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On the embedding of 2-concave Orlicz spaces into L^1

by

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Abstract. In [K-S 1] it was shown that $\text{Ave}_\pi(\sum_{i=1}^n |x_i a_{\pi(i)}|^2)^{1/2}$ is equivalent to an Orlicz norm whose Orlicz function is 2-concave. Here we give a formula for the sequence $a_1, \ldots, a_n$ so that the above expression is equivalent to a given Orlicz norm.

A convex function $M : \mathbb{R} \to \mathbb{R}$ with $M(t) = M(-t)$, $M(0) = 0$, and $M(t) > 0$ for $t \neq 0$ is called an *Orlicz function.* M is said to be 2-*concave* if $M(\sqrt{t})$ is a concave function on $[0, \infty)$, and *strictly* 2-*concave* if $M(\sqrt{t})$ is strictly concave. M is 2-*convex* if $M(\sqrt{t})$ is convex, and *strictly* 2-*convex* if $M(\sqrt{t})$ is strictly convex. If M' is invertible on $(0, \infty)$ then the *dual function* is given by

$$M^*(t) = \int_0^t M'^{-1}(s)\, ds.$$

We define the *Orlicz norm* of a sequence $\{x_i\}_{i=1}^\infty$ by

$$\|x\|_M = \sup\Big\{\sum_{i=1}^\infty x_i y_i \,\Big|\, \sum_{i=1}^\infty M^*(y_i) \le 1\Big\}.$$

In [K-S 1, K-S 2] we have used a different expression for the definition of the Orlicz norm: x has norm equal to 1 if and only if $\sum_{i=1}^\infty M(x_i) = 1$. But it turns out that the above definition gives slightly better estimates.

Bretagnolle and Dacunha-Castelle [B-D] showed that an Orlicz space l^M is isomorphic to a subspace of L^1 if and only if M is equivalent to a 2-concave Orlicz function. As a corollary we get the same result here. In [K-S 1] a variant of the following result was obtained.

THEOREM 1. *Let $a_1 \ge \ldots \ge a_n > 0$ and let M be an Orlicz function with*

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$$(1)\qquad M^{*-1}\left(\frac{l}{n}\right)=\left\{\left(\frac{1}{n}\sum_{i=1}^{l}a_i\right)^2+\frac{l}{n}\left(\frac{1}{n}\sum_{i=l+1}^{n}|a_i|^2\right)\right\}^{1/2}$$

for all $l=1,\ldots,n$ *and such that* M^{*-1} *is an affine function between the given values. Then for all* $x\in\mathbb{R}^n$ *we have*

$$(2)\qquad \frac{1}{2\sqrt{5}}\cdot\frac{(n-1)^2}{n^2+(n-1)^2}\|x\|_M\le \operatorname*{Ave}_{\pi}\left(\sum_{i=1}^{n}|x_i a_{\pi(i)}|^2\right)^{1/2}\le\frac{2\sqrt{2}}{c_n}\|x\|_M$$

where $c_n=1-\frac{1}{2!}+\frac{1}{3!}-\ldots+(-1)^{n+1}\frac{1}{n!}$.

We present here those arguments of the proof of Theorem 1 that are different from the arguments in [K–S 1, K–S 2].

There is always an Orlicz function M satisfying the assumptions of Theorem 1. In order to verify this we show that

$$\left\{\left(\int_0^x f(t)\,dt\right)^2+x\int_x^1|f(t)|^2\,dt\right\}^{1/2}$$

is a concave function of x. We may assume that f is differentiable. The second derivative of the above expression is

$$\frac{f'(x)(\int_0^x f(t)\,dt-xf(x))}{\{(\int_0^x f(t)\,dt)^2+x\int_x^1|f(t)|^2\,dt\}^{1/2}}-\frac{1}{4}\cdot\frac{(2f(x)\int_0^x f(t)\,dt+\int_x^1|f(t)|^2\,dt-xf(x))^2}{\{(\int_0^x f(t)\,dt)^2+x\int_x^1|f(t)|^2\,dt\}^{3/2}}.$$

The first summand is nonpositive since f is decreasing.

It follows from Theorem 1 that an Orlicz function M has to be equivalent to a 2-concave Orlicz function if l^M is isomorphic to a subspace of L^1 [K–S 1, K–S 2]. We compute here how we have to choose the sequence $a_1,\ldots,a_n$ to get (1) for a given 2-concave Orlicz function M. From this it also follows that l^M is isomorphic to a subspace of L^1 if M is 2-concave.

THEOREM 2. *Let M be a strictly convex, twice differentiable Orlicz function that is strictly 2-concave. Assume that* $M^*(1)=1$ *and let*

$$(3)\qquad a_l=-\frac{n}{2}\int_{(l-1)/n}^{l/n}\left(\int_t^1\frac{((M^{*-1})^2)''(s)}{\sqrt{(M^{*-1})^2(s)-s((M^{*-1})^2)'(s)}}\,ds+1-\sqrt{1-((M^{*-1})^2)'(1)}\right)dt$$

for $l=1,\ldots,n$. *Then for all* $x\in\mathbb{R}^n$ *we have*

$$\frac{1}{c}\|x\|_M\le\operatorname*{Ave}_{\pi}\left(\sum_{i=1}^{n}|x_i a_{\pi(i)}|^2\right)^{1/2}\le c\|x\|_M$$

where c is a constant that does not depend on n and M.

Since $\mathbb{R}^n$ with the norm

$$\|x\|=\operatorname*{Ave}_{\pi}\left(\sum_{i=1}^{n}|x_i a_{\pi(i)}|^2\right)^{1/2}$$

is isometric to a subspace of L^1, we get the following corollary.

COROLLARY 3. *Let M be a 2-concave Orlicz function. Then l^M is isomorphic to a subspace of L^1.*

LEMMA 4 ([K–S 1]). *For all $n\in\mathbb{N}$ and all $n\times n$ matrices A with nonnegative entries we have*

$$c_n\frac{1}{n}\sum_{k=1}^{n}s(k)\le\operatorname*{Ave}_{\pi}\max_{1\le i\le n}|a(i,\pi(i))|\le\frac{1}{n}\sum_{k=1}^{n}s(k)$$

where $c_n=1-\frac{1}{2!}+\frac{1}{3!}-\ldots+(-1)^{n+1}\frac{1}{n!}$ *and* $s(k)$, $k=1,\ldots,n^2$, *is the nonincreasing rearrangement of the numbers* $a(i,j)$, $i,j=1,\ldots,n$.

LEMMA 5 ([K–S 2]). *For all $n\in\mathbb{N}$ and all nonnegative numbers* $a(i,j,k)$, $i,j,k=1,\ldots,n$, *we have*

$$\frac{(n-1)^2}{n^2+(n-1)^2}\cdot\frac{1}{n^2}\sum_{k=1}^{n^2}s(k)\le\operatorname*{Ave}_{\pi,\sigma}\max_{1\le i\le n}|a(i,\pi(i),\sigma(i))|\le\frac{1}{n^2}\sum_{k=1}^{n^2}s(k)$$

where the average is taken over all permutations π,σ of $\{1,\ldots,n\}$ and $\{s(k)\}_{k=1}^{n^3}$ *is the nonincreasing rearrangement of the numbers* $a(i,j,k)$, $i,j,k=1,\ldots,n$.

LEMMA 6 ([K–S 1]). *Let* $b_1\ge\ldots\ge b_s>0$, $n\le s$, *and*

$$(4)\qquad \|x\|_b=\max_{\Sigma k_j=s}\sum_{i=1}^{n}\left(\sum_{j=1}^{k_i}b_j\right)|x_i|.$$

Then for all Orlicz functions M with $M^*(\sum_{j=1}^{l}b_j)=l/s$, $l=1,\ldots,s$, *and all* $x\in\mathbb{R}^n$ *we have*

$$\|x\|_b\le\|x\|_M\le 2\|x\|_b.$$

The proof of the right hand inequality of Lemma 6 is the same as in [K–S 2]. The left hand inequality follows from the definition of the norm.

Proof of Theorem 1. We choose the sequence b_j, $j=1,\dots,n$, with

$$\sum_{j=1}^{k} b_j = \sqrt{nk}, \qquad k=1,\dots,n.$$

Then by Lemmata 4 and 6 we get

$$\frac{c_n}{2}\operatorname*{Ave}_{\pi}\Big(\sum_{i=1}^{n}|x_i a_{\pi(i)}|^2\Big)^{1/2} \le \operatorname*{Ave}_{\pi,\sigma}\max_{1\le i\le n}|x_i a_{\pi(i)} b_{\sigma(i)}|$$
$$\le \operatorname*{Ave}_{\pi}\Big(\sum_{i=1}^{n}|x_i a_{\pi(i)}|^2\Big)^{1/2}.$$

By Lemma 5,

$$\frac{(n-1)^2}{n^2+(n-1)^2}\cdot\frac{1}{n^2}\sum_{k=1}^{n^2}s(k) \le \Big(\sum_{i=1}^{n}|x_i a_{\pi(i)}|^2\Big)^{1/2} \le \frac{2}{c_n}\cdot\frac{1}{n^2}\sum_{k=1}^{n^2}s(k),$$

where $s(k)$, $k=1,\dots,n^3$, is the nonincreasing rearrangement of $|x_i a_j b_k|$, $i,j,k=1,\dots,n$. We apply Lemma 6 again with $s=n^2$ and the Orlicz function N such that

$$N^*\Big(\frac{1}{n^2}\sum_{j=1}^{l}t(j)\Big) = \frac{l}{n^2}, \qquad l=1,\dots,n^2,$$

where $t(j)$, $j=1,\dots,n^2$, is the nonincreasing rearrangement of $|a_i b_k|$, $i,k=1,\dots,n$, and such that N^* is an affine function between the given values. We get

$$\frac{1}{2}\cdot\frac{(n-1)^2}{n^2+(n-1)^2}\|x\|_N \le \Big(\sum_{i=1}^{n}|x_i a_{\pi(i)}|^2\Big)^{1/2} \le \frac{2}{c_n}\|x\|_N.$$

For some integers k_i with $k_i\le n$ and $\sum k_i = ln$ we have

$$N^{*-1}\Big(\frac{l}{n}\Big) = \frac{1}{n^2}\sum_{j=1}^{ln}t(j) = \frac{1}{n^2}\sum_{i=1}^{n}a_i\sum_{j=1}^{k_i}b_j = \frac{1}{n^2}\sum_{i=1}^{n}a_i\sqrt{nk_i}.$$

Since $a_1\ge\ldots\ge a_n\ge 0$ we also have $k_1\ge\ldots\ge k_n$. Therefore

$$N^{*-1}\Big(\frac{l}{n}\Big) \le \frac{1}{n^{3/2}}\Big(\sqrt{k_1}\sum_{i=1}^{l}a_i + \sum_{i=l+1}^{n}a_i\sqrt{k_i}\Big)$$
$$\le \frac{1}{n^{3/2}}\Big(\Big|\sum_{i=1}^{l}a_i\Big|^2 + l\sum_{i=l+1}^{n}|a_i|^2\Big)^{1/2}\Big(k_1+\frac{1}{l}\sum_{i=l+1}^{n}k_i\Big)^{1/2}$$
$$\le \frac{\sqrt{2}}{n}\Big(\Big|\sum_{i=1}^{l}a_i\Big|^2 + l\sum_{i=l+1}^{n}|a_i|^2\Big)^{1/2} = \sqrt{2}M^{*-1}\Big(\frac{l}{n}\Big).$$

We get immediately

$$N^{*-1}\Big(\frac{l}{n}\Big) = \frac{1}{n^2}\sum_{j=1}^{ln}t(j) \ge \frac{1}{n}\sum_{i=1}^{l}a_i$$

and as in [K-S 2],

$$N^{*-1}\Big(\frac{l}{n}\Big) \ge \frac{\sqrt{l}}{2n}\Big(\sum_{i=1}^{l}|a_i|^2\Big)^{1/2}.$$

Therefore we have

$$M^{*-1}\Big(\frac{l}{n}\Big) = \frac{1}{n}\Big(\Big|\sum_{i=1}^{l}a_i\Big|^2 + l\sum_{i=l+1}^{n}|a_i|^2\Big)^{1/2} \le \sqrt{5}N^{*-1}\Big(\frac{l}{n}\Big).$$

Altogether, for $l=1,\dots,n$ we have

$$\frac{1}{\sqrt{2}}N^{*-1}\Big(\frac{l}{n}\Big) \le M^{*-1}\Big(\frac{l}{n}\Big) \le \sqrt{5}N^{*-1}\Big(\frac{l}{n}\Big).$$

Since M^* and N^* are affine functions for the other values, the above inequalities extend to arbitrary values and we get

$$\frac{1}{\sqrt{2}}\|x\|_N \le \|x\|_M \le \sqrt{5}\|x\|_N. \ \blacksquare$$

LEMMA 7. *Let H be a concave, increasing function on $[0,1]$ that is twice continuously differentiable on $(0,1]$, continuous on $[0,1]$ and satisfies $H(0)=0$. Assume that $(H(t)/t)'\ne 0$ for all $t\in(0,1]$. Then*

(i) $$\lim_{t\to 0} t\Big(-\frac{d}{dt}\Big(\frac{H(t)}{t}\Big)\Big)^{1/2} = 0.$$

(ii) *The function f given by*

$$f(t) = -\frac{1}{2}\int_t^1 \frac{H''(s)}{\sqrt{H(s)-sH'(s)}}\,ds + \sqrt{H(1)} - \sqrt{H(1)-H'(1)} \tag{5}$$

is well defined, nonnegative, decreasing, and differentiable on $(0,1]$.

(iii) $\lim_{t\to 0} tf(t) = 0$.

(iv) *The integral $\int_0^1 f(t)\,dt$ is finite and for all $t\in[0,1]$,*

$$H(t) = \Big(\int_0^t f(s)\,ds\Big)^2 + t\int_t^1 |f(s)|^2\,ds.$$

Proof. (i) We have

$$\lim_{t\to 0} t\Big(-\frac{d}{dt}\Big(\frac{H(t)}{t}\Big)\Big)^{1/2} = \lim_{t\to 0} t\Big(\frac{H(t)}{t^2}-\frac{H'(t)}{t}\Big)^{1/2} = \lim_{t\to 0}(H(t)-tH'(t))^{1/2}.$$

We use the fact that $0\le H'(t)\le H(t)/t$.

(ii) Since $(H(t)/t)' \neq 0$ and H is concave, we have $H(t) - tH'(t) > 0$. Again, by the concavity of H the integrand is nonpositive and therefore f is nonnegative and decreasing.

(iii) We have

$$\frac{1}{t} \cdot \frac{d}{dt}\left(t\sqrt{-\frac{d}{dt}\left(\frac{H(t)}{t}\right)}\right) = -\frac{H''(t)}{2\sqrt{H(t) - tH'(t)}}. \tag{6}$$

Integration by parts gives us

$$tf(t) = t\left[\sqrt{-\frac{d}{ds}\left(\frac{H(s)}{s}\right)}\right]_t^1 + t\int_t^1 \frac{1}{s}\sqrt{-\frac{d}{ds}\left(\frac{H(s)}{s}\right)}\,ds + t(\sqrt{H(1)} - \sqrt{H(1) - H'(1)}).$$

The first summand tends to 0 because of (i), and the third trivially. The second summand also tends to 0: If the integral is bounded this is trivial. If the integral is not bounded we apply l'Hôpital's rule and (i).

(iv) In general f is unbounded in a neighborhood of 0. We have

$$\int_0^t f(s)\,ds = \lim_{\varepsilon\to 0}\left\{[sf(s)]_\varepsilon^t - \int_\varepsilon^t sf'(s)\,ds\right\}.$$

By (ii), the definition (5) of f, and (6) we get

$$\int_0^t f(s)\,ds = tf(t) + t\sqrt{-\frac{d}{dt}\left(\frac{H(t)}{t}\right)} - \lim_{\varepsilon\to 0}\varepsilon\sqrt{-\frac{d}{d\varepsilon}\left(\frac{H(\varepsilon)}{\varepsilon}\right)}.$$

By (i),

$$\int_0^t f(s)\,ds = tf(t) + t\sqrt{-\frac{d}{dt}\left(\frac{H(t)}{t}\right)} \tag{7}$$

or

$$\left\{\frac{1}{t}\int_0^t f(s)\,ds - f(t)\right\}^2 = -\frac{d}{dt}\left(\frac{H(t)}{t}\right).$$

Therefore we have

$$\frac{H(x)}{x} - H(1) = -\int_x^1 \frac{d}{dt}\left(\frac{H(t)}{t}\right)dt = \int_x^1 \left|\frac{1}{t}\int_0^t f(s)\,ds - f(t)\right|^2 dt.$$

With

$$\frac{d}{dt}\left\{\frac{1}{t}\Big(\int_0^t f(s)\,ds\Big)^2 + \int_t^1 |f(s)|^2\,ds\right\} = -\left|\frac{1}{t}\int_0^t f(s)\,ds - f(t)\right|^2$$

we get

$$\frac{H(x)}{x} - H(1) = \frac{1}{x}\Big(\int_0^x f(s)\,ds\Big)^2 + \int_x^1 |f(s)|^2\,ds - \Big(\int_0^1 f(s)\,ds\Big)^2. \tag{8}$$

By (7) we have

$$\int_0^1 f(s)\,ds = f(1) + \sqrt{H(1) - H'(1)}.$$

By the definition (5) of f we get $f(1) = \sqrt{H(1)} - \sqrt{H(1) - H'(1)}$ and therefore $\int_0^1 f(s)\,ds = \sqrt{H(1)}$. Thus from (8) we obtain

$$\frac{H(x)}{x} - H(1) = \frac{1}{x}\Big(\int_0^x f(s)\,ds\Big)^2 + \int_x^1 |f(s)|^2\,ds - H(1),$$

or

$$H(x) = \Big(\int_0^x f(s)\,ds\Big)^2 + x\int_x^1 |f(s)|^2\,ds. \ \blacksquare$$

Proof of Theorem 2. Since M is strictly convex, M'^{-1} exists and $M^{*\prime}(t) = M'^{-1}(t)$. Since M is twice differentiable, so is M^{*-1}. Since $M(\sqrt{t})$ is strictly concave, $(M^{*-1}(t))^2$ is also strictly concave. Therefore,

$$0 > ((M^{*-1})^2)'(s) - \frac{(M^{*-1})^2(s)}{s} = s\frac{d}{ds}\left(\frac{(M^{*-1}(s))^2}{s}\right).$$

We put $H(t) = (M^{*-1}(t))^2$ and apply Lemma 7. Therefore $a_1, \ldots, a_n$ given by (3) is a positive, decreasing sequence with

$$a_l = n\int_{(l-1)/n}^{l/n} f(s)\,ds.$$

We get

$$M^{*-1}\Big(\frac{l}{n}\Big) = \bigg(\Big(\int_0^{l/n} f(s)\,ds\Big)^2 + \frac{l}{n}\Big(\int_{l/n}^1 |f(s)|^2\,ds\Big)\bigg)^{1/2}$$
$$= \bigg(\Big(\frac{1}{n}\sum_{j=1}^{l} a_j\Big)^2 + \frac{l}{n}\Big(\sum_{j=l}^{n-1}\int_{j/n}^{(j+1)/n} |f(s)|^2\,ds\Big)\bigg)^{1/2}.$$

Since $f(l/n) \le a_l \le f((l-1)/n)$ we get

$$M^{*-1}\left(\frac{l}{n}\right) \le \left(\left(\frac{1}{n}\sum_{j=1}^{l} a_j\right)^2 + \frac{l}{n}\left(\frac{1}{n}\sum_{j=l}^{n-1} a_j^2\right)\right)^{1/2},$$

$$M^{*-1}\left(\frac{l}{n}\right) \ge \left(\left(\frac{1}{n}\sum_{j=1}^{l} a_j\right)^2 + \frac{l}{n}\left(\frac{1}{n}\sum_{j=l+1}^{n} a_j^2\right)\right)^{1/2}.$$

Now it remains to apply Theorem 1. ■

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On the behaviour of Jordan-algebra norms on associative algebras

by

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Abstract. We prove that for a suitable associative (real or complex) algebra which has many nice algebraic properties, such as being simple and having minimal idempotents, a norm can be given such that the mapping $(a,b) \mapsto ab + ba$ is jointly continuous while $(a,b) \mapsto ab$ is only separately continuous. We also prove that such a pathology cannot arise for associative simple algebras with a unit. Similar results are obtained for the so-called "norm extension problem", and the relationship between these results and the normed versions of Zel'manov's prime theorem for Jordan algebras are discussed.

Introduction. If A is an associative algebra with the product ab, then its symmetrization A^+, which has the same vector space as A and new product $a.b := \frac{1}{2}(ab+ba)$, becomes a model for the so-called Jordan algebra. Jordan algebras are a well-known class of nonassociative algebras defined by a suitable identity. Our general reference for them is Jacobson's book [11]. Not all Jordan algebras are of the form A^+. Another example of such an algebra arises when A has an involution $*$. Then the space of hermitian elements $H(A,*) := \{x \in A : x^* = x\}$ is closed for the product "." so it can be naturally considered as a subalgebra of A^+. In many cases A^+ and $H(A,*)$ will not be isomorphic. Other standard examples can be constructed from bilinear forms (see Section 1) and octonion matrices. In the algebraic theory of Jordan algebras one of the most powerful results is the recent Zel'manov prime theorem.

Zel'manov's theorem [27] classifies prime nondegenerate Jordan algebras into four types which, roughly speaking, are the following: simple exceptional ones, simple Jordan algebras of a symmetric bilinear form, prime associative algebras regarded as Jordan algebras, and Jordan algebras of hermitian elements in prime associative algebras with a (linear) involution. In this way, an attempt to obtain a reasonable normed variant of Zel'manov's theorem has to involve the following:

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