[14] S. L. Sobolev, *The density of C_0^∞ finite functions in the $L_p^{(m)}$ space*, Sibirsk. Mat. Zh. 4 (1963), 673–682 (in Russian).

[15] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
BANACHA 2
02-097 WARSZAWA, POLAND
E-mail: HAJLASZ@MIMUW.EDU.PL
KALAMAJS@MIMUW.EDU.PL

Received September 10, 1993
Revised version August 16, 1994 (3161)

STUDIA MATHEMATICA 113 (1) (1995)

On automatic boundedness of Nemytskiĭ set-valued operators

by

S. ROLEWICZ (Warszawa) and WEN SONG (Harbin)

Abstract. Let X, Y be two separable F-spaces. Let (Ω,Σ,μ) be a measure space with μ complete, non-atomic and σ-finite. Let N_F be the Nemytskiĭ set-valued operator induced by a sup-measurable set-valued function $F:\Omega\times X\to 2^Y$. It is shown that if N_F maps a modular space $(N(L(\Omega,\Sigma,\mu;X)),\varrho_{N,\mu})$ into subsets of a modular space $(M(L(\Omega,\Sigma,\mu;Y)),\varrho_{M,\mu})$, then N_F is automatically modular bounded, i.e. for each set $K\subset N(L(\Omega,\Sigma,\mu;X))$ such that $r_K=\sup\{\varrho_{N,\mu}(x):x\in K\}<\infty$ we have $\sup\{\varrho_{M,\mu}(y): y\in N_F(K)\}<\infty$.

In 1933–1934 V. Nemytskiĭ [10], [11] considered the operator $F:L^2[a,b]\to L^2[a,b]$, $y(\cdot)=F(x(\cdot))$, where $y(t)=f(t,x(t))$. Nemytskiĭ proved that if F maps $L^2[a,b]$ into itself, then it is automatically continuous. He also used the obtained results to prove the existence and uniqueness of solutions of Hammerstein equations. Since that time the operator F has been generalized in several ways and there are many papers devoted to this subject. Operators of this type are now called *Nemytskiĭ operators.*

In the last years a new important extension of Nemytskiĭ operators appeared.

Let (Ω,Σ,μ) be a measure space. We assume that the measure μ is complete and σ-finite. A function $x(\cdot)$ mapping Ω into a Banach space X is called *measurable* if for each open set $Q\subset X$ the inverse image $x^{-1}(Q)=\{t\in\Omega:x(t)\in Q\}$ is measurable, $x^{-1}(Q)\in\Sigma$. The set of all measurable functions defined on Ω with values in X is denoted by $S(\Omega,X)$.

A function $F(\cdot)$ mapping Ω into subsets of X is called *measurable* if for each open set $Q\subset X$ the inverse image $F^{-1}(Q)=\{t\in\Omega:F(t)\cap Q\neq\emptyset\}$ is measurable, $F^{-1}(Q)\in\Sigma$. By a *measurable selection* of $F(\cdot)$ we mean a (single-valued) function $x_F(\cdot)$ such that $x_F(t)\in F(t)$ for all $t\in\Omega$.

1991 *Mathematics Subject Classification*: 47H99, 28B20, 46A06.
Key words and phrases: Nemytskiĭ set-valued operators, superposition measurable set-valued operators, automatic boundedness, modular spaces.
Research of the first author was partially supported by the Polish Committee for Scientific Research under grant no. 2 2009 91 02.

Let $F(\cdot,\cdot)$ be a function mapping $\Omega\times X$ into subsets of an F-space Y. We say that F is *sup-measurable* if for any measurable function $x(\cdot):\Omega\to X$, the set-valued function $s\to F(s,x(s)):\Omega\to 2^Y$ is measurable.

Every sup-measurable closed-valued map $F:\Omega\times X\to 2^Y$ induces the set-valued operator $N_F: S(\Omega,X)\to 2^{S(\Omega,Y)}$ defined by

$$N_F(x(\cdot))=\{y(\cdot): y(\cdot) \text{ is a measurable selection of } F(\cdot,x(\cdot))\}.$$

The set-valued operator N_F is called the *superposition operator* (or *Nemytskiĭ operator*) generated by F.

In the last years set-valued Nemytskiĭ operators are extensively used in the theory of differential inclusions (see for example [2]).

Recently Appell, Nguyen and Zabreĭko [1] proved the following extension of the classical Nemytskiĭ theorem. Let as before (Ω,Σ,μ) be a measure space. Let U,V be spaces of measurable functions with values in $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). We assume that U,V are so-called ideal spaces. We shall not give the definition here; we only want to mention that Orlicz spaces are ideal spaces. We assume, moreover, that V is reflexive. Appell, Nguyen and Zabreĭko [1] proved that if a Nemytskiĭ operator N_F maps an open set $\Omega\subset U$ into V, then for each $u_0\in\Omega$ there is a neighbourhood U_0 such that $N_F(U_0)$ is a bounded set.

Then a natural questions arises. Is the Nemytskiĭ type theorem also valid for spaces of functions with values in infinite-dimensional spaces?

The aim of this paper is to show that this is true for a large class of spaces called Musielak–Orlicz spaces. The considered spaces need not be locally convex.

Let X be a separable F-space (i.e. complete linear metric space) ([1]) with an F-norm $\|\cdot\|_X$. We say that a set $A\subset X$ is *bounded* in X if for each neighbourhood U of 0, there is a constant $k>0$ such that $A\subset kU$. The space X is called *locally bounded* if there is a bounded neighbourhood of 0. An operator (resp. a set-valued operator) $T(\cdot)$ mapping an F-space X into an F-space Y (resp. into subsets of Y) is called *bounded* if for each bounded set $K\subset X$ the set $T(K)=\{y\in Y: y=Tx,\ x\in K\}$ (resp. $T(K)=\{y\in Y: y\in Tx,\ x\in K\}$) is bounded.

Let $\mathcal{F},\mathcal{G}$ be two classes of F-spaces. We say that an operator (resp. a set-valued operator) is *automatically bounded with respect to* $(\mathcal{F},\mathcal{G})$ if for each $X\in\mathcal{F}$, $Y\in\mathcal{G}$, whenever T maps X into Y (resp. into subsets of Y), then it is bounded from X to Y. If T is automatically bounded with respect to $(\mathcal{F},\mathcal{F})$ then we say that T is *automatically bounded in* $\mathcal{F}$.

In this paper we shall show that set-valued Nemytskiĭ operators (and thus also single-valued Nemytskiĭ operators) are automatically bounded in locally bounded (thus in particular normed) Musielak–Orlicz F-spaces of functions with values in locally bounded F-spaces.

([1]) The basic properties of F-spaces can be found in [13].

In fact, we shall prove a stronger theorem for all Musielak–Orlicz spaces, which gives the result mentioned above for locally bounded Musielak–Orlicz spaces.

Firstly we define a set-valued Nemytskiĭ operator for functions with values in an F-space as at the beginning of this paper. It is enough to assume that X is an F-space.

Let $N(\cdot,\cdot)$ be a real-valued measurable function on $\Omega\times\mathbb{R}$ such that for each $t\in\Omega$ the function $N(t,\cdot)$ is continuous increasing and $N(t,0)=0$ for all $t\in\Omega$. We can define on $S(\Omega,X)$ a metrizing modular

$$\varrho_{N,\mu}(x(\cdot))=\int_\Omega N(t,\|x(t)\|_X)\,d\mu \tag{1}$$

(see Nakano [7]–[9], Musielak [4], Rolewicz [13], p. 6). The set of those $x(\cdot)\in S(\Omega,X)$ such that there is a positive k such that $\varrho_{N,\mu}(kx(\cdot))<\infty$ is denoted by $N(L(\Omega,\Sigma,\mu;X))$.

Observe that if $x_1(\cdot),x_2(\cdot),\ldots$ have disjoint supports, then

$$\varrho_{N,\mu}(x_1(\cdot))+\varrho_{N,\mu}(x_2(\cdot))+\ldots=\varrho_{N,\mu}(x_1(\cdot)+x_2(\cdot)+\ldots). \tag{2}$$

We recall that a *metrizing modular* on a linear space X is a function $\varrho: X\to[0,\infty]$ such that

(*md* 1) $\varrho(x)=0$ if and only if $x=0$,

(*md* 2) $\varrho(ax)=\varrho(x)$ provided $|a|=1$,

(*md* 3) $\varrho(ax+by)\le\varrho(x)+\varrho(y)$ provided $a,b\ge 0,\ a+b=1$,

(*md* 4) $\varrho(a_nx)\to 0$ provided $a_n\to 0,\ \varrho(x)<\infty$,

(*md* 5) $\varrho(ax_n)\to 0$ provided $\varrho(x_n)\to 0$.

A linear space X with a modular ϱ is denoted by (X,ϱ) and called a *modular space*. Let (X,ϱ) be a modular space with metrizing modular ϱ. It is known that ϱ induces an F-norm $\|\cdot\|_X$ in X by

$$\|x\|_X=\inf\{\varepsilon>0:\varrho(x/\varepsilon)<\varepsilon\}. \tag{3}$$

The norm $\|x\|_X$ is equivalent to the modular ϱ in the sense that $\|x_n\|_X\to 0$ if and only if $\varrho(x)\to 0$ (Musielak and Orlicz [5], [6], Musielak [4]; see also Rolewicz [13], p. 8).

Let (X,ϱ_X) and (Y,ϱ_Y) be two modular spaces. An operator (resp. a set-valued operator) $T(\cdot)$ mapping (X,ϱ_X) into (Y,ϱ_Y) (resp. into subsets of (Y,ϱ_Y)) is called *modular bounded* if for each set $K\subset X$ such that

$$r_K=\sup\{\varrho_X(x): x\in K\}<\infty \tag{4}$$

we have

$$\sup\{\varrho_Y(y) : y \in T(K)\} < \infty, \tag{5}$$

where $T(K) = \{y \in Y : y = Tx,\ x \in K\}$ (resp. $T(K) = \{y \in Y : y \in Tx,\ x \in K\}$).

Of course the definition of a modular bounded operator depends on the modulars ϱ_X, ϱ_Y and it is not a topological invariant.

However, just from the definitions it follows that if the spaces (X, ϱ_X) and (Y, ϱ_Y) are locally bounded and the modulars ϱ_X and ϱ_Y satisfy (4) for all bounded sets K, then boundedness and modular boundedness coincide.

THEOREM 1. *Let X, Y be two separable F-spaces. Let (Ω, Σ, μ) be a measure space with μ complete, non-atomic and σ-finite. Let N_F be the Nemytskiĭ set-valued operator induced by a sup-measurable set-valued function $F(\cdot,\cdot) : \Omega \times X \to 2^Y$. Then N_F is automatically modular bounded from a modular space $(N(L(\Omega, \Sigma, \mu; X)), \varrho_{N,\mu})$ into subsets of a modular space $(M(L(\Omega, \Sigma, \mu; Y)), \varrho_{M,\mu})$*

Proof. The proof will be done in three steps:

(s1) If μ is finite then there is a neighbourhood U of 0 in $N(L(\Omega, \Sigma, \mu; X))$ such that

$$\sup\{\varrho_{M,\mu}(y(\cdot)) : y(\cdot) \in F(x(\cdot)), x(\cdot) \in U\} < \infty. \tag{6}$$

(s2) The same as in (s1) for σ-finite measures.

(s3) Each operator as in (s2) is automatically modular bounded.

Proof. (s1). Suppose that (s1) does not hold. For every $\delta > 0$ we can find sequences $\{x_n\}$, $x_n \in N(L(\Omega, \Sigma, \mu; X))$, and $\{y_n\}$, $y_n \in M(L(\Omega, \Sigma, \mu; Y))$, such that $y_n(\cdot) \in N_F(x_n(\cdot))$,

$$\|x_n(\cdot)\|_{N,\mu} \le 2^{-n}\delta \tag{7}$$

and

$$\varrho_{M,\mu}(y_n(\cdot)) \ge n2^n. \tag{8}$$

Since we have assumed that the measure μ is non-atomic and finite, we can find a partition $\{D_{n,j}\}$, $j = 1, \ldots, 2^n$, of Ω such that $\mu(D_{n,j}) = 2^{-n}\mu(\Omega)$ for $j = 1, \ldots, 2^n$. For at least one index $j(n)$ we have

$$\int_{D_{n,j(n)}} M(t, \|y_n(t)\|_Y)\, d\mu \ge n, \tag{9}$$

since otherwise $\varrho_{M,\mu}(y_n(\cdot)) < n2^n$ contradicting (8). Set

$$\Omega_{n,m} = D_{n,j(n)} \setminus \bigcup_{k=m}^{\infty} D_{k,j(k)}.$$

Since

$$\mu\Big(\bigcup_{k=m}^{\infty} D_{k,j(k)}\Big) \le \sum_{k=m}^{\infty} 2^{-k}\mu(\Omega) \to 0$$

as $m \to \infty$ and $y_n \in M(L(\Omega, \Sigma, \mu; Y))$ we have

$$\int_{D_{n,j(n)}} M(t, \|y_n(t)\|_Y)\, d\mu - \int_{\Omega_{n,m}} M(t, \|y_n(t)\|_Y)\, d\mu = \int_{\bigcup_{k=m}^{\infty} D_{k,j(k)}} M(t, \|y_n(t)\|_Y)\, d\mu \to 0 \tag{10}$$

as $m \to \infty$. Thus

$$\int_{\Omega_{n,m}} M(t, \|y_n(t)\|_Y)\, d\mu \to \int_{D_{n,j(n)}} M(t, \|y_n(t)\|_Y)\, d\mu \tag{11}$$

as $m \to \infty$. This means that with each $n \in \mathbb{N}$, we can associate an $h(n) \in \mathbb{N}$, $h(n) > n$, such that

$$\int_{\Omega_{n,h(n)}} M(t, \|y_n(t)\|_Y)\, d\mu \ge n. \tag{12}$$

Thus by induction we construct a sequence $n_1, \ldots, n_k = h(n_{k-1}), \ldots$ of natural numbers such that the sets $\Omega_k = \Omega_{n_k, n_{k+1}}$ are mutually disjoint and satisfy

$$\mu(\Omega_k) \le 2^{-k}\mu(\Omega) \tag{13}$$

and

$$\int_{\Omega_k} M(t, \|y_{n_k}(t)\|_Y)\, d\mu \ge n_k. \tag{14}$$

Define

$$x_*(s) = \begin{cases} x_{n_k}(s) & \text{if } s \in \Omega_k,\ k = 1, 2, \ldots, \\ 0 & \text{if } s \notin \bigcup_{k=1}^{\infty} \Omega_k, \end{cases} \tag{15}$$

$$y_*(s) = \begin{cases} y_{n_k}(s) & \text{if } s \in \Omega_k,\ k = 1, 2, \ldots, \\ v(s) & \text{if } s \notin \bigcup_{k=1}^{\infty} \Omega_k, \end{cases} \tag{16}$$

where $v(\cdot)$ belongs to $N_F(0)$.

By (2) and (7), $x_* \in N(L(\Omega, \Sigma, \mu; X))$. It is obvious that $y_*(\cdot) \in N_F(x_*(\cdot))$. On the other hand,

$$\int_{\Omega} M(t, \|y_*(t)\|_Y)\, d\mu \ge \sum_{k=1}^{\infty} \int_{\Omega_k} M(t, \|y_{n_k}(t)\|_Y)\, d\mu \ge \sum_{k=1}^{\infty} n_k \to \infty, \tag{17}$$

contrary to the assumption that N_F maps $N(L(\Omega, \Sigma, \mu; X))$ into subsets of $M(L(\Omega, \Sigma, \mu; Y))$. This finishes the proof of (s1).

Proof of (s2). Since μ is σ-finite, we can represent Ω as a countable union of sets Ω_n of finite measure. Without loss of generality we may assume that the sets Ω_n are disjoint and that $\mu(\Omega_n) \le 1$. Now we define a new measure on Ω by

$$\mu_1(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mu(A \cap \Omega_n) \quad \text{for } A \in \Sigma.$$

Observe that μ_1 is complete, non-atomic and finite. Now we consider the functions

$$N_1(t,u) := N(t, 2^n u) \quad \text{for } t \in \Omega_n, \tag{18}$$

$$M_1(t,u) := M(t, 2^n u) \quad \text{for } t \in \Omega_n. \tag{19}$$

It is easy to see that $x(\cdot) \in N(L(\Omega,\Sigma,\mu;X))$ if and only if $x(\cdot) \in N_1(L(\Omega,\Sigma,\mu_1;X))$, and similarly $y(\cdot) \in M(L(\Omega,\Sigma,\mu;Y))$ if and only if $y(\cdot) \in M_1(L(\Omega,\Sigma,\mu_1;Y))$. Moreover,

$$\varrho_{N,\mu}(x(\cdot)) = \varrho_{N_1,\mu_1}(x(\cdot)) \tag{20}$$

and

$$\varrho_{M,\mu}(y(\cdot)) = \varrho_{M_1,\mu_1}(y(\cdot)). \tag{21}$$

Note that if N_F maps $N(L(\Omega,\Sigma,\mu;X))$ into subsets of $M(L(\Omega,\Sigma,\mu;Y))$, then it also maps $N_1(L(\Omega,\Sigma,\mu_1;X))$ into subsets of $M_1(L(\Omega,\Sigma,\mu_1;Y))$. Thus applying the previous step we trivially obtain (s2).

Proof of (s3). By (s2) there exist $r > 0$ and $R > 0$ such that for all $x(\cdot) \in N(L(\Omega,\Sigma,\mu;X))$ such that $\varrho_{N,\mu}(x(\cdot)) \le r$ we have

$$\sup\{\varrho_{M,\mu}(y(\cdot)) : y(\cdot) \in F(x(\cdot))\} < R < \infty. \tag{22}$$

Let $K \subset N(L(\Omega,\Sigma,\mu;X))$ be an arbitrary set such that

$$r_K = \sup\{\varrho_{N,\mu}(x) : x \in K\} < \infty. \tag{23}$$

Let k be the smallest integer greater than r_K/r. Then each $x(\cdot) \in K$ can be represented as a sum of k elements $x_1(\cdot),\ldots,x_k(\cdot)$ with disjoint supports and such that $\varrho_{N,\mu}(x_i(\cdot)) \le r$, $i = 1,\ldots,k$. Observe that the $N_F(x_i(\cdot))$ also have disjoint supports. Thus by (22) we trivially obtain

$$\sup\{\varrho_{M,\mu}(y(\cdot)) : y(\cdot) \in F(x(\cdot)),\ x(\cdot) \in K\} \le \sum_{i=1}^{k} \sup\{\varrho_{M,\mu}(y(\cdot)) : y_i(\cdot) \in F(x_i(\cdot))\} \le kR. \ \blacksquare \tag{24}$$

For single-valued operators we obtain

COROLLARY 2. *Under the assumptions of Theorem 1, let G be the Nemytskiĭ operator induced by a sup-measurable function $g(\cdot,\cdot) : \Omega \times X \to Y$ by means of the formula $G(x(\cdot)) = g(t, x(t))$. Then G is automatically modular bounded from a modular space $(N(L(\Omega,\Sigma,\mu;X)), \varrho_{N,\mu})$ into a modular space $(M(L(\Omega,\Sigma,\mu;Y)), \varrho_{M,\mu})$.*

As mentioned above, if the spaces (X, ϱ_X) and (Y, ϱ_Y) are locally bounded and the modulars ϱ_X and ϱ_Y satisfy (4) for all bounded sets K, then boundedness and modular boundedness coincide. Hence we obtain

COROLLARY 3. *Let X, Y be two separable locally bounded F-spaces (in particular, normed spaces). Let (Ω,Σ,μ) be a measure space with μ complete, non-atomic and σ-finite. Let $N(\cdot,\cdot)$ be a non-negative measurable function such that for all $t \in \Omega$, $N(t,0) = 0$, $N(t,\cdot)$ is increasing and there is $p \ge 1$ such that $N_0(t,u) = N(t,u^p)$ is a convex function of u. Let N_F be the Nemytskiĭ set-valued (resp. single-valued) operator from $N(L(\Omega,\Sigma,\mu;X))$ into subsets of $M(L(\Omega,\Sigma,\mu;Y))$ (resp. into $M(L(\Omega,\Sigma,\mu;Y))$) induced by a sup-measurable set-valued function $F(\cdot,\cdot) : \Omega \times X \to 2^Y$ (resp. a measurable function $F : \Omega \times X \to Y$). Then the operator N_F is automatically bounded.*

In particular, we obtain

COROLLARY 4. *Let X, Y, (Ω,Σ,μ) and F be as in Corollary 3. Then the operator N_F is automatically bounded from $L^p(\Omega,\Sigma,\mu;X)$ into subsets of $L^q(\Omega,\Sigma,\mu;Y)$ (resp. into $L^q(\Omega,\Sigma,\mu;Y)$), $0 < p,q < \infty$.*

As an immediate consequence of Corollary 4 and Theorem 2 of [14], one sees that the composition TN_F of the superposition operator N_F with some completely continuous linear operator T is upper semicontinuous and compact provided $F : \Omega \times X \to 2^Y$ has closed images, for each $t \in \Omega$ the mapping $F(t,\cdot)$ is upper semicontinuous and the Nemytskiĭ operator N_F maps $L^p(\Omega,\Sigma,\mu;X)$ into subsets of $L^q(\Omega,\Sigma,\mu;Y)$, $0 < p,q < \infty$.

Such results are useful in proving existence of solutions to boundary value problems for differential inclusions [12].

References

[1] J. Appell, Nguyen Hong Tai and P. P. Zabrejko [P. P. Zabreĭko], *Multivalued superposition operators in ideal spaces of vector functions. I, II*, Indag. Math. (N.S.) 2 (1991), 385–395, 397–409.

[2] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[3] M. C. Joshi and R. K. Bose, *Some Topics in Nonlinear Functional Analysis*, Halsted Press, New York, 1985.

[4] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. 1034, Springer, 1983.

[5] J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. 18 (1959), 49–65.

[6] —, —, *Some remarks on modular spaces*, Bull. Acad. Polon. Sci. 7 (1959), 661–668.

[7] H. Nakano, *Modulared linear spaces*, J. Fac. Sci. Univ. Tokyo Sect. I 6 (1950), 85–131.
[8] —, *Modulared Semi-Ordered Linear Spaces*, Maruzen, Tokyo, 1950.
[9] —, *Topology and Linear Topological Spaces*, Maruzen, Tokyo, 1951.
[10] V. Niemytzki [V. Nemytskiĭ], *Sur les équations intégrales non linéaires*, C. R. Acad. Sci. Paris 196 (1933), 836–838.
[11] —, *Théorèmes d'existence et d'unicité des solutions de quelques équations intégrales non-linéaires*, Mat. Sb. 41 (1934), 421–438.
[12] T. Pruszko, *Topological degree methods in multi-valued boundary value problems*, Nonlinear Anal. 5 (1981), 959–973.
[13] S. Rolewicz, *Metric Linear Spaces*, Reidel and PWN, 1985.
[14] W. Song, *Multivalued superposition operators in $L^p(\Omega, X)$*, preprint.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
P.O. BOX 137
00-950 WARSZAWA, POLAND

DEPARTMENT OF MATHEMATICS
HARBIN NORMAL UNIVERSITY
HARBIN, P.R. CHINA

Received October 15, 1993
Revised version March 14, 1994 (3171)

STUDIA MATHEMATICA 113 (1) (1995)

On the embedding of 2-concave Orlicz spaces into L^1

by

CARSTEN SCHÜTT (Stillwater, Okla., and Kiel)

Abstract. In [K-S 1] it was shown that $\mathrm{Ave}_\pi(\sum_{i=1}^n |x_i a_{\pi(i)}|^2)^{1/2}$ is equivalent to an Orlicz norm whose Orlicz function is 2-concave. Here we give a formula for the sequence $a_1, \ldots, a_n$ so that the above expression is equivalent to a given Orlicz norm.

A convex function $M : \mathbb{R} \to \mathbb{R}$ with $M(t) = M(-t)$, $M(0) = 0$, and $M(t) > 0$ for $t \neq 0$ is called an *Orlicz function*. M is said to be 2-*concave* if $M(\sqrt{t})$ is a concave function on $[0, \infty)$, and *strictly* 2-*concave* if $M(\sqrt{t})$ is strictly concave. M is 2-*convex* if $M(\sqrt{t})$ is convex, and *strictly* 2-*convex* if $M(\sqrt{t})$ is strictly convex. If M' is invertible on $(0, \infty)$ then the *dual function* is given by

$$M^*(t) = \int_0^t M'^{-1}(s)\,ds.$$

We define the *Orlicz norm* of a sequence $\{x_i\}_{i=1}^\infty$ by

$$\|x\|_M = \sup\Big\{\sum_{i=1}^\infty x_i y_i \,\Big|\, \sum_{i=1}^\infty M^*(y_i) \le 1\Big\}.$$

In [K-S 1, K-S 2] we have used a different expression for the definition of the Orlicz norm: x has norm equal to 1 if and only if $\sum_{i=1}^\infty M(x_i) = 1$. But it turns out that the above definition gives slightly better estimates.

Bretagnolle and Dacunha-Castelle [B-D] showed that an Orlicz space l^M is isomorphic to a subspace of L^1 if and only if M is equivalent to a 2-concave Orlicz function. As a corollary we get the same result here. In [K-S 1] a variant of the following result was obtained.

THEOREM 1. *Let $a_1 \ge \ldots \ge a_n > 0$ and let M be an Orlicz function with*

1991 *Mathematics Subject Classification*: 46B07, 46B09, 46E30.
Supported by NSF-grant DMS-9301506.