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A sharp correction theorem

by

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Abstract. Under certain conditions on a function space X, it is proved that for every L^∞-function f with $\|f\|_\infty \le 1$ one can find a function φ, $0 \le \varphi \le 1$, such that $\varphi f \in X$, $\operatorname{mes}\{\varphi \ne 1\} \le \varepsilon \|f\|_1$ and $\|\varphi f\|_X \le \operatorname{const}(1+\log \varepsilon^{-1})$. For X one can take, e.g., the space of functions with uniformly bounded Fourier sums, or the space of L^∞-functions on $\mathbb{R}^n$ whose convolutions with a fixed finite collection of Calderón–Zygmund kernels are also bounded.

In 1979 (see [5]), the author obtained the following refinement of the classical correction theorem of D. E. Men'shov:

Let F be a continuous function on the unit circle $\mathbb{T}$, $\|F\|_\infty \le 1$, and $0 < \varepsilon \le 1$. Then there exists a function G with uniformly convergent Fourier series such that $m\{F \ne G\} \le \varepsilon$ and the absolute values of the partial sums of the Fourier series of G are uniformly bounded by $\operatorname{const}(1+\log \varepsilon^{-1})$. (Here and in the sequel, m is normalized Lebesgue measure on $\mathbb{T}$.)

The logarithmic majorant is sharp, as can easily be derived from the logarithmic growth of the L^1-norms of the Dirichlet kernels (see [5]). However, much later, motivated by a question of B. S. Kashin, the author was able to obtain a much stronger statement:

(∗) In the above theorem, one can replace the inequality $m\{F \ne G\} \le \varepsilon$ by $\operatorname{mes}\{F \ne G\} \le \varepsilon \|F\|_{L^1}$ and, moreover, ensure the pointwise majorization $|G| + |F - G| \le (1+\delta)|F|$.

So, the result of 1979 had provided a correct order of magnitude for the Fourier sums of G, but under improper scaling.

The proof of (∗) (in a slightly weaker form, with $m\{F \ne 0\}$ in place of $\|F\|_{L^1}$) can be found in [6]. However, the statement becomes much more spectacular if we renounce all continuity and convergence conditions, i.e., demand only that the Fourier sums of G be uniformly bounded. In this

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case, as we shall see, an analog of $(*)$ with $\delta = 0$ is true; moreover, F can be an arbitrary L^∞-function ($\|F\|_\infty \le 1$).

This analog of $(*)$ can be regarded as a strong existence theorem. Indeed, since $|G| + |F - G| \le |F|$ (where in fact we have equality), G is obtained from F by means of multiplication by a positive function not exceeding 1; in particular, F is supported on the set $\{F \ne 0\} = E$. We note that the existence of functions supported on a given set of positive measure and having uniformly bounded Fourier sums is a nontrivial but well-known fact; however, the "discontinuous" version of $(*)$ yields many interesting new details about this matter.

In this paper we focus on this "discontinuous" version of $(*)$ and its generalizations. The latter are mainly related to the fact that the proof of this version is based on an easily formulated (though nontrivial) property of the space of functions with uniformly bounded Fourier sums, and so can be carried over to any space with a similar property.

The paper is organized as follows. In §1 we formulate the main result for abstract function spaces satisfying certain "axioms". In §2 we give a series of examples of such spaces and formulate the corresponding existence theorems. In particular, we revisit the framework of the paper [2], where an existence theorem for analytic functions with a fixed set of singularities was discussed (in [2] the "abstract" part of [5] was used to produce a rich family of examples). We obtain some additional information in the context of Men'shov's theorem as well. In §3 we prove the main result. Doing that, we deviate from the outline of [6] at many points. Even in the main decomposition lemma, where the calculations are nearly the same as in [6], the accompanying words are quite different. The reason is that the setting we have chosen requires great care in applying duality. Finally, in §4 we prove an analog of the decomposition lemma for the space of functions with Fourier series of power type and uniformly bounded Fourier sums.

Some preceding work in the spirit of this paper should be mentioned. In [3] the reader can find "axioms" close to those introduced in §1, and in [2]–[4] some examples of existence proofs via duality, of the same nature as here.

1. The main theorem. Let (Ω, μ) be a σ-finite measure space and X a Banach space of locally summable functions on Ω ("locally summable" means "summable on every set of finite measure"). Defining

$$L_0^\infty(\mu) = \{f \in L^\infty(\mu) : \mu(\operatorname{supp} f) < \infty\},$$

we see that every function $g \in L_0^\infty(\mu)$ generates a functional Φ_g on X:

$$\Phi_g(x) = \int_\Omega xg\, d\mu, \quad x \in X.$$

We assume that the following two basic properties (A1 and A2) are satisfied.

A1. The natural embedding $X \hookrightarrow L^1_{\mathrm{loc}}(\mu)$ is continuous, and the unit ball of X is weakly compact in $L^1_{\mathrm{loc}}(\mu)$.

(In particular, A1 implies that all functionals Φ_g are continuous).

A2. For every $g \in L_0^\infty(\mu)$, we have the following "weak type $(1, X^*)$-estimate":

$$\mu\{|g| > t\} \le ct^{-1}\|\Phi_g\|_{X^*}, \quad t > 0,$$

where c is a constant depending only on X.

THEOREM 1. *Let $F \in L^\infty(\mu) \cap L^1(\mu)$, $\|F\|_\infty \le 1$. Then for every $0 < \varepsilon \le 1$, there is a function $G \in X$ such that $|G| + |F - G| = |F|$,*

$$\mu\{F \ne G\} \le \varepsilon\|F\|_1 \quad \text{and} \quad \|G\|_X \le C(1 + \log \varepsilon^{-1})$$

(*the constant C depends only on c in* A2).

We note that the set $E = \{F \ne 0\}$ implicitly plays a very important part in Theorem 1. In general, even the existence of at least one nonzero function from X supported on E is nontrivial (though much easier to prove than Theorem 1).

Since $\|F\|_1 \le \mu E$, we deduce that $\mu\{F \ne G\} \le \varepsilon\mu E$ (of course, this is of interest only if $\mu E < \infty$).

Though we are going to prove Theorem 1 only in §3, one small step in this direction is in order now. Let Y be the closure of the set $\{\Phi_g : g \in L_0^\infty(\mu)\}$ in the norm of X^*.

LEMMA 1. *Assume that X satisfies* A1 (A2 *is not needed here*). *Then, under the natural duality of X and X^*, the dual of Y is X (in particular, X is a conjugate space). On the ball of X, the topology $\sigma(X, Y)$ coincides with the weak topology of $L^1_{\mathrm{loc}}(\mu)$.*

The proof is in fact well known. Let φ be a functional on Y of norm at most 1. This functional is completely determined by its values on the functionals of the form Φ_g, so it can be regarded as an element of the algebraic dual to the space $L_0^\infty(\mu)$, which is, in its turn, dual to $L^1_{\mathrm{loc}}(\mu)$. The condition $\|\varphi\| \le 1$ means that φ lies in the bipolar (taken in the algebraic dual to $L_0^\infty(\mu)$) of the unit ball B of X. Since B is weakly compact in $L^1_{\mathrm{loc}}(\mu)$, it follows that the bipolar coincides with B, hence $\varphi \in B$. Now it is easy to see that on B the topology $\sigma(X, Y)$ is stronger than the weak topology of $L^1_{\mathrm{loc}}(\mu)$. Since B is compact for the former topology, the two topologies must coincide. ■

2. Examples

2.1. *Calderón–Zygmund kernels.* In this example, (Ω, μ) is the space $\mathbb{R}^n$ with Lebesgue measure. By a *Calderón–Zygmund kernel* we shall mean a tempered distribution K on $\mathbb{R}^n$ such that $\widehat{K} \in L^\infty$ and off $\{0\}$ K is a measurable function with the following properties:

(i) $|K(x)| \le c|x|^{-n}$, $x \in \mathbb{R}^n \setminus \{0\}$;
(ii) $|K(x-y) - K(x)| \le c|y|/|x|^{n+1}$ for $|y| \le |x|/2$.

It is well known that the convolution with a Calderón–Zygmund kernel is a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and has weak type $(1,1)$:

$$\operatorname{mes}\{|K * f| > \tau\} \le \operatorname{const} \tau^{-1} \|f\|_1;$$

see, e.g., [1].

Roughly speaking, we want to consider uniformly bounded functions for which $K * f$ is also uniformly bounded. However, for $f \in L^\infty(\mathbb{R}^n)$ the convolution $K * f$ is well defined only as an element of the space BMO. Since BMO consists of equivalence classes modulo constants rather than of functions, we need to work with the quotient space $L^\infty(\mathbb{R}^n)/\mathbb{C}$, which naturally embeds in BMO. We denote the norm in $L^\infty(\mathbb{R}^n)/\mathbb{C}$ by $\|\cdot\|$.

Now let $K_1, \ldots, K_l$ be Calderón–Zygmund kernels on $\mathbb{R}^n$. Consider the space $X = \{f \in L^\infty(\mathbb{R}^n) : K_1 * f, \ldots, K_l * f \in L^\infty(\mathbb{R}^n)/\mathbb{C}\}$ and endow it with the norm

$$\|f\| = \max\{\|f\|_\infty, \|K_1 * f\|, \ldots, \|K_l * f\|\}.$$

LEMMA 2. *The space X satisfies properties* A1 *and* A2.

Proof. For simplicity, we consider the case where the collection $K_1, \ldots$ $\ldots, K_l$ consists of a single kernel K (the arguments in the general case are the same). Clearly, $X \subset L^1_{\text{loc}}$. To check A1, it suffices to prove that if $\|f_n\|_\infty \le 1$, $\|Kf_n\| \le 1$ and $f_n \to f$ weakly in L^1_{loc}, then $\|Kf\| \le 1$. But on the ball of L^∞ the weak convergence in L^1_{loc} is the same as the weak* convergence in L^∞. The convolution with K is a weak*-continuous operator from L^∞ to $\text{BMO} = (H^1)^*$ (indeed, this is an adjoint operator, because any Calderón–Zygmund operator maps H^1 to L^1). Therefore, $K * f_n \to K * f$ in the weak* topology of BMO. Now, the unit ball of $L^\infty(\mathbb{R}^n)/\mathbb{C}$ is compact in this topology. Indeed, it is compact in the weak* topology of $L^\infty(\mathbb{R}^n)/\mathbb{C}$, and the canonical embedding $L^\infty(\mathbb{R}^n)/\mathbb{C} \to \text{BMO}$ is weak* continuous (because every function in H^1 is summable and of mean 0).

So, we have checked A1 for X. We pass to A2. By Lemma 1, X is a conjugate space. We note that the above argument actually shows that the natural embedding

$$X \to (L^\infty \oplus (L^\infty/\mathbb{C}))_\infty, \qquad f \mapsto (f, K * f),$$

is a w^*-continuous isometry. Since for every $g \in L^\infty_0$ the functional Φ_g is in the predual of X, it follows that for every $\varepsilon > 0$ there exist two functions $u, v \in L^1(\mathbb{R}^n)$, $\int v = 0$, such that for every $f \in X$,

$$(1) \qquad \int fg = \Phi_g(f) = \int fu + \int (K * f)v,$$

and

$$(2) \qquad \|u\|_1 + \|v\|_1 \le \|\Phi_g\|_{X^*} + \varepsilon.$$

We claim that $g = u + \widetilde{K} * v$, where $\widetilde{K}(t) = K(-t)$. (Once this is established, A2 for X follows from (2) and the weak type $(1,1)$ inequality for the operator of convolution with $\widetilde{K}$.)

To prove the claim, we put $f = \varphi * \psi$ in (1), where $\varphi, \psi \in D(\mathbb{R}^n)$ (such an f is clearly in X). Since $K * \varphi$ is a uniformly bounded C^∞-function, we can write

$$\int [(K*\varphi)*\psi]v = \int_{\mathbb{R}^n} \int_{\operatorname{supp}\psi} (K*\varphi)(t-s)\psi(s)\,ds\,v(t)\,dt = \int [(K*\varphi)^\sim * v]\psi,$$

and so (1) leads to

$$\int [\widetilde{\varphi} * g - \widetilde{\varphi} * u - (\widetilde{K} * \widetilde{\varphi}) * v]\psi = 0, \qquad \psi \in D(\mathbb{R}^n).$$

Thus, $\widetilde{\varphi} * g = \widetilde{\varphi} * u + (\widetilde{K} * \widetilde{\varphi}) * v$ for every $\varphi \in D(\mathbb{R}^n)$. Approximating v in L^1 by functions belonging to $D(\mathbb{R}^n)$ and using the weak type $(1,1)$ inequality for $\widetilde{K}$, we conclude that $(\widetilde{K} * \widetilde{\varphi}) * v = \widetilde{K} * (\widetilde{\varphi} * v)$. Now the claim follows (again by using weak type $(1,1)$) if we let φ run through the elements of some approximate unity. ∎

The following corollary is a specification of Theorem 1 for the space X.

COROLLARY 1. *Let $K_1, \ldots, K_l$ be Calderón–Zygmund kernels on $\mathbb{R}^n$, and let $E \subset \mathbb{R}^n$ be a set of finite positive measure. Then for every $F \in L^\infty(E)$ with $\|F\|_\infty \le 1$ and every $0 < \varepsilon \le 1$, there exists a function G vanishing on $\mathbb{R}^n \setminus E$ such that $\operatorname{mes}\{F \ne G\} \le \varepsilon \|F\|_1$, $|G| + |F - G| = |F|$, and the convolutions $K_1 * G, \ldots, K_l * G$ do not exceed* $\operatorname{const}(1 + \log \varepsilon^{-1})$ *in modulus.* (*The constant depends only on $K_1, \ldots, K_l$.*)

Proof. By Lemma 2, Theorem 1 implies the existence of a function G vanishing off E and satisfying $|G| + |F - G| = |F|$, $\operatorname{mes}\{F \ne G\} \le \varepsilon \|F\|_1$ and $\|K_j * G\| \le c(1 + \log \varepsilon^{-1})$. Since $\operatorname{mes} E < \infty$, G is in L^2, and so $K_j * G$ can be regarded as an L^2-function. Thus, there exist some constants c_j such that $\|K_j * G - c_j\|_\infty \le c(1 + \log \varepsilon^{-1})$. Since

$$R^{-n} \int_{|x| \le R} |K_j * G| \le \operatorname{const} R^{-n/2} \|G\|_2 \to 0 \quad \text{as } R \to \infty,$$

we see that $|c_j| \le c(1 + \log \varepsilon^{-1})$. ∎

Now we are going to discuss two specific versions of the space X.

2.2. *Lipschitz functions.* In [2] a certain function space was employed to give a simple proof of the fact that for every plane compact set E of positive measure there exist nontrivial functions analytic off E and Lipschitzian on $\mathbb{C}$. This was the space X from the preceding subsection generated by a single Calderón–Zygmund kernel z^{-2} on $\mathbb{C}$. Corollary 1 says that there are "very many" bounded functions G supported on E for which $z^{-2} * G$ is also bounded. Setting $\Phi = c(z^{-1} * G)$ for such a G (c is an appropriate universal constant), we obtain $\bar{\partial}\Phi = G$ and $\partial\Phi = cz^{-2} * G$ in the sense of the theory of distributions. Thus, Φ is analytic off E and Lipschitzian everywhere (because its first derivatives are in L^∞). We obtain the following theorem.

THEOREM 2. *Let E be a plane compact set of positive measure. For every essentially bounded function F of norm at most 1 supported on E and every $0 < \varepsilon < 1$, there exists a function Φ analytic off E, satisfying the estimate $|\Phi(z_1) - \Phi(z_2)| \le \text{const}(1 + \log \varepsilon^{-1})|z_1 - z_2|$ $(z_1, z_2 \in \mathbb{C})$, and such that $|\bar{\partial}\Phi| + |\bar{\partial}\Phi - F| = |F|$, $\text{mes}\{\bar{\partial}\Phi \ne F\} \le \varepsilon\|F\|_1$.*

It should be noted that [2] contains a weaker result of the same type based on the results of [5].

2.3. *Harmonic functions with Lipschitz derivatives.* In a similar way, we can establish the following theorem.

THEOREM 3. *Let E be a compact subset of positive measure in $\mathbb{R}^n$ $(n \ge 2)$. For every essentially bounded function F of norm at most 1 supported on E and every $0 < \varepsilon \le 1$, there exists a function Φ on $\mathbb{R}^n$ harmonic off E and differentiable everywhere such that the first derivatives of Φ are Lipschitzian with constant $c(1 + \log \varepsilon^{-1})$, $\text{mes}\{\Delta\Phi \ne G\} \le \varepsilon\|F\|_1$, and $|\Delta\Phi| + |F - \Delta\Phi| = |F|$.*

Hint: We take the space X (as in 2.1) generated by the following collection of Calderón–Zygmund kernels: $K_{ij}(x) = x_i x_j |x|^{-n-2}$, $i, j = 1, \ldots, n$, $i \ne j$; $K_{jj}(x) = |x|^{-n-2} \sum_{1 \le i \le n}(x_i^2 - x_j^2)$, $j = 1, \ldots, n$. Then we apply Corollary 1 and convolve the resulting function G with the fundamental solution of the Laplace equation to obtain Φ. It can easily be seen that, in the sense of the theory of distributions,

$$\partial^2\Phi/\partial x_i \partial x_j = c_1 K_{ij} * G \quad (i \ne j);$$
$$\partial^2\Phi/\partial x_j^2 = c_2 K_{jj} * G + c_3 G$$

(c_1, c_2, c_3 are some constants). So, the theorem follows.

2.4. We can play the same game with some other "elliptic" differential operators. It is probably easier to work with Fourier transforms than with fundamental solutions, as we did in 2.2 and 2.3. Omitting certain details and avoiding accurate statements, we give only a general outline. Consider

a differential operator L of the form

$$L = \sum_{k_1+\ldots+k_n=l} a_{k_1\ldots k_n}(\partial/\partial x_1)^{k_1} \ldots (\partial/\partial x_n)^{k_n}$$

with constant coefficients. Assume that the characteristic polynomial

$$p(\xi) = \sum_{k_1+\ldots+k_n=l} a_{k_1\ldots k_n}(i\xi_1)^{k_1} \ldots (i\xi_n)^{k_n}$$

is nonzero off $\xi = 0$. If $L\Phi = G$, then any derivative of Φ of order l, i.e., any function

$$\Psi = \left(\frac{\partial}{\partial x_1}\right)^{s_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{s_n} \Phi, \qquad s_1 + \ldots + s_n = l,$$

satisfies $\widehat{\Psi} = k\widehat{G}$, where $k(\xi) = (i\xi_1)^{s_1} \ldots (i\xi_n)^{s_n}/p(\xi)$. Since k is of class C^∞ off $\{0\}$ and homogeneous of degree 0, it is the Fourier transform of a Calderón–Zygmund kernel (probably involving a multiple of the δ-function at 0). Therefore, Corollary 1 guarantees the existence of a rich family of functions Φ supported on a given compact set E of positive measure such that $L\Phi = 0$ off E and the derivatives of order $l-1$ of Φ are Lipschitz. "Rich" means that on E the function $L\Phi$ can "mimic" any prescribed L^∞-function F, in the same way as in Theorems 2 and 3.

2.5. *The space U^∞.* Here $(\Omega, \mu) = (\mathbb{T}, m)$, where m is normalized Lebesgue measure on the unit circle. We denote by U^∞ the space of functions $f \in L^\infty(\mathbb{T})$ for which the following norm is finite:

$$\|f\|_U = \sup\Big\{\Big|\sum_{k \le j \le l} \widehat{f}(n)\zeta^n\Big| : \zeta \in \mathbb{T},\ k, l \in \mathbb{Z},\ k \le l\Big\}.$$

It is rather easy to check A1 for U^∞ (see, however, 2.6 for a more general statement). Moreover, by a result of Vinogradov [8], U^∞ satisfies A2 (we note that the proof of this relies upon the Carleson almost everywhere convergence theorem; see 2.6 for more details). So, Theorem 1 specializes as follows.

THEOREM 4. *For every $F \in L^\infty(\mathbb{T})$ with $\|F\|_\infty \le 1$ and every $0 < \varepsilon \le 1$ there exists a function $G \in U^\infty$ with the following properties: $|G| + |F - G| = |F|$; $m\{F \ne G\} \le \varepsilon\|F\|_1$; $\|G\|_{U^\infty} \le \text{const}(1 + \log \varepsilon^{-1})$.*

2.6. *Pointwise control of the Fourier sums.* As before, the set $E = \{F \ne 0\}$ implicitly plays an important part in Theorem 4. In view of the fact that we have a very good pointwise control of G off E ($G = 0$ there), the uniform estimate $O(1 + \log \varepsilon^{-1})$ for the Fourier sums may seem not quite satisfactory. It would be desirable to produce G whose Fourier sums are controlled better at the points where F vanishes. Of course, we cannot expect too much (e.g., const $|\widehat{F}(0)|$ must minorize the pointwise supremum of the Fourier sums of G). However, something can be said.

Let φ be a weight (positive summable function) on $\mathbb{T}$ satisfying the Muckenhoupt condition A_1. We recall that this means that $M\varphi \leq \text{const}\,\varphi$, where M is the Hardy–Littlewood maximal operator:

$$Mg(\zeta) = \sup |I|^{-1} \int_I |g|,$$

where the supremum is taken over all arcs I containing ζ.

For the basic measure space we now take $(\mathbb{T}, \varphi dm)$. Define

$$(S_k f)(\zeta) = \begin{cases} \sum_{0\leq j\leq k} \widehat{f}(j)\zeta^j & \text{for } k \geq 0,\\ \sum_{k\leq j<0} \widehat{f}(j)\zeta^j & \text{for } k < 0.\end{cases}$$

Consider the space X consisting of all functions x for which

$$\|x\|_X := \sup_{k\in\mathbb{Z}} \|\varphi^{-1} S_k(\varphi x)\|_\infty < \infty.$$

(It is probably worth noting that every A_1-weight on $\mathbb{T}$ is bounded away from 0.)

LEMMA 3. *On the measure space* $(\mathbb{T}, \varphi dm)$, *the space* X *satisfies conditions* A1 *and* A2.

Proof. The space X contains all functions of the form p/φ, where p is a trigonometric polynomial. Consider the natural embedding $\alpha : X \to (\ldots \oplus L^\infty \oplus L^\infty \oplus L^\infty \oplus \ldots)_\infty$,

$$\alpha(x) = \{\varphi^{-1} S_k(\varphi x)\}_{k\in\mathbb{Z}}.$$

Let a sequence $\{x_n\}$ from the unit ball of X converge weakly to some x in $L^1(\varphi dm)$. We claim that x also lies in the unit ball of X and $\alpha x_n \to \alpha x$ in the weak* topology of the space $(\sum_{\mathbb{Z}} \oplus L^\infty)_\infty$ (regarded as the dual of $(\sum_{\mathbb{Z}} \oplus L^1(\mathbb{T}, m))_1$). Indeed, for every fixed l we have

$$(\varphi x_n)^\wedge(l) = \int x_n \bar{z}^l \varphi\, dm \to \int x \bar{z}^l \varphi\, dm = (\varphi x)^\wedge(l) \quad \text{as } n \to \infty,$$

whence $\lim_{n\to\infty} S_k(\varphi x_n) = S_k(\varphi x)$ pointwise for every fixed k, and the claim follows.

So, we have checked A1 for X. By Lemma 1, X is a conjugate space, $X = Y^*$, and we see that α is a weak*-continuous isometric embedding. So, $\alpha = \beta^*$, where $\beta : (\sum_{\mathbb{Z}} \oplus L^1(\mathbb{T}, m))_1 \to Y$ is a quotient map.

The latter fact constitutes the first step in checking A2 for X. Taking a trigonometric polynomial y on $\mathbb{T}$, we obtain the following representation for the functional Φ_y (a notation from §1):

$$\Phi_y = \beta\{u_n\}_{n\in\mathbb{Z}},$$

where $u_n \in L^1(\mathbb{T}, m)$, $\sum \int |u_n| \leq \|\Phi_y\| + \varepsilon$. This formula for Φ_y can be rewritten as follows:

$$\int xy\varphi\, dm = \sum_{n\in\mathbb{Z}} \int S_n(x\varphi) u_n \varphi^{-1}\, dm, \quad x \in X.$$

Take N satisfying $\operatorname{spec} y \subset [-N, N]$. We put $x = z^l/\varphi$ in the latter formula, where $l \in [-N, N]$. If $l \geq 0$, we have $S_n(z^l) = z^l$ for $n \geq N$ and $S_n(z^l) = 0$ for $n < -N$ (in fact for $n < 0$), whence

$$\int yz^l = \int z^l \sum_{-N\leq n\leq N} \overline{S_n(\overline{u}_n\varphi^{-1})} + \int z^l \sum_{n>N} u_n\varphi^{-1}.$$

Keeping in mind a similar formula for $l < 0$, we obtain

$$\overline{y} = \sum_{-N\leq n\leq N} S_n(\overline{u}_n\varphi^{-1}) + S_{-N}\Big(\sum_{n>N} \overline{u}_n\varphi^{-1}\Big) + S_N\Big(\sum_{n<-N} \overline{u}_n\varphi^{-1}\Big).$$

Recall that the estimate to be proved is

$$\int_{|y|>t} \varphi \leq ct^{-1}\|\Phi_y\|$$

with c independent of y. Denoting by $\mathbb{P}$ the Riesz projection,

$$\mathbb{P}f = \sum_{n\geq 0} \widehat{f}(n) z^n,$$

and rewriting the partial sum operators S_n in terms of $\mathbb{P}$, we obtain

$$\begin{aligned}\overline{y} = &- \sum_{0\leq n\leq N} z^{n+1}\mathbb{P}(\bar{z}^{n+1}\overline{u}_n\varphi^{-1}) + \sum_{-N\leq n<0} z^n \mathbb{P}(\bar{z}^n\overline{u}_n\varphi^{-1})\\ &+ \Big[\mathbb{P}\Big(\sum_{0\leq n\leq N} \overline{u}_n\varphi^{-1} - \sum_{-N\leq n<0} \overline{u}_n\varphi^{-1}\Big)\\ &+ S_{-N}\Big(\sum_{n>N} \overline{u}_n\varphi^{-1}\Big) + S_N\Big(\sum_{n<-N} \overline{u}_n\varphi^{-1}\Big)\Big].\end{aligned}$$

Now, $\mathbb{P}$ is a Calderón–Zygmund singular integral operator, and so it is of weak type $(1,1)$ with respect to any A_1-weight; see [1]. Consequently, S_N has the same property uniformly in N. Thus, the terms in the square brackets are of right order (we recall that, by the choice of u_n's, $\|\sum |\overline{u}_n|\varphi^{-1}\|_{L^1(\varphi dm)} \leq \|\Phi_y\| + \varepsilon$).

It remains to estimate the two sums outside the square brackets. Of course, they are treated in one and the same way, so we consider only the first of them. After an evident change of variables ($\bar{z}^{n+1}\overline{u}_n\varphi = v_n$), the inequality we need takes the form

$$\int_{A_\lambda} \varphi \leq c\lambda^{-1} \sum_{0\leq n\leq N} \int |v_n|\varphi\, dm, \tag{3}$$

where

$$A_\lambda = \Big\{ \Big| \sum_{0\le n\le N} z^{n+1}\mathbb{P}_+(v_n) \Big| > \lambda \Big\},$$

with c independent of N and the v_n's. This latter estimate is known. See, e.g., Theorem 2.3 in part III of [7], where a similar fact for the Hilbert transform on $\mathbb{R}$ (in place of the operator $\mathbb{P}$) was established. The same proof as in [7] works in our situation.

It should be noted that (3) is highly nontrivial. The basic fact used to prove it is the Carleson–Hunt estimate $\int(\sup|S_k f|)^p \le C_p^p \int |f|^p$, $1<p<\infty$ (we emphasize that this is needed for p arbitrarily close to 1). To derive (3) from this estimate, the modern machinery of weighted inequalities for singular integrals is needed.

Anyway, we have checked A2 for X. It should be noted that the result of Vinogradov [8] leading to A2 for U^∞ was precisely inequality (3) for $\varphi\equiv 1$. ∎

Now, Lemma 3 allows us to apply Theorem 1 to the space X: if $\|F\|_\infty \le 1$ and $0<\varepsilon\le 1$, then there exists a function $G\in X$ such that $\|G\|_X = \sup_n \|\varphi^{-1}S_n(\varphi x)\|_\infty \le c(1+\log\varepsilon^{-1})$, $\int_{F\ne G}\varphi \le \varepsilon\int|F|\varphi$, and $|G|+|F-G| = |F|$. Multiplying F and G by φ (note that $f = F\varphi$ can be an arbitrary function with $|f|\le\varphi$) and slightly changing the attitude (it is natural to start with f, not with φ), we obtain the following statement.

THEOREM 5. *Let f be a measurable function on $\mathbb{T}$. For any A_1-weight φ satisfying $|f|\le\varphi$ and any $\varepsilon>0$, one can find a function g with the following properties*:

(i) $|g|+|f-g| = |f|$,
(ii) $|S_n g| \le \mathrm{const}(1+\log\varepsilon^{-1})\varphi$,
(iii) $\int_{\{f\ne g\}}\varphi \le \varepsilon \int |f|$.

The constant in (ii) *depends on the A_1-constant for φ.*

Setting $\varphi\equiv 1$, we regain Theorem 4. In general, we see that conditions (ii) and (iii) compete (making φ smaller, we get a better majorant for the Fourier sums, but a worse control of the set $\{f\ne g\}$). We give two samples of the choice of φ in the case $\|f\|_\infty\le 1$ (they are, in a sense, extreme). Recall that if $h\ge 0$ is (say) a bounded measurable function and $p>1$, then the function $\varphi = (Mh^p)^{1/p}$ is an A_1-weight (with A_1-constant depending only on p); see, e.g., [1]. Moreover, to a certain extent, φ "mimics" the behaviour of h (in the mean, at least: for $q>p$ we have $\|\varphi\|_q \le C_{p,q}\|h\|_q$).

So, let $\|f\|_\infty\le 1$. We can put $\varphi = (M|f|^p)^{1/p}$ ($p>1$) in Theorem 5. This provides a "relatively good" pointwise control of the partial Fourier sums of g. As to the set $\{f\ne g\}$, inequality (iii) (and even a weaker one $\int_{\{f\ne g\}}|f| \le \varepsilon\int|f|$) tells us something, but in general not too much. If "almost all" of the mass of f is concentrated on a small interval I, f is "very small" on the set $\{f\ne 0\}\setminus I$, and this latter set is "relatively big", we learn almost nothing about the Lebesgue measure of the set $\{f\ne g\}$. (Of course, we know from (i) that g vanishes where f does.)

Another possibility is to take $\varphi = (M\chi_E)^{1/p}$, where $E=\{f\ne 0\}$. It is easily seen that $\varphi = 1$ a.e. on E (and $\varphi\le 1$ off E). Since $g=0$ off E, (iii) turns into $m\{f\ne g\}\le\varepsilon\int|f|$, i.e., the same as in Theorem 4. At the same time, (ii) still provides some control for the S_n's (though not so sharp as under the former choice of φ) (we recall that $\varphi\chi_{\mathbb{T}\setminus E}$ is "small in the mean" provided mE is small).

3. Proof of the main theorem. We have already seen that any space X with property A1 (see §1) is a dual space. By Lemma 1, its predual Y is the norm closure of the set Y_0 of all functionals of the form Φ_g, $g\in L_0^\infty(\mu)$. Now, A2 implies that the mapping α, $\alpha(\Phi_g)=g$, is a bounded linear operator from Y_0 to the Lorentz space $L^{1,\infty}(\mu)$. (The latter is also called weak-$L^1(\mu)$.) Therefore, α extends by continuity to the whole of Y. In the sequel we shall write Φ_* for $\alpha(\Phi)$ ($\Phi\in Y$).

It would be desirable to identify Φ and Φ_* (and to regard Y as a function space). Unfortunately, in the generality we have adopted, there seems to be no reason for the implication $\Phi_*=0\Rightarrow\Phi=0$. However, this is not really an obstacle. We do obtain a function space passing to an appropriate quotient. We consider at once a more general setting.

Let $E\subset\Omega$ be a measurable subset of nonzero measure. We define $V(E)=\{\Phi\in Y:\Phi_*|_E=0 \text{ a.e.}\}$ (A2 implies that $V(E)$ is closed in Y) and $Y(E)=Y/V(E)$. For every $y\in Y(E)$ and any representative Φ of Y, the function $\Phi_*|_E$ does not depend on the particular choice of Φ. Moreover, y is uniquely determined by this function (indeed, if $\Phi_*|_E=\Psi_*|_E$, then $\Phi-\Psi\in V(E)$).

This enables us to identify $Y(E)$ with a space of functions on E (in the sequel we always treat the elements of $Y(E)$ as functions). The dual space $X_E = Y(E)^*$ coincides with the annihilator of $V(E)$ in X. It follows that every function from X_E vanishes on $\Omega\setminus E$. Indeed, $\Phi_g\in V(E)$ for every $g\in L_0^\infty(\mu)$ vanishing on E; hence for every $G\in X_E$,

$$0 = \langle\Phi_g, G\rangle = \int Gg\,d\mu.$$

(It should be noted that in specific examples the question as to whether X_E coincides with the set $\{G\in X: G|_{\Omega\setminus E}=0 \text{ a.e.}\}$ leads to difficult problems. There seems to be no reason for a positive answer in the general setting.)

Eventually we shall see that there are a lot of functions in X_E, but to emphasize that our results are of "pure existence" nature we show at once that $X_E \neq \{0\}$. Indeed, otherwise we would have $V(E) = Y$, which is clearly false (any functional Φ_g with g satisfying $\mu\{\operatorname{supp} g \cap E\} \neq 0$ is not in $V(E)$). This argument is extremely simple, but does not exhibit any particular function lying in X_E.

We are going to prove the following (apparently stronger) version of Theorem 1.

THEOREM 1′. *Let $E \subset \Omega$ be a set of positive measure, $F \in L^\infty(\mu) \cap L^1(\mu)$ a function supported on E, $\|F\|_\infty \le 1$, and $0 < \varepsilon \le 1$. Then there exists a function $G \in X_E$ such that $|G| + |F - G| = |F|$, $\mu\{F \neq G\} \le \varepsilon \|F\|_1$, and $\|G\|_X \le C(1 + \log \varepsilon^{-1})$ (with C depending only on c in A2).*

Before passing to the proof, we note that, under the above identification of $Y(E)$ with a function space, we can use (with some precautions) the ordinary formula for the pairing of X_E and $Y(E)$:

$$\langle F, g \rangle = \int F g \, d\mu. \tag{4}$$

To be rigorous, this formula works if g is an L_0^∞-function supported on E. In the examples of §2 (where all functions belonging to X were bounded) one could extend the formula to $g \in L^1(\mu) \cap Y(E)$ with $\operatorname{supp} g \subset E$. However, beyond this limit we can only say that g can be approximated by functions for which the formula makes sense and is true.

In the sequel we shall use the duality (4) for all function spaces on E that will occur (even for weighted L^p).

The following lemma is in the core of the proof of Theorem 1′.

LEMMA 4. *Let f be a nonzero function belonging to $L^\infty(\mu) \cap L^1(\mu)$, $\|f\|_\infty \le 1$, and let $e = \{f \neq 0\}$. There exists a constant A independent of f such that f is representable in the form $f = g + h$, where $g \in X_e$, $\|g\|_X \le A$, $(\int_e |h|^2 |f|^{-2} \, d\mu)^{1/2} \le 8^{-1} \|f\|_1^{1/2}$ and $|f| = |g| + |h|$.*

Remark. By homogeneity, the condition $\|f\|_\infty \le 1$ can be lifted. If $f \in L^\infty(\mu) \cap L^1(\mu)$ and $\|f\|_\infty \le B$, then $f = g + h$, with $|f| = |g| + |h|$,

$$\|g\|_X \le AB, \quad \Big(\int |h|^2 |f|^{-2} \, d\mu \Big)^{1/2} \le 8^{-1} (B^{-1} \|f\|_1)^{1/2}.$$

(We note that the choice $B = \|f\|_\infty$ is not necessarily the best possible, because the last two estimates compete.)

We postpone the proof of Lemma 4 and derive Theorem 1′ from this lemma. This will be done by iterations. First, we claim that, given F (as in Theorem 1′), there exists a function $G_0 \in X_E$ for which

$$\mu\{F \neq G_0\} \le 2^{-1} \|F\|_1, \quad \|G_0\|_X \le 2A, \quad |G_0| + |F - G_0| = |F|.$$

To prove the claim, we construct three sequences of functions $\{g_n\}_{n\ge0}$, $\{v_n\}_{n\ge0}$, and $\{u_n\}_{n\ge0}$ such that

$$F = g_0 + \ldots + g_n + v_0 + \ldots + v_n + u_n,$$
$$|F| = |g_0| + \ldots + |g_n| + |v_0| + \ldots + |v_n| + |u_n|,$$
$$g_n \in X_E, \quad \|g_n\|_X \le 2^{-n} A,$$
$$\mu(\operatorname{supp} v_n) \le 4^{-1} 2^{-n} \|F\|_1, \quad |u_n| \le 4^{-n-1} |F|.$$

This is done by induction. At the zero step, we apply Lemma 4 to F and obtain $F = g + h, \ldots$, etc. Then we set $g_0 = g$, $u_0 = h\chi_{\{|h| \le 4^{-1}|F|\}}$, $v_0 = h - u_0$. Clearly, $|F| = |g_0| + |v_0| + |u_0|$ because u_0 and v_0 are disjoint. Now

$$\mu(\operatorname{supp} v_0) \le 16 \int |h|^2 |F|^{-2} \le 4^{-1} \|F\|_1,$$

as required.

To pass from n to $n+1$, we apply the remark after Lemma 4 to u_n, setting $B = 2^{-n-1}$ (this choice of B is possible, because $|u_n| \le 4^{-n-1} |F| \le 4^{-n-1}$). We obtain $u_n = g + h$, where $|u_n| = |g| + |h|$ and

$$g \in X_{\operatorname{supp} u_n} \subset X_E, \quad \|g\|_X \le A 2^{-n-1}, \quad \int |h|^2 |u_n|^{-2} \le 64^{-1} 2^{n+1} \|u_n\|_1.$$

Since $|u_n| \le 4^{-n-1} |F|$, the latter inequality yields

$$4^{2n+2} \int |h|^2 |F|^{-2} \le 64^{-1} 2^{-n-1} \|F\|_1.$$

Now we set $g_{n+1} = g$, $u_{n+1} = h\chi_{\{|h| \le 4^{-n-2}|F|\}}$, $v_{n+1} = h - u_{n+1}$. Then, by the preceding estimate,

$$4^{-2n-4} 4^{2n+2} \mu(\operatorname{supp} v_{n+1}) \le 64^{-1} 2^{-n-1} \|F\|_1,$$

whence $\mu(\operatorname{supp} v_{n+1}) \le 4^{-1} 2^{-n-1} \|F\|_1$, and this finishes the induction step.

Clearly, the series $\sum_{n\ge0} g_n$ converges in X_E to some function G_0 with $\|G_0\|_X \le 2A$. We have

$$|g_0 + \ldots + g_n| + |F - g_0 - \ldots - g_n| \le |g_0 + \ldots + g_n| + |v_0 + \ldots + v_n + u_n| \le |F|,$$

whence $|G_0| + |F - G_0| \le |F|$ (we recall that convergence in X implies convergence in L^1_{loc}), i.e., $|G_0| + |F - G_0| = |F|$. Finally, $u_n \to 0$ uniformly, whence

$$\mu\{G_0 \neq F\} \le \mu\Big(\bigcup_{n\ge0} \operatorname{supp} v_n \Big) \le 2^{-1} \|F\|_1,$$

and the claim is established.

Now we iterate once again. We apply the above claim to $F - G_0$ (we have $|F - G_0| \le |F|$, so that $\|F - G_0\|_\infty \le 1$). This yields $G_1 \in X_E$

with $|F-G_0| = |G_1| + |F-G_0-G_1|$, $\|G_1\|_X \le 2A$, and

$$\mu\{G_0+G_1\ne F\} = \mu\{G_1 \ne F-G_0\} \le 2^{-1}\|F-G_0\|_1 \le 2^{-1}\int_{\{F\ne G_0\}} |F| \le 2^{-1}\mu\{F\ne G_0\} \le 4^{-1}\|F\|_1.$$

Repeating this N times, we obtain some functions $G_0,\dots,G_N \in X_E$, all of norm at most $2A$, such that

$$|G_0|+\dots+|G_N|+|F-G_0-\dots-G_N| = |F|$$

and

$$\mu\{G_0+\dots+G_N\ne F\} \le 2^{-N}\|F\|_1.$$

Now, $\|G_0+\dots+G_N\|_X \le 2(N+1)A$. So, putting $G = G_0+\dots+G_N$, we obtain the statement of Theorem 1′ for $\varepsilon = 2^{-N}$. ∎

Proof of Lemma 4. We consider two spaces Z_1 and Z_2 of pairs of measurable functions on e:

$$Z_1 = \{(g,h) : g\in X_e,\ h\in L^2(e,|f|^{-2}d\mu)\},$$
$$Z_2 = \{(g,h) : |g|+|h| \le \text{const}\,|f|\}.$$

The norms are given by

$$\|(g,h)\|_{Z_1} = \|(g,h)\|^{(1)} = \max\Big\{A^{-1}\|g\|_X,\ 8\|f\|_1^{-1/2}\Big(\int |h|^2|f|^{-2}d\mu\Big)^{1/2}\Big\},$$
$$\|(g,h)\|_{Z_2} = \|(g,h)\|^{(2)} = \|(|g|+|h|)|f|^{-1}\|_\infty.$$

We must prove that for some A (independent of f) the intersection of the unit balls of Z_1 and Z_2 contains a pair (g,h) with $g+h=f$.

Clearly, Z_1 and Z_2 are dual spaces. We need to write explicitly their preduals W_1 and W_2 (we keep in mind the pairing $\langle (u,v),(g,h)\rangle = \int (ug+vh)\,d\mu$, in accordance with (4) and remarks after it). We have

$$W_1 = \{(y,v) : y\in Y(e),\ v\in L^2(e,|f|^2\,d\mu)\},$$
$$W_2 = \{(y,v) : y,v \in L^1(e,|f|\,d\mu)\},$$

the norms being given by

$$\|(y,v)\|^{(1)} = A\|y\|_{Y(e)} + 8^{-1}\|f\|_1^{1/2}\|v\|_{L^2(e,|f|^2\,d\mu)},$$
$$\|(y,v)\|^{(2)} = \int (|y|\vee|v|)|f|\,d\mu,$$

respectively. The main duality trick (justifying the subsequent calculations) will be based on the following statement.

SUBLEMMA. *The intersection of the spaces W_1 and W_2 is dense in each of them.*

Proof. Already the set of pairs (a,b), where a and b are L^∞-functions with support of finite measure contained in e, is dense both in W_1 and in W_2. This is nearly obvious. Only the density of the a's of the above form in $Y(e)$ is probably not quite immediate. But if $y\in Y(e)$ and Φ is any representative of the class y, we can approximate y by a functional of the form Φ_z, where $z\in L_0^\infty(\mu)$. Then the function $\chi_e z$ approximates y within the same accuracy, because $\Phi_{z-\chi_e z}\in V(e)$. ∎

We endow the spaces $Z_1\cap Z_2$ and W_1+W_2 with the standard norms of intersection and sum, respectively. The sublemma implies that

$$(W_1+W_2)^* = Z_1\cap Z_2,$$

with equality of norms. (We note that *a priori* it was not clear whether $Z_1\cap Z_2$ contains any pair (g,h) with $g\ne 0$. The only proof of this fact known to the author is via the above formula.)

With respect to our standard duality, the dual M of the space $L^1(e,|f|\,d\mu)$ consists of all functions w for which $w|f|^{-1}\in L^\infty(e,\mu)$ (and for the norm we have the formula $\|w\|_M = \|w|f|^{-1}\|_\infty$). Define an operator $T : Z_1\cap Z_2\to M$ by setting $T(g,h) = g+h$. We shall show that, under a proper choice of A, the image of the unit ball of $Z_1\cap Z_2$ under T contains the unit ball of M (once this is done, Lemma 4 follows, because the latter ball contains f).

Clearly, $T = S^*$, where $S : L^1(e,|f|\,d\mu)\to W_1+W_2$ acts as follows: $S(w) = (w,w)$. To prove the above claim, it suffices to show that S is an isometric embedding. The formula for the norm in W_2 immediately yields $\|S\|\le 1$.

Let us check that $\|Sw\|\ge\|w\|$ for all w. We must prove that, whenever $(w,w) = (y_1,v_1)+(y_2,v_2)$ ($(y_i,v_i)\in W_i$, $i=1,2$), we have

$$\|(y_1,v_1)\|^{(1)} + \|(y_2,v_2)\|^{(2)} \ge \int |w|\,|f|\,d\mu.$$

We put $a = \{|y_1|>\lambda\}$ (λ is a number to be chosen later). Since $w = y_1+y_2 = v_1+v_2$, we obtain

$$\int |y|\,|f|\,d\mu = \int_a |v_1+v_2|\,|f|\,d\mu + \int_{e\setminus a}|y_1+y_2|\,|f|\,d\mu \le \int_a |v_1|\,|f|\,d\mu + \Big[\int_a |v_2|\,|f|\,d\mu + \int_{e\setminus a}|y_2|\,|f|\,d\mu\Big] + \int_{e\setminus a}|y_1|\,|f|\,d\mu. \tag{5}$$

Since the sets a and $e\setminus a$ are disjoint, the middle term (i.e., the sum in the square brackets) is dominated by

$$\int_e (|y_2|\vee|v_2|)|f|\,d\mu = \|(y_2,v_2)\|^{(2)}.$$

For the first term we can write

$$\int_a |v_1|\,|f|\,d\mu \le (\mu a)^{1/2}\|v_1\|_{L^2(|f|^2\,d\mu)} \le (c\lambda^{-1}\|y_1\|_{Y(e)})^{1/2}\|v_1\|_{L^2(|f|^2\,d\mu)},$$

by A2. Let us estimate the third term. Fixing ε, $0<\varepsilon<\lambda$, we obtain

$$\int_{e\setminus a} |y_1|\,|f|\,d\mu = \int_{e\setminus a} (|y_1|\wedge\varepsilon)|f|\,d\mu + \int_{e\setminus a} (|y_1|-|y_1|\wedge\varepsilon)|f|\,d\mu$$

$$\le \varepsilon\|f\|_1 + \int_{e\setminus(a\cup\{|y_1|\le\varepsilon\})} (|y_1|-\varepsilon)\,d\mu.$$

The last integral can be rewritten in the form $\int_0^{\lambda-\varepsilon}\alpha(\tau)\,d\tau$, where

$$\alpha(\tau)=\mu\{\omega\in e\setminus(a\cup\{|y_1|\le\varepsilon\}) : |y_1(\omega)|-\varepsilon>\tau\}.$$

Since

$$\alpha(\tau)\le\mu\{|y_1|>\tau+\varepsilon\}\le c\|y_1\|_{Y(e)}(\tau+\varepsilon)^{-1},$$

we obtain

$$\int_{e\setminus a}|y_1|\,|f|\,d\mu\le\varepsilon\|f\|_1+c\|y_1\|_{Y(e)}\log\frac{\lambda}{\varepsilon}.$$

The expression on the right attains its minimum for $\varepsilon=c\|y_1\|_{Y(e)}/\|f\|_1$ (so, when fixing λ at the end, we must not forget to choose it greater than the latter quantity).

Collecting the estimates, we obtain

$$\int |w|\,|f|\,d\mu\le c\|y_1\|_{Y(e)}\Big(1+\log\frac{\lambda\|f\|_1}{c\|y_1\|_{Y(e)}}\Big)$$
$$+\Big(\frac{c\|y_1\|_{Y(e)}}{\lambda}\Big)^{1/2}\|v_1\|_{L^2(|f|^2\,d\mu)}+\|(y_2,v_2)\|^{(2)}.$$

Now, we choose λ in such a way that the coefficient of $\|v_1\|_{L^2(|f|^2\,d\mu)}$ be equal to $8^{-1}\|f\|_1^{1/2}$ (as dictated by the formula for the norm in W_1): $\lambda=64c\|y_1\|_{Y(e)}/\|f\|_1$ (we indeed get $\lambda>\varepsilon$, as required). It follows that if A is taken to be $c(1+\log 64)$, the last estimate turns into the desired inequality $\|w\|\le\|(y_1,v_1)\|^{(1)}+\|(y_2,v_2)\|^{(2)}$. ∎

4. Decomposition of analytic functions. In this section we consider a space X satisfying conditions A1 and A2 on the measure space $(\mathbb{T},m)$. It is implicit in the proof of Lemma 4 that, for any $\eta>0$, any function $f\in L^\infty(\mathbb{T})$ with $\|f\|_\infty\le 1$ can be represented in the form $f=g+h$, where $g\in X$, $|f|=|g|+|h|$ and

$$\|g\|_X\le\eta,\qquad\Big(\int|h|^2|f|^{-2}\,dm\Big)^{1/2}\le\|f\|_1^{1/2}c_1\exp(-c_2\eta).$$

(An alternative way to derive this sharpened version of Lemma 4 is to apply Theorem 1′ and to note that $\int|F-G|^2|F|^{-2}\le m\{F\ne G\}\le\varepsilon\|F\|_1$.)

Now a question arises: If f is analytic (i.e., $f\in H^\infty$), can one ensure that g and h also be analytic (i.e., satisfy $\widehat{g}(u)=\widehat{h}(u)=0$ for $n<0$)? It is rather easy to see that one cannot expect as much as the equality $|f|=|g|+|h|$ in this setting. Nevertheless, the following result is true.

THEOREM 6. *Let $f\in H^\infty$, $\|f\|_\infty\le1$, $\eta>0$. Then there exist functions $g\in X_{\mathbb{T}}$ and h such that $f=g+h$ and*

$$g/f\in H^\infty,\quad |g/f|\le1,\quad \|g\|_X\le\eta;$$
$$h/f\in H^2,\quad \Big(\int|h|^2|f|^{-2}\,dm\Big)^{1/2}\le C\|f\|_1^{1/2}\exp(-C'\eta).$$

We recall that, in the notation of §3, $X_{\mathbb{T}}$ is the annihilator of the set $\{\Phi\in Y:\Phi_*=0\}$.

This result is very much in the spirit of the preceding part of the paper. Namely, g and h have zeros where f has (no matter where the zeros are: in the disc or on the boundary). To be more precise, g and h are analytic functions divisible by the inner part of f (we note that the "metric" conditions on g and h, except $\|g\|_X\le\eta$, are expressed in fact in terms of the outer part of f).

Proceeding to the proof, we note first that, quite unexpectedly, we do not need to take any special care of the analyticity properties of g. Indeed, let $f\in L^\infty(\mathbb{T})$ be an arbitrary function with $f\ne0$ a.e. Then multiplication by f is an isometry of $L^2(m)$ onto $L^2(|f|^{-2}dm)$. In particular, it follows that $fH^2=\{h:h=fg,\ g\in H^2\}$ is a closed subspace of $L^2(|f|^{-2}dm)$. The following proposition is true.

PROPOSITION 1. *For any $\eta>0$, any f as above with $\|f\|_\infty\le1$ can be represented in the form $f=g+h$, where $g\in X_{\mathbb{T}}$, $h\in fH^2$, $|g|\le|f|$, $\|g\|_X\le\eta$, and*

$$\Big(\int|h|^2|f|^{-2}\,dm\Big)^{1/2}\le C\|f\|_1\exp(-C'\eta).$$

This statement readily implies Theorem 6. Indeed, if $f\in H^\infty$, then $f\ne0$ a.e. So we can apply Proposition 1 to get $f=g+(h/f)f$ with $h/f\in H^2$. Thus, g belongs to H^2 and is divisible by the inner part of f (in the sense that the quotient is still in H^2). Now the property $g/f\in H^\infty$ follows from the boundary estimate $|g|\le|f|$. ∎

Proof of Proposition 1. Since $|f|^{-2}\ge1$ a.e., the identity mapping of $L^2(|f|^{-2}dm)$ into $L^1(m)$ (even into $L^2(m)$, in fact) is continuous. Consequently, the unit ball of $L^2(|f|^{-2}dm)$ is weakly compact in $L^1(m)$, and the same is true for the unit ball of fH^2.

We consider the space $Z=\{g\in X_{\mathbb{T}}:|g|\le \mathrm{const}\,|f|\}$ and endow it with the norm

$$\|g\|_Z=\max\{\eta^{-1}\|g\|_X,\|g/f\|_\infty\}.$$

As in the preceding section, we easily see that

$$Z=(\eta Y(\mathbb{T})+L^1(|f|dm))^*$$

(because the intersection of the summands on the right is dense in each of them). Clearly, the unit ball B_1 of Z is weakly compact in $L^1(m)$. Denoting by B_2 the ball of radius t and centered at 0 in fH^2, we see that the statement to be proved is

$$f\in B_1+B_2,$$

for some $t\sim\|f\|_1^{1/2}\exp(-C\eta)$. If this inclusion fails, we derive from the weak compactness of B_1+B_2 in $L^1(m)$ that f can be separated from B_1+B_2 by a functional, i.e., there exists a function $w\in L^\infty$ such that

$$\Big|\int fw\,dm\Big|=1+\varrho\quad \text{for some } \varrho>0,$$

(6)

$$\sup\Big\{\Big|\int(g+h)w\,dm\Big|:g\in B_1,\ h\in B_2\Big\}\le 1.$$

Defining $\alpha=\sup\{|\int gw\,dm|:g\in B_1\}$ and $\beta=\sup\{|\int hw\,dm|:h\in B_2\}$, we see that $\alpha+\beta\le 1$. In accordance with the above description of Z as a dual space, for every $\delta>0$ the function w admits a representation $w=y+v$, where $y\in Y(\mathbb{T})$, $v\in L^1(|f|\,dm)$ and

$$\eta\|y\|_{Y(\mathbb{T})}+\int|v|\,|f|\,dm\le(1+\delta)\alpha.$$

Now we fix $\lambda>0$ (to be specified later) and put $E=\{|y|>\lambda\}$, $a=1\vee(\lambda^{-1}|y|)$, $\varphi=\exp(-\log a-iH(\log a))$, where H is the harmonic conjugation operator. (In other words, φ is an outer function with $|\varphi|=a^{-1}$.) We have

(7) $$\int wf\,dm=\int wf\varphi\,dm+\int wf(1-\varphi)\,dm.$$

We estimate the integrals on the right separately. For the second one, we use the fact that $f(1-\varphi)\in fH^2$, so, by the definition of the number β,

$$\Big|\int wf(1-\varphi)\,dm\Big|\le\beta t^{-1}\|f(1-\varphi)\|_{L^2(|f|^{-2}dm)}=\beta t^{-1}\Big(\int|1-\varphi|^2\,dm\Big)^{1/2}.$$

Now $|1-\varphi|\le\mathrm{const}(|\log a|+|\log Ha|)$, and by the L^2-continuity of H,

$$\Big(\int|1-\varphi|^2\,dm\Big)^{1/2}\le C\Big(\int_E(\log a)^2\,dm\Big)^{1/2}$$

(we recall that $\log a=0$ off E). Introducing the distribution function $s(\tau)=m\{\tau\in E:\log a>\tau\}$ and using the estimate $s(\tau)\le m\{|y|>\lambda e^\tau\}\le c\|y\|_{Y(\mathbb{T})}\lambda^{-1}e^{-\tau}$, we obtain

$$\int_E(\log a)^2\,dm=2\int_0^\infty\tau s(\tau)\,d\tau\le\mathrm{const}\,\lambda^{-1}\|y\|_{Y(\mathbb{T})}.$$

Finally,

(8) $$\Big|\int wf(1-\varphi)\,dm\Big|\le C\beta t^{-1}(\|y\|_{Y(\mathbb{T})}\lambda^{-1})^{1/2}.$$

Now we pass to estimating the first term on the right in (7). We use the representation $w=y+v$ (see above) and the fact that $|y\varphi|\le|y|\wedge\lambda$:

$$\Big|\int wf\varphi\,dm\Big|\le\int(|y|\wedge\lambda)\,dm+\int|v|\,|f|\,dm.$$

As we shall see, the second summand on the right is already good. The first one is treated in a manner entirely similar to the estimation of the summand $\int_{e\setminus a}|y_1|\,|f|\,dm$ in (5) (see §3). This will lead to

$$\Big|\int wf\varphi\,dm\Big|\le\|y\|_{Y(\mathbb{T})}c\Big(1+\log\frac{\lambda\|f\|_1}{c\|y\|_{Y(\mathbb{T})}}\Big)+\int|v|\,|f|\,dm.$$

Now we choose λ so that the coefficient of $\|y\|_{Y(\mathbb{T})}$ in the last expression be η:

$$\lambda=c\|y\|_{Y(\mathbb{T})}\|f\|^{-1}\exp(\eta/c-1).$$

This yields $|\int wf\varphi\,dm|\le(1+\delta)\alpha$, by the choice of y and v. Combining this with (8) (where we use the value of λ chosen above), we obtain

$$\Big|\int wf\,dm\Big|\le(1+\delta)\alpha+\beta,$$

if $t=\mathrm{const}\,\|f\|_1^{1/2}\exp(2^{-1}-\eta/(2c))$. This contradicts (6) if δ is small. ■

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Summability "au plus petit terme"

by

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Abstract. There is a curious phenomenon in the theory of Gevrey asymptotic expansions. In general the asymptotic formal power series is divergent, but there is some partial sum which approaches the value of the function very well. In this note we prove that there exists a truncation of the series which comes near the function in an exponentially flat way.

A *polysector* is a subset of $\mathbb{C}^n$ of the type

$$V = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : 0 < |z_j| < r_j,\ \arg z_j \in (a_j, b_j),\ 1 \le j \le n\},$$

where $|b_j - a_j| < 2\pi$, $j = 1, \ldots, n$.

Let $s \in [0, \infty)$. Let f be a holomorphic function in V. We say that the formal power series $\sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$ is the *weak asymptotic expansion of Gevrey type s of the function f* if the following condition is satisfied:

There is a constant $K > 0$ such that for any $t \in \mathbb{N}$,

$$\Big| f(z) - \sum_{|\alpha| < t} a_\alpha z^\alpha \Big| < K t!^s |z|^t \quad \text{for } z \in V.$$

THEOREM. *Let $s > 0$. Let $\sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$ be the weak asymptotic expansion of Gevrey type s of a holomorphic function f in a polysector V. Then there exist constants $A > 0$ and $B > 0$ such that for every z in V,*

$$\Big| f(z) - \sum_{|\alpha|=0}^{p_z} a_\alpha z^\alpha \Big| < A \exp(-B/|z|^{1/s})$$

whith some $p_z \in \mathbb{N}$.

Proof. We take $|z| = \max(|z_1|, \ldots, |z_n|)$ as norm. First assume that

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