$$M^{*-1}\left(\frac{l}{n}\right) \le \left(\left(\frac{1}{n}\sum_{j=1}^{l} a_j\right)^2 + \frac{l}{n}\left(\frac{1}{n}\sum_{j=l}^{n-1} a_j^2\right)\right)^{1/2},$$

$$M^{*-1}\left(\frac{l}{n}\right) \ge \left(\left(\frac{1}{n}\sum_{j=1}^{l} a_j\right)^2 + \frac{l}{n}\left(\frac{1}{n}\sum_{j=l+1}^{n} a_j^2\right)\right)^{1/2}.$$

Now it remains to apply Theorem 1. ■

References

[B–D] J. Bretagnolle et D. Dacunha-Castelle, *Application de l'étude de certaines formes linéaires aléatoires au plongement d'espaces de Banach dans les espaces* L^p, Ann. Sci. École Norm. Sup. 2 (1969), 437–480.

[K–S 1] S. Kwapień and C. Schütt, *Some combinatorial and probabilistic inequalities and their application to Banach space theory*, Studia Math. 82 (1985), 91–106.

[K–S 2] —,—, *Some combinatorial and probabilistic inequalities and their application to Banach space theory II*, ibid. 95 (1989), 141–154.

DEPARTMENT OF MATHEMATICS
OKLAHOMA STATE UNIVERSITY
STILLWATER, OKLAHOMA 74078
U.S.A.

MATHEMATISCHES SEMINAR
CHRISTIAN ALBRECHTS UNIVERSITÄT
24118 KIEL, GERMANY

Received February 22, 1994 (3233)

STUDIA MATHEMATICA 113 (1) (1995)

On the behaviour of Jordan-algebra norms on associative algebras

by

MIGUEL CABRERA GARCIA, ANTONIO MORENO GALINDO, and ANGEL RODRÍGUEZ PALACIOS (Granada)

Abstract. We prove that for a suitable associative (real or complex) algebra which has many nice algebraic properties, such as being simple and having minimal idempotents, a norm can be given such that the mapping $(a,b)\mapsto ab+ba$ is jointly continuous while $(a,b)\mapsto ab$ is only separately continuous. We also prove that such a pathology cannot arise for associative simple algebras with a unit. Similar results are obtained for the so-called "norm extension problem", and the relationship between these results and the normed versions of Zel'manov's prime theorem for Jordan algebras are discussed.

Introduction. If A is an associative algebra with the product ab, then its symmetrization A^+, which has the same vector space as A and new product $a.b := \frac{1}{2}(ab+ba)$, becomes a model for the so-called Jordan algebra. Jordan algebras are a well-known class of nonassociative algebras defined by a suitable identity. Our general reference for them is Jacobson's book [11]. Not all Jordan algebras are of the form A^+. Another example of such an algebra arises when A has an involution $*$. Then the space of hermitian elements $H(A,*) := \{x \in A : x^* = x\}$ is closed for the product "." so it can be naturally considered as a subalgebra of A^+. In many cases A^+ and $H(A,*)$ will not be isomorphic. Other standard examples can be constructed from bilinear forms (see Section 1) and octonion matrices. In the algebraic theory of Jordan algebras one of the most powerful results is the recent Zel'manov prime theorem.

Zel'manov's theorem [27] classifies prime nondegenerate Jordan algebras into four types which, roughly speaking, are the following: simple exceptional ones, simple Jordan algebras of a symmetric bilinear form, prime associative algebras regarded as Jordan algebras, and Jordan algebras of hermitian elements in prime associative algebras with a (linear) involution. In this way, an attempt to obtain a reasonable normed variant of Zel'manov's theorem has to involve the following:

1991 *Mathematics Subject Classification*: 46H70, 17C10, 17C65, 47A30.

QUESTION 1. *Let A be a real or complex associative algebra and let $\|\cdot\|$ be a norm on the vector space A making the Jordan product $(a,b)\mapsto \frac{1}{2}(ab+ba)$ continuous. Does the norm $\|\cdot\|$ make the associative product continuous?*

QUESTION 2 (the norm extension problem). *Let A be a real or complex associative algebra with an involution $*$ and let $H(A,*)$ denote the Jordan algebra of all hermitian elements in A. Assume that A is generated by $H(A,*)$ and that every nonzero $*$-ideal of A meets $H(A,*)$. Let $\|\cdot\|$ be an algebra norm on $H(A,*)$. Is there an algebra norm on A whose topology extends the one of $\|\cdot\|$ on $H(A,*)$?*

As far as we know, Question 1 arose for the first time in the literature in the paper of S. Shirali [24] where an affirmative answer is given under very strong additional algebraic and analytical assumptions. Later these assumptions were ostensibly relaxed in [18] (see also [20; Proposition 3]) where the following was proved:

THEOREM 1. *Question 1 has an affirmative answer whenever A is semiprime and $\|\cdot\|$ is complete.*

It is also shown in [18] that in Theorem 1 semiprimeness cannot be dropped in general. Question 2 has been considered in [4], as a tool for the classification of simple normed Jordan algebras with a unit, and more recently in [23] as a topic of its own interest. In the last paper the obviously necessary condition for an affirmative answer to Question 2, namely the continuity of the "tetrad mapping" $(x,y,z,t)\mapsto \frac{1}{2}(xyzt+tzyx)$ from $(H(A,*),\|\cdot\|)\times(H(A,*),\|\cdot\|)\times(H(A,*),\|\cdot\|)\times(H(A,*),\|\cdot\|)$ into $(H(A,*),\|\cdot\|)$, is shown to be also sufficient. Then the automatic continuity of the tetrad mapping is proved under reasonable algebraic and analytic assumptions, leading to:

THEOREM 2. *Question 2 has an affirmative answer whenever $H(A,*)$ is semiprime and the norm $\|\cdot\|$ on $H(A,*)$ is complete.*

Examples showing that in Theorems 1 and 2 the completeness of $\|\cdot\|$ cannot be dropped in general can be found in [1] and [23] respectively. However, the algebras in these examples are very far from being prime (they are in fact infinite direct sums of finite-dimensional simple ideals). Particuliar affirmative answers to Questions 1 and/or 2 under extra assumptions can be found in [20], [6], and [5]. A variant of Question 2, in the setting of topological algebras over strange fields, was answered in the negative in [25].

In this paper we are dealing with Questions 1 and 2. Our contributions are divided into three sections. In Section 1 a discussion of the total nonreversibility of high-dimensional simple Jordan algebras of a symmetric bilinear form (Lemmas 1 and 2), together with one of the main results in

[3], allows us to obtain some new positive results about the two questions, namely:

THEOREM 3. *Question 1 has an affirmative answer whenever A is simple and has a unit.*

THEOREM 4. *Question 2 has an affirmative answer if $H(A,*)$ is simple and has a unit. Moreover, if in addition the norm $\|\cdot\|$ on $H(A,*)$ is complete, then the algebra norm on A extending the topology of $\|\cdot\|$ can be chosen complete.*

In proving Theorems 3 and 4, the result in [3] commented above can be replaced by the arguments from [4], but we prefer to apply the first reference because it allows us to derive Theorems 3 and 4 from new interesting "almost" affirmative answers to the two questions (Theorems 5 and 6). We also emphasize that, if A in Question 1 is assumed to be simple but not necessarily unital, then $\|\cdot\|$ makes the associative product of A separately continuous (Corollary 1).

Section 2 contains the main result of the paper, namely an example showing simultaneously that neither the assumption of completeness of $\|\cdot\|$ in Theorems 1 and 2 nor that of the existence of a unit in Theorems 3 and 4 can be dropped in general. In fact, we show that, if $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$, the algebra $M_\infty(\mathbb{K})$ of all countably infinite matrices over $\mathbb{K}$ with only a finite number of nonzero entries can be endowed with a norm for which the Jordan product is continuous while the associative product is not. Moreover, $M_\infty(\mathbb{K})$ has a linear involution $*$ such that the topology of the restriction of the above norm to $H(M_\infty(\mathbb{K}),*)$ cannot be extended to the topology of any algebra norm on $M_\infty(\mathbb{K})$. Note that the associative algebra $M_\infty(\mathbb{K})$ and the Jordan algebra $H(M_\infty(\mathbb{K}),*)$ are central simple algebras over $\mathbb{K}$ coinciding with their socles ([10], [15]), and that $H(M_\infty(\mathbb{K}),*)$ generates $M_\infty(\mathbb{K})$ [9].

If J is a nonsimple prime nondegenerate Jordan algebra with nonzero socle, then the socle of J is either of the form A^+, where A is a simple associative algebra coinciding with its socle, or of the form $H(A,*)$ where A is a simple associative algebra A coinciding with its socle and $*$ is an involution on A [15]. Therefore, if J is a nonsimple prime nondegenerate Jordan–Banach algebra with nonzero socle, and if we restrict the norm of J to the socle of J, then we are in very suggestive particular situations of Questions 1 or 2 above. Note that neither $M_\infty(\mathbb{K})$ nor $H(M_\infty(\mathbb{K}),*)$ can be the socle of a prime nondegenerate Jordan–Banach algebra (otherwise, by [15] and [16], the vector space of countably infinite algebraic dimension over $\mathbb{K}$ would be completely normable, contradicting Baire's theorem). By choosing with some precision the pathological norm on $M_\infty(\mathbb{K})$ built in Section 2, we prove in Section 3 that the Jordan–Banach algebra J obtained by completing $M_\infty(\mathbb{K})^+$ (respectively, $H(M_\infty(\mathbb{K}),*)$) in this norm is prime nondegenerate and has a nonzero socle. Moreover, the socle of J is of the form A^+ for a central simple associative algebra A over $\mathbb{K}$ coinciding with its socle (respectively, of the form $H(A,*)$, where A is a central simple associative algebra over $\mathbb{K}$ coinciding with its socle and $*$ is an involution on A), and nevertheless the restriction of the norm of J to its socle makes the associative product of A discontinuous (respectively, the topology of the restriction of the norm of J to its socle cannot be extended to the topology of any algebra norm on A).

1. Some affirmative answers. Let $\mathbb{F}$ be a field of characteristic different from 2, X a vector space over $\mathbb{F}$ and f a symmetric bilinear form on X. Then the vector space $\mathbb{F}\mathbf{1} \oplus X$ endowed with the product

$$(\lambda\mathbf{1} + x).(\mu\mathbf{1} + y) := (\lambda\mu + f(x,y))\mathbf{1} + (\lambda y + \mu x)$$

becomes a Jordan algebra called the *Jordan algebra of the symmetric bilinear form* f and denoted by $J(X,f)$. These Jordan algebras are "special", that is, they can be regarded as Jordan subalgebras of associative algebras. If A is an associative algebra and if J is a Jordan subalgebra of A, then J is said to be *reversible* in A if the tetrad $\{xyzt\} := \frac{1}{2}(xyzt + tzyx)$ lies in J whenever x, y, z, t are in J. We emphasize the obvious fact that, if A is an associative algebra with an involution $*$, then $H(A,*)$ is a reversible Jordan subalgebra of A.

LEMMA 1. *Let $\mathbb{F}$ be a field of characteristic different from* 2, *X a vector space over $\mathbb{F}$, f a nondegenerate symmetric bilinear form on X, A an associative algebra over $\mathbb{F}$ containing $J(X,f)$ as a Jordan subalgebra and assume that either* $\dim(X) = 4$ *or* $\dim(X) \geq 6$. *Then $J(X,f)$ is not reversible in A.*

Proof. Clearly we may assume that A is generated by $J := J(X,f)$, so the unit of J is also a unit for A. We argue by contradiction, so assume that $\{xyzt\}$ lies in J whenever x,y,z,t are in J. In particular, taking x_1,x_2,x_3,x_4 in X with $f(x_i,x_j) = 0$ if $i \neq j$ and $f(x_i,x_i) \neq 0$, we must have that $t := \{x_1x_2x_3x_4\}$ belongs to J. Since x_1,x_2,x_3,x_4 pairwise anticommute in A, we obtain $x_1x_2x_3x_4 = x_4x_3x_2x_1 = t$ and so t^2 $(= f(x_1,x_1)f(x_2,x_2)f(x_3,x_3)f(x_4,x_4)\mathbf{1})$ belongs to $\mathbb{F}\mathbf{1} \setminus \{0\}$. But t cannot belong to $\mathbb{F}\mathbf{1}$ because, from $t = \lambda\mathbf{1}$ with λ in $\mathbb{F} \setminus \{0\}$, we would deduce that x_4 is invertible in A with $x_4^{-1} = \lambda^{-1}x_1x_2x_3 = \lambda^{-1}x_3x_2x_1$, which is impossible since $x_1x_2x_3 = -x_3x_2x_1$.

Now from the intrinsic algebraic characterization of X in J given by $X = \{0\} \cup \{y \in J \setminus \mathbb{F}\mathbf{1} : y^2 \in \mathbb{F}\mathbf{1}\}$ we conclude that t is actually in X. Using again the fact that x_1,x_2,x_3,x_4 pairwise anticommute in A, for $i = 1,2,3,4$ we have

$$f(t,x_i)\mathbf{1} = t.x_i = \tfrac{1}{2}(tx_i + x_it) = \tfrac{1}{2}(x_1x_2x_3x_4x_i + x_ix_1x_2x_3x_4) = 0,$$

and therefore t is a nonzero element in X which is orthogonal to the subspace generated by x_1, x_2, x_3 and x_4. This is already a contradiction in the case $\dim(X) = 4$.

Let us therefore assume $\dim(X) \geq 6$ and continue our argument by writing $x_5 := t$ and choosing x_6 in X with $f(x_6,x_6) \neq 0$ and $f(x_6,x_i) = 0$ for $i = 1,\ldots,5$. Then, for $i,j = 1,\ldots,6$, we have $f(x_i,x_j) = 0$ if $i \neq j$ and $f(x_i,x_j) \neq 0$ if $i = j$, so the linear hull of $\{x_1,\ldots,x_6\}$ in X (say Y) is a 6-dimensional f-nonisotropic subspace of X and $\mathbb{F}\mathbf{1} \oplus Y$ is a subalgebra of J isomorphic to $J(Y,g)$, where g denotes the restriction of f to $Y \times Y$.

Now we have natural inclusions $J(Y,g) \hookrightarrow A$ and $J(Y,g) \hookrightarrow C(Y,g)$, where $C(Y,g)$ denotes the Clifford algebra of (Y,g) which becomes in this way a unital special universal envelope for $J(Y,g)$ [11; pp. 74–75]. By the universal property of such an envelope, there exists a unique associative homomorphism $\Phi : C(Y,g) \to A$ extending the inclusion $J(Y,g) \hookrightarrow A$ and, since $C(Y,g)$ is simple [11; Theorem 2, p. 263], Φ is injective. Therefore, regarding Φ as an inclusion, the situation is the following: $C(Y,g)$ is an associative subalgebra of A, J is a Jordan subalgebra of A, and $J(Y,g)$ is a Jordan subalgebra of $C(Y,g)$ contained in J. Now the contradiction is flagrant because we know that $x_1x_2x_3x_4 = x_5$ in A whereas such an equality is impossible in $C(Y,g)$ in view of [12; Theorem 4.12]. ■

Remark 1. The above lemma need not remain true if $\dim(X) = 1, 2, 3$, or 5 (see [8; Theorem 6.2.5]).

LEMMA 2. *Let X be a complex vector space, f a nondegenerate symmetric bilinear form on X, A a real associative algebra containing $J(X,f)$ as a Jordan subalgebra and assume that either* $\dim_{\mathbb{C}}(X) = 4$ *or* $\dim_{\mathbb{C}}(X) \geq 6$. *Then $J(X,f)$ is not reversible in A.*

Proof. As in the proof of Lemma 1, we may assume that A is generated by $J := J(X,f)$, and then the unit $\mathbf{1}$ of J is also a unit for A. Consider the complexification $\mathbb{C} \otimes_{\mathbb{R}} A$ of A, which contains the complexification $\mathbb{C} \otimes_{\mathbb{R}} J$ of J as a Jordan subalgebra. Then $e := \frac{1}{2}(1 \otimes \mathbf{1} - i \otimes (i\mathbf{1}))$ and $f := \frac{1}{2}(1 \otimes \mathbf{1} + i \otimes (i\mathbf{1}))$ are mutually complementary central idempotents in $\mathbb{C} \otimes_{\mathbb{R}} J$, so that $e.(\mathbb{C} \otimes_{\mathbb{R}} J)$ and $f.(\mathbb{C} \otimes_{\mathbb{R}} J)$ are mutually complementary ideals of $\mathbb{C} \otimes_{\mathbb{R}} J$. The mappings $u \mapsto e.(1 \otimes u)$ from J into $e.(\mathbb{C} \otimes_{\mathbb{R}} J)$ and $v \mapsto f.(1 \otimes v)$ from J into $f.(\mathbb{C} \otimes_{\mathbb{R}} J)$ are (complex-linear and complex-conjugate-linear respectively) surjective algebra isomorphisms.

Since A is generated by J as a real algebra, $\mathbb{C} \otimes_{\mathbb{R}} A$ is generated by $\mathbb{C} \otimes_{\mathbb{R}} J$ as a complex algebra, and therefore e and f are (mutually complementary) central idempotents in $\mathbb{C} \otimes_{\mathbb{R}} A$. [Indeed, for every α in $\mathbb{C} \otimes_{\mathbb{R}} J$ we have

$e.(e.\alpha) = e.\alpha$ (since e is central in $\mathbb{C}\otimes_{\mathbb{R}} J$), which can be rewritten in terms of the associative product of $\mathbb{C}\otimes_{\mathbb{R}} A$ as $2e\alpha e = e\alpha + \alpha e$. Multiplying this last equality from the left (respectively, from the right) by e, we obtain $e\alpha e = e\alpha$ (respectively, $e\alpha e = \alpha e$), so e is in the commutant of $\mathbb{C}\otimes_{\mathbb{R}} J$ in $\mathbb{C}\otimes_{\mathbb{R}} A$ and therefore e is central in $\mathbb{C}\otimes_{\mathbb{R}} A$ because $\mathbb{C}\otimes_{\mathbb{R}} J$ generates $\mathbb{C}\otimes_{\mathbb{R}} A$.]

Now, if J were reversible in A, then $\mathbb{C}\otimes_{\mathbb{R}} J$ would be obviously reversible in $\mathbb{C}\otimes_{\mathbb{R}} A$, and therefore $e.(\mathbb{C}\otimes_{\mathbb{R}} J)$ would be reversible in $e(\mathbb{C}\otimes_{\mathbb{R}} A)$ (because $e(\mathbb{C}\otimes_{\mathbb{R}} A)$ and $f(\mathbb{C}\otimes_{\mathbb{R}} A)$ are mutually complementary ideals of $\mathbb{C}\otimes_{\mathbb{R}} A$ containing $e.(\mathbb{C}\otimes_{\mathbb{R}} J)$ and $f.(\mathbb{C}\otimes_{\mathbb{R}} J)$, respectively). But this is not possible because $e.(\mathbb{C}\otimes_{\mathbb{R}} J)$ is a copy of J and Lemma 1 applies. ∎

Recall that a Jordan algebra J is said to be a *central order* in the Jordan algebra of a symmetric bilinear form if its centre $Z(J)$ is not zero, the nonzero elements in $Z(J)$ are not zero-divisors in J and the central localization $Z(J)^{-1}J$ is isomorphic to the Jordan algebra of a symmetric bilinear form on some vector space over the field of fractions $Z(J)^{-1}Z(J)$.

THEOREM 5. *Let A be a real or complex associative algebra with an involution $*$ and let $H(A,*)$ denote the Jordan algebra of all hermitian elements in A. Assume that A is generated by $H(A,*)$, that every nonzero $*$-ideal of A meets $H(A,*)$ and that $H(A,*)$ is simple. Then for every algebra norm $\|\cdot\|$ on $H(A,*)$ there exists an algebra norm $|\!|\!|\cdot|\!|\!|$ on A with the following properties*:

(i) $\|h\| \le |\!|\!|h|\!|\!|$ *for all h in $H(A,*)$*,

(ii) *for all a in A the mappings $h\mapsto ah$ and $h\mapsto ha$ from $(H(A,*),\|\cdot\|)$ into $(A,|\!|\!|\cdot|\!|\!|)$ are continuous*,

(iii) $|\!|\!|a^*|\!|\!| = |\!|\!|a|\!|\!|$ *for all a in A, and*

(iv) *if $(\widehat{A},*)$ denotes the completion of $(A,*)$ with respect to the norm $|\!|\!|\cdot|\!|\!|$, then every nonzero $*$-ideal of $\widehat{A}$ meets $H(\widehat{A},*)$.*

Proof. Write $J := H(A,*)$. If J is not a central order in the Jordan algebra of a bilinear form, then, taking into account that simple Jordan algebras are prime and nondegenerate [26], our result is nothing but the specialization of [3; Theorem 1] to simple normed Jordan algebras.

Assume, therefore, that J is a central order in the Jordan algebra of a symmetric bilinear form. Since J is a simple algebra with a unit, $Z(J)$ is a field extension of the base field $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$), hence the passing to the central localization trivializes, and so J is actually the Jordan algebra of a nondegenerate symmetric bilinear form on a vector space over $Z(J)$. On the other hand, by the Gelfand–Mazur Theorem, we have either $Z(J)=\mathbb{C}$, if $\mathbb{K}=\mathbb{C}$, or $Z(J)\in\{\mathbb{R},\mathbb{C}\}$, if $\mathbb{K}=\mathbb{R}$. If $Z(J)=\mathbb{K}$, then we apply Lemma 1, whereas if $Z(J)=\mathbb{C}$ and $\mathbb{K}=\mathbb{R}$ we apply Lemma 2, obtaining in both cases $\dim_{\mathbb{K}}(J)<\infty$. Since for all h_1,h_2,h_3,h_4 in $H(A,*)$ we have

$$\begin{aligned} h_1h_2h_3h_4 = \{h_1h_2h_3h_4\} &+ h_1h_2(h_3.h_4) - h_1(h_2.h_4)h_3 \\ &+ (h_1.h_4)h_2h_3 - h_4(h_1.h_2)h_3 + h_4h_2(h_1.h_3) - h_4(h_2.h_3)h_1 \end{aligned}$$

and A is generated by $J=H(A,*)$, we get $A = H(A,*)H(A,*)H(A,*)$ and therefore also $\dim_{\mathbb{K}}(A)<\infty$. Now every norm $|\!|\!|\cdot|\!|\!|$ on the vector space of A, multiplied if necessary by a suitable (sufficiently large) positive number, becomes an algebra norm on A satisfying property (i) from the statement. By replacing $|\!|\!|\cdot|\!|\!|$ by $\max\{|\!|\!|\cdot|\!|\!|, |\!|\!|\cdot^*|\!|\!|\}$ if necessary, we have additionally property (iii), whereas (ii) and (iv) are automatic by the finite-dimensionality of A. ∎

Proof of Theorem 4. It follows directly from Theorem 5 that Question 2 has an affirmative answer whenever $H(A,*)$ is simple with unit, because the unit $\mathbf{1}$ of $H(A,*)$ must also be a unit for A (since $H(A,*)$ generates A) and then, applying property (i) and (ii) with $a=\mathbf{1}$, we see that the restriction to $H(A,*)$ of the norm $|\!|\!|\cdot|\!|\!|$ in that theorem is equivalent to $\|\cdot\|$.

Now we assume that the algebra norm $\|\cdot\|$ on the simple Jordan algebra $H(A,*)$ is complete and we show that the algebra norm on A extending the topology of $\|\cdot\|$ can be chosen complete. If A is commutative, then $H(A,*)$ is an associative subalgebra of A, hence $A=H(A,*)$ (because A is generated by $H(A,*)$), and there is nothing to prove. Therefore we may assume additionally that A is not commutative. Again, let $|\!|\!|\cdot|\!|\!|$ be the norm on A given by Theorem 5 and let $(\widehat{A},*)$ denote the completion of $(A,*)$ with respect to this norm. Then $H(\widehat{A},*) = H(A,*)$ (by the completeness of $\|\cdot\|$ and the equivalence of $|\!|\!|\cdot|\!|\!|$ and $\|\cdot\|$ on $H(A,*)$) and therefore every nonzero $*$-ideal P of $\widehat{A}$ contains $H(A,*)$ (by the simplicity of $H(A,*)$ and property (iv) in Theorem 5). Now, for such P, $P\cap A$ is a subalgebra of A containing $H(A,*)$, hence $P\supseteq A$ (again because $H(A,*)$ generates A) and so P is dense in $\widehat{A}$. Since dense ideals in a Banach algebra with unit must coincide with the whole algebra [2; Theorem 9.3(ii)], we have proved in fact that $\widehat{A}$ is (algebraically) $*$-simple. As a consequence, $\widehat{A}$ is semiprime and we may apply [9; Example 2 in p. 59 and Theorem 2.1.2] to conclude that A contains a nonzero $*$-ideal of $\widehat{A}$. Since $\widehat{A}$ is $*$-simple, we have $A=\widehat{A}$. ∎

THEOREM 6. *Let A be a real or complex simple associative algebra and let $\|\cdot\|$ be a norm on the vector space of A making the Jordan product continuous. Then there exists an algebra norm $|\!|\!|\cdot|\!|\!|$ on A with the following properties*:

(i) $\|a\| \le |\!|\!|a|\!|\!|$ *for all a in A, and*

(ii) *for every a in A the mappings $x\mapsto ax$ and $x\mapsto xa$ from $(A,\|\cdot\|)$ into $(A,|\!|\!|\cdot|\!|\!|)$ are continuous.*

Proof. If A is commutative, then there is nothing to prove. Therefore we assume that A is not commutative. Let B denote the direct product $A\times A^{\mathrm{op}}$, where A^{op} is the opposite algebra of A, and let $*$ be the exchange involution on B. Then B is a $*$-simple associative algebra (hence every nonzero $*$-ideal of B meets $H(B,*)$) and $H(B,*)$ is not a commutative subset of B. Therefore, by [9; Example 2 on p. 59 and Theorem 2.1.2], B is generated by $H(B,*)$. On the other hand, the mapping $\varphi: a\mapsto (a,a)$ is an isomorphism from the Jordan algebra A^+ onto $H(B,*)$. As a first consequence, $H(B,*)$ is a simple Jordan algebra because A is simple and [9; Corollary on p. 57] applies. Also the mapping $x\mapsto \|\varphi^{-1}(x)\|$ from $H(B,*)$ into $\mathbb{R}$ becomes an algebra norm on $H(B,*)$.

Now we are in a position to apply Theorem 5 to obtain an algebra norm $|||\cdot|||$ on B with the following properties:

(1) $\|\varphi^{-1}(h)\|\le |||h|||$ for all h in $H(B,*)$,

(2) for every b in B the mappings $h\mapsto bh$ and $h\mapsto hb$ from $(H(B,*), \|\varphi^{-1}(\cdot)\|)$ into $(B, |||\cdot|||)$ are continuous, and

(3) $|||b^*||| = |||b|||$ for all b in B.

Then the mapping $a\mapsto |a| := 2|||(a,0)|||$ from A into $\mathbb{R}$ is clearly an algebra norm on A, and, by (1) and (3), has property (i) in the statement of the theorem.

Let a be in A. Then, by (2), there exists a nonnegative number M_a satisfying $|||(a,0)h|||\le M_a\|\varphi^{-1}(h)\|$ and $|||h(a,0)|||\le M_a\|\varphi^{-1}(h)\|$ for all h in $H(B,*)$. Therefore, for every x in A we have

$$|ax| = 2|||(ax,0)||| = 2|||(a,0)(x,x)||| \le 2M_a\|\varphi^{-1}((x,x))\| = 2M_a\|x\|$$

and analogously $|xa|\le 2M_a\|x\|$, thus concluding the proof of (ii). ■

COROLLARY 1. *Let A be a real or complex simple associative algebra and let $\|\cdot\|$ be a norm on the vector space of A making the Jordan product continuous. Then $\|\cdot\|$ makes the associative product separately continuous.*

Proof. Let $|\cdot|$ be the norm on A given by Theorem 6 and fix a in A. Then, by property (ii) in that theorem, there exists a nonnegative number K_a satisfying $|ax|\le K_a\|x\|$ and $|xa|\le K_a\|x\|$ for all x in A. Applying property (i), we obtain $\|ax\|\le K_a\|x\|$ and $\|xa\|\le K_a\|x\|$ for all x in A. ■

Proof of Theorem 3. It follows directly from Theorem 6 (apply (i) and (ii) with $a=\mathbf{1}$) that Question 1 has an affirmative answer whenever A is simple with unit. ■

Remark 2. It follows easily from Theorem 3 (respectively, Theorem 4) that Question 1 (respectively, Question 2) has an affirmative answer whenever A (respectively, $H(A,*)$) is a finite direct sum of simple ideals with a unit. The counterexample from [1] (respectively [23]) shows that this need not remain true in the case of infinite direct sums.

We conclude this section with an affirmative answer to Question 1, showing again the parallelism between the answers to Questions 1 and 2. It is known that Question 2 has an affirmative answer whenever there exists a positive number K satisfying $K\|h\|^2\le\|h^2\|$ for all h in $H(A,*)$ [23; Proposition 2], the result being uninteresting if the algebra A is complex because then the Jordan algebra $H(A,*)$ is associative [19; Proposition 31].

PROPOSITION 1. *Question 1 has an affirmative answer whenever there exists a positive number K satisfying $K\|a\|^2\le\|a^2\|$ for all a in A. Moreover, if A is complex, then it is commutative.*

Proof. We may assume that $\|a.b\|\le\|a\|\,\|b\|$ for all a,b in A. Since for a and b in A,

$$aba = 2a.(a.b) - a^2.b \quad\text{and}\quad [a,b]^2 = 2a.(bab) - ab^2a - ba^2b,$$

we have in fact

$$K\|[a,b]\|^2\le\|[a,b]^2\|\le 12\|a\|^2\|b\|^2.$$

Therefore $\|ab\| = \left\|a.b+\frac12[a,b]\right\|\le M\|a\|\,\|b\|$, where $M := 1+(3/K)^{1/2}$, hence the associative product of A is continuous. Now $|\cdot| := M\|\cdot\|$ is an algebra norm on A satisfying $(K/M)|a|^2\le|a^2|$ for all a in A, hence, if A is complex, then A is commutative by [2; Corollary 15.8]. ■

2. The monster.

For an algebra B, we denote by $M_\infty(B)$ the algebra of all countably infinite matrices over B with a finite number of nonzero entries. For n in $\mathbb{N}$, we will identify the algebra $M_n(B)$ of all $n\times n$ matrices over B with the subalgebra of $M_\infty(B)$ of those matrices $(b_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ in $M_\infty(B)$ satisfying $b_{ij}=0$ whenever either $i>n$ or $j>n$.

PROPOSITION 2. *Let $(B,\|\cdot\|)$ be an associative normed algebra. Then there exists an algebra norm on $M_\infty(B)$ (also denoted by $\|\cdot\|$) extending the norm of $B=M_1(B)$ and satisfying*

$$\left\|\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right\|_{n+m} = \max\{\|\alpha\|_n, \|\beta\|_m\}$$

for all n,m in $\mathbb{N}$, α in $M_n(B)$ and β in $M_m(B)$, where $\|\cdot\|_n$ denotes the restriction of $\|\cdot\|$ to $M_n(B)$.

Proof. It is well known that B can be isometrically imbedded in the normed algebra $BL(X)$ of all bounded linear operators on a suitable normed space X [17; p. 4]. Then we can convert the vector space Y of all quasi-null sequences in X into a normed space by fixing $1\le p\le\infty$ and defining, for

$y = \{x_n\}$ in Y, $\|y\| := (\sum_n \|x_n\|^p)^{1/p}$ if $p < \infty$ and $\|y\| := \max\{\|x_n\| : n \in \mathbb{N}\}$ if $p = \infty$. Finally, the imbedding $B \hookrightarrow BL(X)$ induces naturally an algebraic imbedding $M_\infty(B) \hookrightarrow BL(Y)$ and it is enough to restrict to $M_\infty(B)$ the norm of $BL(Y)$ to obtain an algebra norm on $M_\infty(B)$ with the required properties. ∎

Let B be an algebra with an involution $*$. Then $M_\infty(B)$ has a "canonical" involution (also denoted by $*$) consisting in transposing a given matrix and then applying the original involution to each entry.

PROPOSITION 3. *Let $(B, \|\cdot\|)$ be an associative normed algebra containing a closed Jordan subalgebra J which is not an associative subalgebra. Then there exists a norm $|||\cdot|||$ on the vector space $M_\infty(B)$ for which the Jordan product is continuous but the associative product is discontinuous. If J is nonreversible in B, then the above norm $|||\cdot|||$ can be chosen in such a way that it even makes the tetrad mapping $(\alpha,\beta,\gamma,\delta) \mapsto \{\alpha\beta\gamma\delta\}$ from $M_\infty(B)\times M_\infty(B)\times M_\infty(B)\times M_\infty(B)$ to $M_\infty(B)$ discontinuous. Moreover, if B has an involution $*$, J is nonreversible in B and J is contained in $H(B,*)$, then the norm $|||\cdot|||$ can be chosen in such a way that the topology of its restriction to $H(M_\infty(B),*)$ cannot be extended to the topology of any algebra norm on $M_\infty(B)$.*

Proof. Let $\|\cdot\|$ be an algebra norm on $M_\infty(B)$ with the properties assured by Proposition 2. For an element α in $M_\infty(B)$ and a (not necessarily $\|\cdot\|$-closed) subspace P of $M_\infty(B)$, write $\|\alpha + P\| := \inf\{\|\alpha+\beta\| : \beta \in P\}$, so that the mapping $\alpha \mapsto \|\alpha + P\|$ is a seminorm on the vector space $M_\infty(B)$. For convenience, we also regard $M_\infty(B)$ as $M_\infty(\mathbb{K}) \otimes_{\mathbb{K}} B$, where $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) denotes the base field of B. Denote by e_n ($n \in \mathbb{N}$) the element (λ_{ij}) in $M_\infty(\mathbb{K})$ given by $\lambda_{ij} = 0$ whenever $(i,j) \neq (n,n)$, and $\lambda_{nn} = 1$. Then, writing $\mathcal{J}_n := M_{n-1}(\mathbb{K}) \otimes B + e_n \otimes J$ ($n \in \mathbb{N}$), where $M_0(\mathbb{K}) := 0$, we can define a norm $|||\cdot|||$ on the vector space $M_\infty(B)$ by

$$|||\alpha||| := \|\alpha\| + \sum_{n=1}^{\infty} 2^{5^n} \|\alpha + \mathcal{J}_n\|$$

for all α in $M_\infty(B)$ (note that, for each α in $M_\infty(B)$, the above series has only a finite number of nonzero terms). For $k = 0,1,2,\ldots$, and α in $M_\infty(B)$, we define inductively $|||\alpha|||_k$ by $|||\alpha|||_0 := \|\alpha\|$ and $|||\alpha|||_{k+1} := |||\alpha|||_k + 2^{5^{k+1}}\|\alpha + \mathcal{J}_{k+1}\|$, so that, for fixed α in $M_\infty(B)$, the sequence $\{|||\alpha|||_k\}_{k\in\mathbb{N}\cup\{0\}}$ is quasi-constant with limit $|||\alpha|||$. Clearly $|||\alpha.\beta|||_0 \le |||\alpha|||_0\, |||\beta|||_0$ for all α,β in $M_\infty(B)$. Assume k is such that $|||\alpha.\beta|||_k \le |||\alpha|||_k\, |||\beta|||_k$ for all α,β in $M_\infty(B)$. Then, since $\mathcal{J}_k$ is a Jordan subalgebra of $M_\infty(B)$, we can argue as in the proof of [23; Lemma 3] to obtain for all α,β in $M_\infty(B)$,

$$
\begin{aligned}
|||\alpha.\beta|||_{k+1} &:= |||\alpha.\beta|||_k + 2^{5^{k+1}}\|\alpha.\beta + \mathcal{J}_{k+1}\| \\
&\le |||\alpha.\beta|||_k + 2^{5^{k+1}}(\|\alpha\|\,\|\beta + \mathcal{J}_{k+1}\| \\
&\quad + \|\alpha + \mathcal{J}_{k+1}\|\,\|\beta\| + \|\alpha + \mathcal{J}_{k+1}\|\,\|\beta + \mathcal{J}_{k+1}\|) \\
&\le |||\alpha|||_k\,|||\beta|||_k + 2^{5^{k+1}}(|||\alpha|||_k\|\beta + \mathcal{J}_{k+1}\| + \|\alpha + \mathcal{J}_{k+1}\|\,|||\beta|||_k) \\
&\quad + (2^{5^{k+1}})^2\|\alpha + \mathcal{J}_{k+1}\|\,\|\beta + \mathcal{J}_{k+1}\| \\
&= (|||\alpha|||_k + 2^{5^{k+1}}\|\alpha + \mathcal{J}_{k+1}\|)(|||\beta|||_k + 2^{5^{k+1}}\|\beta + \mathcal{J}_{k+1}\|) \\
&= |||\alpha|||_{k+1}\,|||\beta|||_{k+1}.
\end{aligned}
$$

In this way we have proved that $|||\alpha.\beta|||_k \le |||\alpha|||_k\,|||\beta|||_k$ for all α,β in $M_\infty(B)$ and every k in $\mathbb{N}\cup\{0\}$. Therefore $|||\alpha.\beta||| \le |||\alpha|||\,|||\beta|||$ for all α,β in $M_\infty(B)$, and $|||\cdot|||$ makes the Jordan product of $M_\infty(B)$ continuous.

Let b be in B and k be in $\mathbb{N}$. Then a straightforward calculation, involving the properties of $\|\cdot\|$ on $M_\infty(B)$ given by Proposition 2, shows that

$$|||e_k \otimes b||| = (1 + 2^5 + \ldots + 2^{5^{k-1}})\|b\| + 2^{5^k}\|b + J\|.$$

Since J is closed in B and is not an associative subalgebra of B, there exist x, y in J with $\|x\| = \|y\| = 1$ and $\|xy + J\| \neq 0$, and, for k in $\mathbb{N}$, we have

$$
\begin{aligned}
\frac{|||(e_k\otimes x)(e_k\otimes y)|||}{|||e_k\otimes x|||\,|||e_k\otimes y|||} &= \frac{|||e_k\otimes (xy)|||}{|||e_k\otimes x|||\,|||e_k\otimes y|||} \\
&= \frac{(1+2^5+\ldots+2^{5^{k-1}})\|xy\| + 2^{5^k}\|xy+J\|}{(1+2^5+\ldots+2^{5^{k-1}})^2} \\
&\ge \frac{2^{5^k}\|xy+J\|}{(1+2^5+\ldots+2^{5^{k-1}})^2} \ge \frac{2^{5^k}}{k^2(2^{5^{k-1}})^2}\|xy+J\| \\
&= \frac{(2^{5^{k-1}})^3}{k^2}\|xy+J\| \to \infty \quad \text{as } k\to\infty.
\end{aligned}
$$

Therefore the associative product of $M_\infty(B)$ is $|||\cdot|||$-discontinuous.

Assume that J is nonreversible in B. Then there exist x,y,z,t in J with $\|x\| = \|y\| = \|z\| = \|t\| = 1$ and $\|\{xyzt\} + J\| \neq 0$, and, for k in $\mathbb{N}$, we have

$$
\begin{aligned}
&\frac{|||\{(e_k\otimes x)(e_k\otimes y)(e_k\otimes z)(e_k\otimes t)\}|||}{|||e_k\otimes x|||\,|||e_k\otimes y|||\,|||e_k\otimes z|||\,|||e_k\otimes t|||} \\
&= \frac{|||e_k\otimes\{xyzt\}|||}{|||e_k\otimes x|||\,|||e_k\otimes y|||\,|||e_k\otimes z|||\,|||e_k\otimes t|||} \\
&= \frac{(1+2^5+\ldots+2^{5^{k-1}})\|\{xyzt\}\| + 2^{5^k}\|\{xyzt\}+J\|}{(1+2^5+\ldots+2^{5^{k-1}})^4} \\
&\ge \frac{2^{5^k}\|\{xyzt\}+J\|}{(1+2^5+\ldots+2^{5^{k-1}})^4} \ge \frac{2^{5^k}}{k^2(2^{5^{k-1}})^2}\|\{xyzt\}+J\| \\
&= \frac{2^{5^{k-1}}}{k^4}\|\{xyzt\}+J\| \to \infty \quad \text{as } k\to\infty.
\end{aligned}
$$

Therefore in this case even the tetrad mapping on $M_\infty(B)$ is $|\cdot|$-discontinuous.

Finally, assume that B has an involution $*$, that J is nonreversible in B and that J is contained in $H(B,*)$. Then, for x,y,z,t as above and for k in $\mathbb{N}$, $e_k\otimes x, e_k\otimes y, e_k\otimes z, e_k\otimes t$ lie in $H(M_\infty(B),*)$, and the last argument shows that the restriction of the tetrad mapping to $H(M_\infty(B),*)\times H(M_\infty(B),*)\times H(M_\infty(B),*)\times H(M_\infty(B),*)$ is $|\cdot|$-discontinuous. This makes it impossible to find an algebra norm on $M_\infty(B)$ extending the topology of the restriction of $|\cdot|$ to $H(M_\infty(B),*)$. ∎

Remark 3. Let A be an associative algebra over $\mathbb{K}$ and $|\cdot|$ a norm on A. If the Jordan product of A is $|\cdot|$-continuous but the associative product is not, then the mapping $(a,b)\mapsto \alpha ab+\beta ba$ is $|\cdot|$-discontinuous for any α,β in $\mathbb{K}$ with $\alpha\neq\beta$. This follows from the equality $\alpha ab+\beta ba = (\alpha-\beta)ab+2\beta a.b$.

THEOREM 7. *Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Then there exists a norm $|\cdot|$ on the vector space $M_\infty(\mathbb{K})$ with the following properties:*

(i) $|\alpha.\beta|\le|\alpha|\,|\beta|$ *for all α,β in $M_\infty(\mathbb{K})$.*

(ii) *There is an involution $*$ on $M_\infty(\mathbb{K})$ such that there is no algebra norm on $M_\infty(\mathbb{K})$ whose restriction to $H(M_\infty(\mathbb{K}),*)$ is equivalent to the restriction of $|\cdot|$ to $H(M_\infty(\mathbb{K}),*)$.*

As a consequence of (i), *the restriction of $|\cdot|$ to the Jordan algebra $H(M_\infty(\mathbb{K}),*)$ is an algebra norm, whereas, as a consequence of* (ii), *the associative product of $M_\infty(\mathbb{K})$ is $|\cdot|$-discontinuous.*

Proof. Let B denote the associative algebra $M_4(\mathbb{K})$ regarded as a normed algebra with respect to an arbitrary algebra norm and let $*$ be the "symplectic" involution of B; namely, for b in B, $b^* := s^{-1}b^ts$, where b^t denotes the transpose of b, and s stands for the element in B given by $s:=\operatorname{diag}\{q,q\}$ with $q:=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. The elements x,y,z,t in $H(B,*)$ given by

$$x=\left(\begin{array}{cc|cc}0&0&1&0\\0&0&0&1\\\hline 1&0&0&0\\0&1&0&0\end{array}\right),\quad y=\left(\begin{array}{cc|cc}0&0&0&1\\0&0&-1&0\\\hline 0&-1&0&0\\1&0&0&0\end{array}\right),$$

$$z=\left(\begin{array}{cc|cc}0&0&1&0\\0&0&0&-1\\\hline -1&0&0&0\\0&1&0&0\end{array}\right),\quad t=\left(\begin{array}{cc|cc}0&0&0&1\\0&0&1&0\\\hline 0&-1&0&0\\-1&0&0&0\end{array}\right)$$

pairwise anticommute and satisfy $x^2=y^2=1$ and $z^2=t^2=-1$. It follows that, with J denoting the linear hull of $\{1,x,y,z,t\}$, J is a Jordan subalgebra of B contained in $H(B,*)$ and is isomorphic to the Jordan algebra of a nondegenerate symmetric bilinear form on a 4-dimensional vector space over $\mathbb{K}$. Moreover, J is closed in B, because B is finite-dimensional, and nonreversible in B in view of (for example) Lemma 1. Now apply Proposition 3 and take into account that $M_\infty(B)$ is isomorphic to $M_\infty(\mathbb{K})$. ∎

Remark 4. The specialization of Zel'manov's prime theorem to simple Jordan algebras [27; Theorem 4] asserts that, if J is a simple Jordan algebra, then J is one of the following:

(i) a simple exceptional Jordan algebra of dimension 27 over its centre,

(ii) a simple Jordan algebra of a symmetric bilinear form on a vector space over the centre of J,

(iii) $J=A^+$, where A is a simple associative algebra,

(iv) $J=H(A,*)$, where A is a simple associative algebra and $*$ is an involution on A.

Moreover, from the "uniqueness of the $*$-tight envelope" for certain special Jordan algebras [14; Theorem 2.3], it is easily seen that, if the simple Jordan algebra J is neither in case (i) nor (ii) but (iii) (respectively, (iv)), then J cannot be in case (iv) (respectively, (iii)). Moreover, A and A^{op} are the only simple associative algebras B satisfying $J=B^+$ (respectively, $(A,*)$ is the only involutive simple associative algebra $(B,*)$ satisfying $J=H(B,*)$).

Now, if J denotes the simple Jordan algebra $M_\infty(\mathbb{K})^+$ (respectively, $H(M_\infty(\mathbb{K}),*)$), then J is neither in case (i) nor (ii) but (iii) (respectively, (iv)), hence, when J is endowed with the pathological norm $|\cdot|$ of Theorem 7, it becomes impossible to obtain the topology of this norm from any algebra norm on the essentially unique simple associative envelope of J. It follows that, as claimed in [22], a "strong" normed version of Zel'manov's theorem, like the one obtained in [4; Theorem 1] for normed simple Jordan algebras with a unit, cannot be true even in the nonunital simple case.

3. Completing the monster.

As announced in the introduction, in this section we study the completion of $M_\infty(\mathbb{K})^+$ in the pathological Jordan-algebra norm constructed above, in order to obtain analytically more interesting negative answers to Questions 1 and 2.

LEMMA 3. *Let A be an associative algebra with zero annihilator and J a simple Jordan subalgebra of A. Assume that J is not isomorphic to the Jordan algebra of a symmetric bilinear form and that A is generated by J. Then either $J=A^+$ and A is simple or there exists an involution $*$ on A such that $J=H(A,*)$ and A is $*$-simple. Moreover, the first possibility arises if and only if there exist x and y in J satisfying $[x,y]\in J\setminus\{0\}$.*

Proof. As in the proof of [23; Corollary 2], if $\mathcal{A}$ denotes the universal special envelope of J, $*$ is the "main" involution on $\mathcal{A}$ and I is the largest (automatically $*$-) ideal of $\mathcal{A}$ contained in the skew part of $\mathcal{A}$, then $J = H(\mathcal{A},*)$ and $IJ = JI = 0$. Since $\mathcal{A}$ is generated by J, it follows that I is contained in the annihilator of $\mathcal{A}$. Since A is also generated by J, the unique associative homomorphism $\Phi : \mathcal{A} \to A$ extending the inclusion of J into A is surjective, and, since A has zero annihilator, $\Phi(I) = 0$. Setting $(\mathcal{B},*) := (\mathcal{A},*)/I$ and taking into account that $J = H(\mathcal{A},*)$ and $J \cap I = 0$, we can view J as $H(\mathcal{B},*)$. Clearly $\mathcal{B}$ is generated by J and every $*$-ideal of $\mathcal{B}$ meets J. Since J is simple, this implies that $\mathcal{B}$ is $*$-simple.

Let Ψ be the homomorphism from $\mathcal{B}$ onto A induced by Φ. If Ψ is injective, then Ψ is an isomorphism from $\mathcal{B}$ onto A whose restriction to J is the identity mapping on J. Therefore we can translate the involution of $\mathcal{B}$ to A in such a way that A becomes $*$-simple and $J = H(A,*)$.

Assume that Ψ is not injective. Then, setting $P := (\mathrm{Ker}(\Psi))^*$ and recalling that $\mathcal{B}$ is $*$-simple, we deduce that P is a simple ideal of $\mathcal{B}$ satisfying $\mathcal{B} = P \oplus P^*$. Now $\Psi_{|P}$ is an isomorphism from P onto A. Let a be in A. Then, writing $p := (\Psi_{|P})^{-1}(a)$, we see that $p + p^*$ belongs to J and hence

$$a = \Psi(p) = \Psi(p+p^*) = p + p^* \in H(\mathcal{B},*) = J.$$

Therefore $A \subseteq J$, hence $A = J$. A is obviously simple because J is a simple Jordan algebra.

If there exist x and y in J with $[x,y] \in J \setminus \{0\}$, then $J = A$ because otherwise J would be of the form $H(A,*)$ for a suitable involution $*$ on A and $[x,y]$ would be a nonzero simultaneously hermitian and skew element in A. Conversely, if $J = A$, then, since J is simple and not isomorphic to the Jordan algebra of a symmetric bilinear form, J is not associative. Hence A is not commutative and the existence of x, y in A with $[x,y] \in J \setminus \{0\}$ is obvious. ■

Notation. In what follows $(B, \|\cdot\|)$ will denote the Banach algebra $M_4(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) regarded as the algebra of all bounded linear operators on the Hilbert space $X := \mathbb{K}^4$. Then, regarding an infinite-dimensional separable Hilbert space H over $\mathbb{K}$ as the ℓ_2-sum of a countably infinite family of copies of X, we can view "the monster" $\mathcal{M} := M_\infty(B)$ as an associative subalgebra of the algebra $BL(H)$ of all bounded linear operators on H (in fact, it consists of bounded linear operators of finite rank). The operator norm on $BL(H)$ will be denoted by $\|\cdot\|$, so that the restriction of $\|\cdot\|$ to $\mathcal{M}$ satisfies the conditions of Proposition 2. According to the proof of Theorem 7, if $*$ denotes the symplectic involution on B parameterized as we did there, then $*$ commutes with the usual operator adjoint on the Hilbert space X, there exists a (closed) nonreversible Jordan subalgebra J of B contained in $H(B,*)$ and J is a self-adjoint subset of the C^*-algebra $B = BL(X)$.

When such a J has been chosen, for every n in $\mathbb{N}$, $\mathcal{J}_n$ will stand for the (finite-dimensional in our case) Jordan subalgebra of $\mathcal{M}$ as defined in the proof of Proposition 3. Finally, we consider the "monstrous norm" $|\!|\cdot|\!|$ on $\mathcal{M}$ given by

$$|\!|\alpha|\!| := \|\alpha\| + \sum_{n=1}^{\infty} 2^{5^n}\|\alpha + \mathcal{J}_n\|$$

for all α in $\mathcal{M}$, so that $|\!|\cdot|\!|$ is an algebra norm on $\mathcal{M}^+$, but the topology of the restriction of $|\!|\cdot|\!|$ to $H(\mathcal{M},*)$ cannot be extended to the topology of any algebra norm on $\mathcal{M}$ (hence, in particular, the associative product of $\mathcal{M}$ is $|\!|\cdot|\!|$-discontinuous). Of course, the involution $*$ on $\mathcal{M}$ now refers to the canonical involution on $\mathcal{M}$ relative to the symplectic involution on B.

FACT 1. *The set $\mathcal{J}$ of those elements F in $BL(H)$ which satisfy $\sum_{n=1}^{\infty} 2^{5^n}\|F + \mathcal{J}_n\| < \infty$ is a selfadjoint Jordan subalgebra of $BL(H)$ containing $\mathcal{M}$ and consisting of compact operators on H. Moreover, if for F in $\mathcal{J}$ we define*

$$|\!|F|\!| := \|F\| + \sum_{n=1}^{\infty} 2^{5^n}\|F + \mathcal{J}_n\|,$$

then $|\!|\cdot|\!|$ becomes a complete algebra norm on $\mathcal{J}$ extending the monstrous norm on $\mathcal{M}$ and satisfying $|\!|F^\#|\!| = |\!|F|\!|$ for all F in $\mathcal{J}$, where $F^\#$ denotes the adjoint operator of F.

Proof. Since J is a selfadjoint subset of B, $\mathcal{J}_n$ is a selfadjoint subset of $BL(H)$, so, for F in $BL(H)$, we have $\|F + \mathcal{J}_n\| = \|F^\# + \mathcal{J}_n\|$ for all n in $\mathbb{N}$. Therefore $F^\#$ lies in $\mathcal{J}$ whenever F is in $\mathcal{J}$ and, if this is the case, the equality $|\!|F^\#|\!| = |\!|F|\!|$ holds.

For $k = 0, 1, 2, \ldots$ and F in $BL(H)$ define inductively $|\!|F|\!|_k$ by

$$|\!|F|\!|_0 := \|F\| \quad\text{and}\quad |\!|F|\!|_{k+1} := |\!|F|\!|_k + 2^{5^{k+1}}\|F + \mathcal{J}_{k+1}\|,$$

so that F is in $\mathcal{J}$ if and only if the increasing sequence $\{|\!|F|\!|_k\}_{k\in\mathbb{N}\cup\{0\}}$ is convergent. If this is the case, then $|\!|F|\!|_k \to |\!|F|\!|$ as $k \to \infty$. As in the proof of Proposition 3, $|\!|F.G|\!|_k \le |\!|F|\!|_k |\!|G|\!|_k$ for all F, G in $BL(H)$ and all k in $\mathbb{N}$. Therefore, if F and G are in $\mathcal{J}$, then $F.G$ lies in $\mathcal{J}$ and $|\!|F.G|\!| \le |\!|F|\!|\,|\!|G|\!|$. Since $\mathcal{J}$ is clearly a subspace of $BL(H)$, this shows that $\mathcal{J}$ is a Jordan subalgebra of $BL(H)$ and that $|\!|\cdot|\!|$ is an algebra norm on $\mathcal{J}$. Of course $\mathcal{M}$ is contained in $\mathcal{J}$ and $|\!|\cdot|\!|$ extends the monstrous norm on $\mathcal{M}$.

For F in $\mathcal{J}$, we have ostensibly $\|F + \mathcal{J}_n\| \to 0$ as $n \to \infty$ and therefore, since $\mathcal{J}_n$ is contained in $\mathcal{M}$ which consists of finite rank operators, F is compact. Let $\{F_n\}$ be a $|\!|\cdot|\!|$-Cauchy sequence in $\mathcal{J}$. Then $\{F_n\}$ is $\|\cdot\|$-Cauchy in $BL(H)$, hence it has a $\|\cdot\|$-limit (say F) in $BL(H)$. For $\varepsilon > 0$, let p be in $\mathbb{N}$ such that $|\!|F_n - F_m|\!| \le \varepsilon$ whenever $n, m \ge p$. Then for every k, n, m in $\mathbb{N}$ with $n, m \ge p$ we have $|\!|F_n - F_m|\!|_k \le \varepsilon$ and, since $|\!|\cdot|\!|_k$ is

$\|\cdot\|$-continuous, by letting $m \to \infty$ we obtain $|F_n - F|_k \leq \varepsilon$. Now, for $n \geq p$, $F_n - F$ belongs to $\mathcal{J}$ and $|F_n - F| \leq \varepsilon$. Therefore F lies in $\mathcal{J}$ and is the $|\cdot|$-limit of $\{F_n\}$. ∎

Remark 5. Note that, in view of Theorem 1 and the fact that selfadjoint associative subalgebras of $BL(H)$ are semiprime, neither the Jordan–Banach algebra $(\mathcal{J}, |\cdot|)$ in Fact 1, nor any of its selfadjoint $|\cdot|$-closed subalgebras containing $\mathcal{M}$ can be an associative subalgebra of $BL(H)$.

From now on, the theory of the socle in semiprime associative algebras and nondegenerate Jordan algebras will become crucial. The reader is referred to [10] and [15] in the associative and Jordan setting, respectively. As usual, for elements x, y in a Jordan algebra J, we define $U_x(y) := 2x.(x.y) - x^2.y$. If J is a Jordan subalgebra of an associative algebra, then $U_x(y)$ has the easier writing $U_x(y) = xyx$.

FACT 2. *Let $\mathcal{K}$ be a selfadjoint Jordan subalgebra of $BL(H)$ containing $\mathcal{M}$. Then $\mathcal{K}$ is prime nondegenerate and has nonzero socle. More precisely, the socle of $\mathcal{K}$ is a simple associative subalgebra of $BL(H)$ containing $\mathcal{M}$. Moreover, if in addition $\mathcal{K}$ consists only of compact operators, then the socle of $\mathcal{K}$ is central simple over $\mathbb{K}$.*

Proof. $\mathcal{K}$ is nondegenerate because, for F in $\mathcal{K} \setminus \{0\}$, we have $F^\# \in \mathcal{K}$ and $U_F(F^\#) = FF^\#F \neq 0$. On the other hand, $\mathcal{M}$ has many nonzero idempotents e satisfying $eBL(H)e = \mathbb{K}e$. Since $\mathcal{K}$ contains $\mathcal{M}$, it follows that $\mathcal{K}$ contains many nonzero idempotents e satisfying $U_e(\mathcal{K}) = \mathbb{K}e$, hence the socle of $\mathcal{K}$, $\mathrm{soc}(\mathcal{K})$, is nonzero. More precisely, for e in $\mathcal{M}$ as above, the ideal Q of $\mathcal{K}$ generated by e is a simple ideal of $\mathcal{K}$ contained in $\mathrm{soc}(\mathcal{K})$. Now $Q \cap \mathcal{M}$ is a nonzero Jordan ideal of the simple associative algebra $\mathcal{M}$, hence $Q \cap \mathcal{M} = \mathcal{M}$ [9; Theorem 2.1.1], and in fact $\mathcal{M}$ is contained in Q (hence in $\mathrm{soc}(\mathcal{K})$).

Let P be an arbitrary nonzero ideal of $\mathcal{K}$. Then $P \cap Q$ is either zero or Q because of the simplicity of Q. But the first possibility cannot occur since, from $P \cap Q = 0$, we would obtain $0 = P.Q \supseteq P.\mathcal{M}$, hence $P = 0$, a fact that contradicts the assumption. In this way we have proved that every nonzero ideal of $\mathcal{K}$ contains Q, which in turn implies the primeness of $\mathcal{K}$.

Finally, since $\mathcal{K}$ is prime and nondegenerate, $\mathrm{soc}(\mathcal{K})$ is a simple Jordan subalgebra of $BL(H)$ and clearly it contains $[\mathcal{M}, \mathcal{M}] \neq 0$. Moreover, the associative subalgebra of $BL(H)$ generated by $\mathrm{soc}(\mathcal{K})$ has zero annihilator in itself because it is $\#$-invariant. It follows from Lemma 3 that $\mathrm{soc}(\mathcal{K})$ is an associative subalgebra of $BL(H)$.

To conclude the proof, assume that $\mathcal{K}$ is contained in the algebra $KL(H)$ of all compact operators on H. Then, since $KL(H)$ is centrally closed over $\mathbb{K}$ [13; Theorem 12] and $\mathrm{soc}(\mathcal{K})$ is $\|\cdot\|$-dense in $KL(H)$ (note that even $\mathcal{M}$ is $\|\cdot\|$-dense in $KL(H)$), we may apply [21; Lemma 4] to conclude that the simple algebra $\mathrm{soc}(\mathcal{K})$ is central over $\mathbb{K}$. ∎

By taking $\mathcal{K}$ in Fact 2 equal to the completion of the monster relative to the monstrous norm, namely the closure of $\mathcal{M}$ in $(\mathcal{J}, |\cdot|)$ (see Fact 1), we obtain immediately the following theorem.

THEOREM 8. *Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Then there exists a nondegenerate topologically simple (hence prime) Jordan–Banach algebra $\mathcal{K}$ over $\mathbb{K}$ whose socle is of the form A^+, where A is a central simple associative algebra over $\mathbb{K}$, and nevertheless the restriction of the norm of $\mathcal{K}$ to A makes the associative product of A discontinuous.*

A *complex Banach pairing* consists of a pair (Y, Z) of complex Banach spaces together with a continuous nondegenerate bilinear mapping $\langle\cdot,\cdot\rangle : Y \times Z \to \mathbb{C}$. Given such a complex Banach pairing, we denote by $L_Z(Y)$ the associative algebra of all linear operators $F : Y \to Y$ having an adjoint $F^* : Z \to Z$ relative to $\langle\cdot,\cdot\rangle$, and by $FL_Z(Y)$ the simple ideal of $L_Z(Y)$ consisting of all finite-rank operators in $L_Z(Y)$. By the closed graph theorem, the operators in $L_Z(Y)$, as well as their adjoints, are automatically continuous, and $L_Z(Y)$ becomes a complex Banach algebra under the norm $|F| := \max\{\|F\|, \|F^*\|\}$ (see for example [2]). By the results of [15] and [16], Theorem 8 has the following

COROLLARY 2. *There exist a complex Banach pairing $(Y, Z, \langle\cdot,\cdot\rangle)$, a Jordan subalgebra $\mathcal{K}$ of $L_Z(Y)$ containing $FL_Z(Y)$, and a complete algebra norm $|\cdot|$ on $\mathcal{K}$ such that the inclusion $(\mathcal{K}, |\cdot|) \hookrightarrow (L_Z(Y), |\cdot|)$ is continuous, but the associative product of $FL_Z(Y)$ is $|\cdot|$-discontinuous.*

In what follows we need to know that the (linear) involution $*$ on $\mathcal{M}$ can be extended to an involution (also denoted by $*$) on $BL(H)$ commuting with the operator adjoint $\#$, as well as the observation that the Jordan algebra $\mathcal{J}$ in Fact 1 is $*$-invariant and $*$ becomes $|\cdot|$-isometric on $\mathcal{J}$. To see this, consider the mapping $j : (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto (-\overline{\lambda}_2, \overline{\lambda}_1, -\overline{\lambda}_4, \overline{\lambda}_3)$ from $X = \mathbb{K}^4$ into X (where, for λ in $\mathbb{K}$, $\overline{\lambda}$ denotes the conjugate of λ) and let k be the mapping from $H = \bigoplus_{n\in\mathbb{N}}^{\ell_2} X_n$ (where $X_n = X$ for all n in $\mathbb{N}$) into H given by $k((x_1, x_2, \ldots)) = (j(x_1), j(x_2), \ldots)$. Then, since $b^* = j^{-1} \circ b^\# \circ j$ for all b in $B = BL(X)$, we have $\alpha^* = k^{-1} \circ \alpha^\# \circ k$ for all α in $\mathcal{M} = M_\infty(B)$, and therefore the mapping $F \mapsto k^{-1} \circ F^\# \circ k$ is an involution on $BL(H)$ coinciding with $*$ on $\mathcal{M}$ and commuting with $\#$. Now the $*$-invariance of $\mathcal{J}$ and the $|\cdot|$-isometry of $*$ on $\mathcal{J}$ follow easily from the definitions of $\mathcal{J}$ and $|\cdot|$.

FACT 3. *Let $\mathcal{K}$ be a selfadjoint Jordan subalgebra of $BL(H)$ contained in $H(BL(H), *)$ and containing $H(\mathcal{M}, *)$. Then $\mathcal{K}$ is prime nondegenerate and has nonzero socle. If A denotes the (automatically $*$-invariant) associa-*

tive subalgebra of $BL(H)$ *generated by* $\mathrm{soc}(\mathcal{K})$, *then* A *is simple*, $\mathcal{M} \subseteq A$, *and* $\mathrm{soc}(\mathcal{K}) = H(A,*)$. *Moreover, if in addition* $\mathcal{K}$ *consists only of compact operators, then* A *is central simple over* $\mathbb{K}$.

Proof. As in the proof of Fact 2, we can verify that $\mathcal{K}$ is nondegenerate. Define

$$e := \left(\begin{array}{cc|ccc} 1 & 0 & 0 & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot \\ \hline 0 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right).$$

Then e is an idempotent in $H(\mathcal{M},*)$ satisfying $eH(BL(H),*)e = \mathbb{K}e$, so $U_e(\mathcal{K}) = \mathbb{K}e$, which implies that $\mathcal{K}$ has a nonzero socle. Now, minor changes in the proof of Fact 2 show that $\mathcal{K}$ is prime (hence $\mathrm{soc}(\mathcal{K})$ is simple) and that $\mathrm{soc}(\mathcal{K}) \supseteq H(\mathcal{M},*)$.

Let A denote the associative subalgebra of $BL(H)$ generated by $\mathrm{soc}(\mathcal{K})$. Then A has zero annihilator in itself (since it is selfadjoint) and it is impossible to find x, y in $\mathrm{soc}(\mathcal{K})$ with $[x,y] \in \mathrm{soc}(\mathcal{K}) \setminus \{0\}$ (since $\mathrm{soc}(\mathcal{K}) \subseteq \mathcal{K} \subseteq H(BL(H),*)$). It follows from Lemma 3 that there exists an involution $\square$ on A such that A is $\square$-simple and $\mathrm{soc}(\mathcal{K}) = H(A,\square)$. But $\square$ and $*$ coincide on A because both involutions fix the elements of $\mathrm{soc}(\mathcal{K})$ and $\mathrm{soc}(\mathcal{K})$ generates A. On the other hand, since $H(A,*)$ contains $H(\mathcal{M},*)$ and generates A, A must contain $\mathcal{M}$ because $H(\mathcal{M},*)$ generates $\mathcal{M}$ [9; Example 2 on p. 59 and Theorem 2.1.2].

Now A is prime (a consequence of the associative specialization of Fact 2) and $*$-simple, hence A is simple. The fact that A is central simple over $\mathbb{K}$ whenever $\mathcal{K} \subseteq KL(H)$ follows from the $\|\cdot\|$-density of A in $KL(H)$ by the same arguments as in the conclusion of the proof of Fact 2. ■

By taking $\mathcal{K}$ in Fact 3 equal to the completion of $H(\mathcal{M},*)$ relative to the monstrous norm, namely the closure of $H(\mathcal{M},*)$ in $(\mathcal{J}, |\!|\cdot|\!|)$ (see Fact 1), we obtain

THEOREM 9. *Let* $\mathbb{K}$ *denote either* $\mathbb{R}$ *or* $\mathbb{C}$. *Then there exists a nondegenerate topologically simple Jordan Banach algebra* $\mathcal{K}$ *over* $\mathbb{K}$ *whose socle is of the form* $H(A,*)$, *where* A *is a central simple associative algebra over* $\mathbb{K}$ *with an involution* $*$, *and nevertheless the topology of the restriction of the norm of* $\mathcal{K}$ *to* $H(A,*)$ *cannot be extended to the topology of any algebra norm on* A.

A *complex self-paired Banach space* is nothing but a complex Banach pairing $(Y, Z, \langle\cdot,\cdot\rangle)$ with $Z = Y$ and such that the continuous nondegenerate bilinear form $\langle\cdot,\cdot\rangle$ on Y becomes either symmetric or alternate. In this case the passing $*$ to the adjoint operator becomes an involution on the algebra $L_Y(Y)$ leaving $FL_Y(Y)$ invariant. Again the results in [15] and [16] allow us to obtain the following

COROLLARY 3. *There exist a complex self-paired Banach space* $(Y, \langle\cdot,\cdot\rangle)$, *a Jordan subalgebra* $\mathcal{K}$ *of* $L_Y(Y)$ *contained in* $H(L_Y(Y),*)$ *and containing* $H(FL_Y(Y),*)$, *and a complete algebra norm* $|\!|\cdot|\!|$ *on* $\mathcal{K}$ *such that the inclusion* $(\mathcal{K}, |\!|\cdot|\!|) \hookrightarrow (L_Y(Y), |\cdot|)$ *is continuous whereas the restriction of* $|\!|\cdot|\!|$ *to the simple Jordan algebra* $H(FL_Y(Y),*)$ *cannot be extended to the topology of any algebra norm on* $FL_Y(Y)$.

Concluding remark. In the spirit of Remark 4, we emphasize that the results obtained above show how a "strong" normed version of Zel'manov's theorem in the case of prime nondegenerate Jordan–Banach algebras cannot be expected to hold in general, even if the existence of a nonzero socle is additionally assumed. This gives special relevance to "light" results on the matter like that in [3; Theorem 2] (on general prime nondegenerate Jordan–Banach algebras) and [16; Theorem 1.1] (on prime nondegenerate Jordan–Banach algebras with nonzero socle). Taking into account that prime nondegenerate Jordan algebras with nonzero socle are primitive [7], it also follows that the recently obtained Zel'manovian classification of primitive Jordan–Banach algebras [3; Theorem 3] cannot be improved in general.

Acknowledgements. The authors wish to express their gratitude to F. Montaner for helpful discussions about the topics developed in this paper. They also want to thank the referee for his very interesting suggestions.

References

[1] R. Arens, M. Goldberg and W. A. J. Luxemburg, *Multiplicativity factors for seminorms II*, J. Math. Anal. Appl. 170 (1992), 401–413.

[2] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, Berlin, 1973.

[3] M. Cabrera, A. Moreno and A. Rodríguez, *Zel'manov's theorem for primitive Jordan–Banach algebras*, preprint, Universidad de Granada, 1993.

[4] M. Cabrera and A. Rodríguez, *Zel'manov's theorem for normed simple Jordan algebras with a unit*, Bull. London Math. Soc. 25 (1993), 59–63.

[5] —, —, *Nondegenerately ultraprime Jordan–Banach algebras: a Zel'manovian treatment*, Proc. London Math. Soc. 69 (1994), 576–604.

[6] A. Fernández, E. Garcia and A. Rodríguez, *A Zel'manov prime theorem for JB*-algebras*, J. London Math. Soc. 46 (1992), 319–335.

[7] A. Fernández and A. Rodríguez, *Primitive noncommutative Jordan algebras with nonzero socle*, Proc. Amer. Math. Soc. 96 (1986), 199–206.

[8] H. Hanche-Olsen and E. Størmer, *Jordan Operator Algebras*, Monographs Stud. Math. 21, Pitman, 1984.

[9] I. N. Herstein, *Rings with Involution*, Chicago Lectures in Math., The University of Chicago Press, Chicago, 1976.

[10] N. Jacobson, *Structure of Rings*, Amer. Math. Soc. Colloq. Publ. 37, Providence, R.I., 1968.

[11] N. Jacobson, *Structure and Representations of Jordan Algebras*, Amer. Math. Soc. Colloq. Publ. 39, Providence, R.I., 1968.
[12] —, *Basic Algebra II*, Freeman, San Francisco, 1980.
[13] W. S. Martindale, 3rd., *Lie isomorphisms of prime rings*, Trans. Amer. Math. Soc. 142 (1969), 437–455.
[14] K. McCrimmon, *The Zel'manov approach to Jordan homomorphisms of associative algebras*, J. Algebra 123 (1989), 457–477.
[15] J. M. Osborn and M. L. Racine, *Jordan rings with nonzero socle*, Trans. Amer. Math. Soc. 251 (1979), 375–387.
[16] J. Pérez, L. Rico, A. Rodríguez and A. R. Villena, *Prime Jordan–Banach algebras with nonzero socle*, Comm. Algebra 20 (1992), 17–53.
[17] C. E. Rickart, *General Theory of Banach Algebras*, Krieger, New York, 1974.
[18] A. Rodríguez, *La continuidad del producto de Jordan implica la del ordinario en el caso completo semiprimo*, in: Contribuciones en Probabilidad, Estadística Matemática, Enseñanza de la Matemática y Análisis, Secretariado de Publicaciones de la Universidad de Granada, Granada, 1979, 280–288.
[19] —, *A Vidav–Palmer theorem for Jordan C^*-algebras and related topics*, J. London Math. Soc. 22 (1980), 318–332.
[20] —, *Jordan axioms for C^*-algebras*, Manuscripta Math. 61 (1988), 297–314.
[21] —, *Continuity of densely valued homomorphisms into H^*-algebras*, Quart. J. Math. Oxford, to appear.
[22] —, *Jordan structures in Analysis*, in: Jordan algebras: Proc. Conf. Oberwolfach, 1992, W. Kaup, K. McCrimmon and H. P. Petersson (eds.), de Gruyter, Berlin, 1994.
[23] A. Rodríguez, A. Slin'ko and E. Zel'manov, *Extending the norm from Jordan–Banach algebras of hermitian elements to their associative envelopes*, Comm. Algebra 22 (1994), 1435–1455.
[24] S. Shirali, *On the Jordan structure of complex Banach $*$-algebras*, Pacific J. Math. 27 (1968), 397–404.
[25] A. M. Slin'ko, *On topological rings with involution*, Uspekhi Mat. Nauk SSSR 41 (1986), 197–198 (in Russian).
[26] E. Zel'manov, *Absolute zero-divisors and algebraic Jordan algebras*, Siberian Math. J. 23 (1982), 100–116.
[27] —, *On prime Jordan algebras II*, ibid. 24 (1983), 89–104.

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
FACULTAD DE CIENCIAS
UNIVERSIDAD DE GRANADA
18071 GRANADA, SPAIN
E-mail: APALACIOS@UGR.ES

Received June 10, 1994
Revised version September 26, 1994 (3290)

INFORMATION FOR AUTHORS

Manuscripts should be typed on one side only, with double or triple spacing and wide margins, and submitted in duplicate, including the original typewritten copy.

An **abstract** of not more than 200 words and the AMS Mathematics Subject Classification are required.

Formulas must be typewritten. A complete **list of all handwritten symbols** with indications for the printer should be enclosed.

Figures must be prepared in a form suitable for direct reproduction. Sending EPS, PCX, TIF or CorelDraw files will be most helpful. The author should indicate on the margin of the manuscript where figures are to be inserted.

References should be arranged in alphabetical order, typed with double spacing, and styled and punctuated according to the examples given below. Abbreviations of journal names should follow Mathematical Reviews. Titles of papers in Russian should be translated into English.

Examples:

[6] D. Beck, *Introduction to Dynamical Systems*, Vol. 2, Progr. Math. 54, Birkhäuser, Basel, 1978.
[7] R. Hill and A. James, *An index formula*, J. Differential Equations 15 (1982), 197–211.
[8] J. Kowalski, *Some remarks on $J(X)$*, in: Algebra and Analysis, Proc. Conf. Edmonton 1973, E. Brook (ed.), Lecture Notes in Math. 867, Springer, Berlin, 1974, 115–124.
[Nov] A. S. Novikov, *An existence theorem for planar graphs*, preprint, Moscow University, 1980 (in Russian).

Authors' **affiliation** should be given at the end of the manuscript.

Authors receive only page **proofs** (one copy). If the proofs are not returned promptly, the article will be printed in a later issue.

Authors receive 50 **reprints** of their articles. Additional reprints can be ordered.

The publisher strongly encourages submission of manuscripts written in TEX. **On acceptance of the paper,** authors will be asked to send discs (preferably PC) plus relevant details to the Editorial Committee, or transmit the file by electronic mail to:

STUDIA@IMPAN.IMPAN.GOV.PL