_{m-1}} \mu_k + \sum_{k=j_{m-2}+1}^{j_{m-3}} \mu_k + \ldots + \sum_{k=j_1+1}^{n} \mu_k + \sum_{k=2n-j_1+1}^{2n-j_2} \mu_k + \ldots + \sum_{k=2n-j_m+1}^{2n} \mu_k$$

if m is odd. So the conditions (6.1′) and (6.1″) are satisfied.

The "only if" part. In the linear algebraic system defined by (6.2) we consider the unknowns $\mu_{n+2}, \ldots, \mu_{2n}$ to be free. We define

$$\mu = \sum_{k=n+2}^{2n} \mu_k$$

and put

$$\mu_k = 2x_{k-1} - 2x_k + \mu_{2n+1-k}, \quad k = 2, 3, \ldots, n-1, \tag{6.4}$$

$$\mu_n = 2x_{n-1} - x_n - \mu, \tag{6.5}$$

$$\mu_{n+1} = x_n - \mu. \tag{6.6}$$

Hence

$$\sum_{k=2}^{2n} \mu_k = 2x_1 - \mu_{2n}. \tag{6.7}$$

The direct substitution of (6.4)–(6.6) into (6.2) shows the validity of (6.2) for all free unknowns $\mu_{n+2}, \ldots, \mu_{2n}$. So we only have to prove that they can be chosen in such a way that both $\mu_k \geq 0$, $k = 2, 3, \ldots, 2n$, and (6.3) hold, provided that (6.1′) or (6.1″) is satisfied. We have

$$\max(0, 2x_n - 1) < \mu < \min(x_n, 2x_{n-1} - x_n), \tag{6.8}$$

the former inequality following from (6.5) and (6.6), and the latter being obtained by summing the inequalities $\mu_{2n+1-k} > 2x_k - 2x_{k-1}$, $k = 2, 3, \ldots, n-1$, resulting from (6.4), and using the condition $\mu_{2n} > 2x_1 - 1$ derived from (6.6). Let μ be chosen in the non-empty interval (6.8). Starting from (6.4) we put

$$\mu_k = \max(0, 2x_{2n+1-k} - 2x_{2n-k}) + h, \quad k = n+2, \ldots, 2n-1, \tag{6.9}$$

with $h > 0$. Then $\mu_k > 0$ for $k = 2, \ldots, 2n-1$. As for μ_{2n}, by summing all the inequalities (6.9) we obtain

$$\mu_{2n} = \mu - 2\sum_{j=1}^{n-2} \max(0, x_{j+1} - x_j) - (n-2)h.$$

Since we must have both $\mu_{2n} \geq 0$ and $\mu_{2n} > 2x_1 - 1$, we see that the condition

$$\mu - 2\sum_{j=1}^{n-2} \max(0, x_{j+1} - x_j) - \max(0, 2x_1 - 1) > (n-2)h$$

is sufficient for this purpose. This last condition can be satisfied by the choice of h if

$$\mu > \max(0, 2x_1 - 1) + 2\sum_{j=1}^{n-2} \max(0, x_{j+1} - x_j) =: A, \tag{6.10}$$

which enables us to choose any $h \in (0, \mu - A)$. Since the inequality $\mu_{2n} > 2x_1 - 1$ is the condition (6.3), we see that the requirement (6.10) guarantees the possibility of choosing μ_k to satisfy (6.3). The condition (6.10) together with (6.8) leads to

$$(6.11)\qquad (2x_1-1)_+ + \sum_{j=1}^{n-1}(2x_{j+1}-2x_j)_+ < x_n,$$

where $t_+ = t$ if $t > 0$ and $t_+ = 0$ if $t \le 0$. It remains to observe that (6.11) is satisfied by (6.1′), (6.1″).

References

[1] A. Benedek and R. Panzone, *The spaces L^P, with mixed norm*, Duke Math. J. 28 (1961), 302–324.

[2] O. V. Besov, V. P. Il'in and S. M. Nikol'skiĭ, *Integral Representations of Functions and Embedding Theorems*, Nauka, Moscow, 1975 (in Russian); English transl.: Scripta Series in Math., Winston and Halsted Press, 1979.

[3] S. Helgason, *The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds*, Acta Math. 113 (1965), 153–180.

[4] —, *The Radon Transform*, Birkhäuser, Boston, 1980.

[5] P. I. Lizorkin, *Generalized Liouville differentiation and the function spaces $L_p^r(E_n)$. Embedding theorems*, Mat. Sb. 60 (1963), 325–353 (in Russian).

[6] —, *Generalized Liouville differentiation and multipliers method in the embedding theory of spaces of differentiable functions*, Trudy Mat. Inst. Steklov. 105 (1969), 89–167 (in Russian).

[7] B. Muckenhoupt, R. L. Wheeden and W.-S. Young, *L^2 multipliers with power weights*, Adv. in Math. 49 (1983), 170–216.

[8] S. G. Samko, *The spaces $L^\alpha_{p,r}(\mathbb{R}^n)$ and hypersingular integrals*, Studia Math. 61 (1977), 193–230 (in Russian).

[9] —, *On test functions vanishing on a given set and on division by functions*, Mat. Zametki 21 (1977), 677–689 (in Russian).

[10] —, *On denseness of the Lizorkin type spaces Φ_V in $L_p(\mathbb{R}^n)$*, ibid. 31 (1982), 855–865 (in Russian).

[11] —, *Hypersingular Integrals and their Applications*, Rostov University Publ. House, Rostov-on-Don, 1984 (in Russian).

[12] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives and Some of their Applications*, Nauka i Tekhnika, Minsk, 1987 (in Russian); English transl.: Gordon and Breach Sci. Publ., 1993.

[13] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, N.J., 1970.

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The Bourgain algebra of the disk algebra $A(\mathbb{D})$ and the algebra QA

by

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Abstract. It is shown that the Bourgain algebra $A(\mathbb{D})_b$ of the disk algebra $A(\mathbb{D})$ with respect to $H^\infty(\mathbb{D})$ is the algebra generated by the Blaschke products having only a finite number of singularities. It is also proved that, with respect to $H^\infty(\mathbb{D})$, the algebra QA of bounded analytic functions of vanishing mean oscillation is invariant under the Bourgain map as is $A(\mathbb{D})_b$.

Introduction. Let $A \subseteq B$ be two commutative Banach algebras. The *Bourgain algebra $(A,B)_b$ of A with respect to B* is the set of all $f \in B$ for which $\operatorname{dist}(ff_n, A) := \inf_{g\in A}\|ff_n+g\|_B \to 0$ whenever f_n is a sequence in A converging weakly to zero (i.e., such that $\varphi(f_n) \to 0$ for every bounded linear functional φ on A). In a recent paper [4] Cima, Stroethoff and Yale characterized the Bourgain algebra $(A(\mathbb{D}), L^\infty(\mathbb{D}))_b$ of the disk algebra with respect to the algebra $L^\infty(\mathbb{D})$ of Lebesgue measurable, essentially bounded functions on the unit disk $\mathbb{D}$. They showed that

$$(A(\mathbb{D}), L^\infty(\mathbb{D}))_b = (H^\infty(\mathbb{D}) \cap W(\mathbb{D})) + UC(\mathbb{D}) + V,$$

where

$$W(\mathbb{D}) = \{f \in L^\infty(\mathbb{D}) : \text{for every } \delta > 0 \text{ the set } \{\zeta \in T : \omega(f,\zeta) \ge \delta\} \text{ is finite}\}$$

is the set of all functions in $L^\infty(\mathbb{D})$ whose essential oscillations

$$\omega(f,\zeta_n) = \lim_{\delta\to 0} \operatorname{ess\,sup}\{|f(z)-f(w)| : z,w \in \mathbb{D},\ |z-\zeta_n| < \delta,\ |w-\zeta_n| < \delta\}$$

converge to zero whenever $\zeta_n \in T$ is a sequence of different points of the unit circle T. Moreover,

$$V = \{f \in L^\infty(\mathbb{D}) : \|f\chi_{\mathbb{D}\setminus r\mathbb{D}}\|_\infty \to 0 \text{ as } r \to 1\}\ (^1)$$

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(1) χ_E is the characteristic function of a subset $E \subseteq \mathbb{D}$.