solution u of the equation $Pu = f$ has the asymptotic expansion

$$u = \sum_{p=0}^{\infty} \sum_{j=1}^{q} w_1^{s_j+p} \sum_{r=0}^{k_j-1} \Big(\sum_{n=r}^{k_j-1} C_{nrp}^{j} (\ln w_1)^{k_j-n-1} \Big) f_{jpr},$$

where $f_{jpr} \in \underset{\sim}{\mathcal{M}}'_{(\omega')}(\overline{I(\widehat{t'}, \widehat{\theta'})})$ for $\omega' < -2$. This means that for every $N \in \mathbb{N}$,

$$u - \sum_{p=0}^{N} \sum_{j=1}^{q} w_1^{s_j+2p} \sum_{r=0}^{k_j-1} \Big(\sum_{n=r}^{k_j-1} C_{nrp}^{j} (\ln w_1)^{k_j-n-1} \Big) f_{jpr}$$
$$\in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{R(\theta_1) \times I(\widehat{t'}, \widehat{\theta'})}) \quad \text{for } \omega \in \overline{\operatorname{Re} \Omega}.$$

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Holomorphic functions and Banach-nuclear decompositions of Fréchet spaces

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Abstract. We introduce a decomposition of holomorphic functions on Fréchet spaces which reduces to the Taylor series expansion in the case of Banach spaces and to the monomial expansion in the case of Fréchet nuclear spaces with basis. We apply this decomposition to obtain examples of Fréchet spaces E for which the τ_ω and τ_δ topologies on $H(E)$ coincide. Our result includes, with simplified proofs, the main known results—Banach spaces with an unconditional basis and Fréchet nuclear spaces with DN [2, 4, 5, 6]—together with new examples, e.g. Banach spaces with an unconditional finite-dimensional Schauder decomposition and certain Fréchet–Schwartz spaces. This gives the first examples of Fréchet spaces, which are not nuclear, with $\tau_0 = \tau_\delta$ on $H(E)$.

In this article we introduce a new decomposition method for holomorphic functions on domains in Fréchet spaces which admit a Banach-nuclear decomposition (Proposition 1). This decomposition reduces to the Taylor series expansion for Banach spaces and to the monomial expansion in the case of Fréchet nuclear spaces with basis. This allows a unified treatment of topological problems on a variety of Fréchet spaces—including Banach spaces and Fréchet nuclear spaces. We apply this decomposition to obtain examples of Fréchet spaces E for which the τ_ω and τ_δ topologies on $H(E)$ coincide. Our result includes, with simplified proofs, the main known results—Banach spaces with an unconditional basis and Fréchet nuclear spaces with DN [2, 4, 5, 6]—together with new examples, e.g. Banach spaces with an unconditional finite-dimensional Schauder decomposition and certain Fréchet–Schwartz spaces (see the examples given below). Combined with results in [7] this gives the first examples of Fréchet spaces, which are not nuclear, with $\tau_0 = \tau_\delta$ on $H(E)$.

The proof is quite technical and we could not avoid some complicated notation. To keep the technicalities to a minimum we confined ourselves in Propositions 3 and 4 to entire functions and indicated afterwards the mod-

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ifications needed to obtain more general results. Our methods also include a description of a fundamental system of compact sets using decreasing sequences of seminorms and the use of polynomials which are homogeneous in each even variable and in the combined odd variables. We refer to [5] for definitions and properties of holomorphic functions on infinite-dimensional spaces.

Let $\lambda := \lambda(A)$ denote a Fréchet nuclear space with Köthe matrix A and let $\{E_n\}_n$ denote a sequence of Banach spaces. We let $E := \lambda(\{E_n\}_n) := \{(x_n)_n : x_n \in E_n$ and $(\|x_n\|)_n \in \lambda(A)\}$ and endow E with the topology generated by the semi-norms

$$\|(x_n)_n\|_k := \sum_{n=1}^{\infty} a_{n,k}\|x_n\|, \qquad k = 1, 2, \ldots \tag{1}$$

It is easily checked that $(E, (\|\cdot\|_k)_{k\ge 1})$ is a Fréchet space and that $\{E_n\}_n$ is an unconditional Schauder decomposition of E.

EXAMPLES. 1. Let $E_1 = E$ and $E_n = \{0\}$ for $n > 1$. Then $\lambda(\{E_n\}_n) \cong E$.

2. If $\dim(E_n) = 1$ for all n then $\lambda(\{E_n\}_n) \cong \lambda(A)$.

3. Let $(b_n)_n$ denote a sequence of positive integers such that $\sum_n b_n a_{n,k}/a_{n,k+j} = \infty$ for all k and j. Let $E_n = \ell^{\infty}_{b_n}$ for all n. Then $\lambda(\{E_n\}_n)$ is a Fréchet–Schwartz space which is not a Fréchet nuclear space. A concrete example is obtained by letting $a_{n,k} = n^{2k}$ and $b_n = n^{2n}$.

4. The space $\lambda(\{E_n\}_n)$ is a Schwartz or Montel space if and only if $\dim(E_n) < \infty$ for all n.

Let π_n denote the canonical projection from E onto E_n for each positive integer n. If B is a subset of $E = \lambda(\{E_n\}_n)$ then the following are easily seen to be equivalent:

(a) B is a relatively compact subset of E,

(b) $\pi_n(B)$ is a relatively compact subset of E_n for each n and B is a bounded subset of E,

(c) $\pi_n(B)$ is a relatively compact subset of E_n for each n and $(\sup_{x\in B}\|\pi_n(x)\|)_n \in \lambda(A)$.

If U_n is a balanced open subset of E_n for each n and $\bigoplus_n U_n := \{(x_n)_n \in E : x_n \in U_n$ for all $n\}$ then U is a balanced open subset of E which we refer to as a *balanced open set of polydisc type*. In Example 1 above all balanced open sets are of polydisc type while in Example 2 the balanced open sets of polydisc type are the open polydiscs.

The equivalent conditions above show that a fundamental system of compact subsets of $\bigoplus_n U_n$ is given by sets of the form

$$\bigoplus_n K_n := \{(x_n)_n \in E : x_n \in K_n \text{ for all } n\}$$

where K_n is a compact balanced subset of U_n for each n and $(\sup_{x\in K_n}\|x\|)_n \in \lambda(A)$.

Hence (see for instance [5, Lemma 5.18]) if K is a compact subset of $U := \bigoplus_n U_n$ then there exists a zero neighbourhood $V := \bigoplus_n V_n$ in E of polydisc type and a sequence $\delta = (\delta_n)_n$ of positive real numbers δ_n with $\delta_n > 1$ and $\sum_{n=1}^{\infty} 1/\delta_n < \infty$ such that δK is a compact subset of U and

$$K \subset \delta(K+V) \subset U. \tag{2}$$

If $n \in \mathbb{N}$ we let $\mathcal{P}(^nE)$ denote the set of all continuous n-homogeneous polynomials on the locally convex space E. If E has a Schauder decomposition $\{E_n\}_n$ and $P \in \mathcal{P}(^nE)$ then we say that P is *k-homogeneous in the j-th variable* if

$$P\Big(\sum_{i\ne j} x_i + \lambda x_j\Big) = \lambda^k P\Big(\sum_{i=1}^{\infty} x_i\Big) \quad \text{for all } \sum_{i=1}^{\infty} x_i \in E \text{ with } x_i \in E_i \text{ for all } i.$$

If $m = (m_1, \ldots, m_n, \ldots) \in \mathbb{N}^{(\mathbb{N})}$ (and so $m_i = 0$ for all i large) we let $|m| = \sum_j m_j$ and denote by $\mathcal{P}(^mE)$ the set of all $P \in \mathcal{P}(^{|m|}E)$ which are m_j-homogeneous in the jth variable for all j. This means that

$$P\Big(\sum_{j=1}^{\infty} \lambda_j x_j\Big) = \lambda^m P\Big(\sum_{j, m_j\ne 0} x_j\Big)$$

where $\lambda = (\lambda_j)_j$ and $\lambda^m = \prod_j \lambda_j^{m_j}$.

The following useful estimate is easily proved.

LEMMA 1. *Let $E = E_1 \times \ldots \times E_n$ where each E_i is a normed linear space and E is normed so that*

$$\Big\|\sum_{i=1}^{n} x_i\Big\| = \sup_{\substack{|\lambda_i|\le 1\\ i=1,\ldots,n}} \Big\|\sum_{i=1}^{n} \lambda_i x_i\Big\|.$$

If $m \in \mathbb{N}^n$, $P \in \mathcal{P}(^mE)$ and $\beta_i > 0$ for $i = 1, \ldots, n$ then $\beta^m\|P\| = \|P\|_{\beta B}$ where

$$\beta B = \Big\{\sum_{i=1}^{n} \beta_i x_i : \Big\|\sum_{i=1}^{n} x_i\Big\| \le 1\Big\}.$$

Now let $f \in \mathcal{H}(U)$ where $U := \bigoplus_n U_n$ is a balanced domain of polydisc type and let $x = (x_n)_n \in U$. For each $m \in \mathbb{N}^{(\mathbb{N})}$ let

$$P_m(x) = \frac{1}{(2\pi i)^n} \int_{|\lambda_i|=1} \frac{f(\sum_{i=1}^{n} \lambda_i x_i)}{\lambda_1^{m_1+1} \ldots \lambda_n^{m_n+1}}\, d\lambda_1 \ldots d\lambda_n \tag{3}$$

where $m = (m_1, \ldots, m_n, 0, \ldots)$.

By [5, Theorem 5.21] applied to the Fréchet nuclear space

$$E_{(x_n)_n} := \{(\gamma_n)_n : \gamma_n \in \mathbb{C} \text{ and } (\gamma_n x_n)_n \in \lambda(\{E_n\}_n)\}$$

we have

$$f(x) = \sum_{m\in\mathbb{N}^{(\mathbb{N})}} P_m(x). \tag{4}$$

We refer to (4) as the *Taylor-monomial expansion* of f since it reduces to the Taylor series expansion when restricted to each U_n and to the monomial expansion when restricted to $E_{(x_n)_n} \cap U$. By uniqueness of the Taylor series and monomial expansions we see that $P_m \in \mathcal{P}(^m E)$ for all $m \in \mathbb{N}^{(\mathbb{N})}$. If K is a compact subset of U then there exists a neighbourhood V of K with $K \subset V \subset U$ such that $\|f\|_V := M < \infty$. By (2) there exists a neighbourhood W of 0 in E and a sequence $\delta := (\delta_n)_n$ with $\delta_n > 1$ for all n and $\sum_{n=1}^{\infty} 1/\delta_n < \infty$ such that

$$K \subset \delta(K+W) \subset V.$$

By (3),

$$\|P_m\|_{K+W} \le M\delta^{-m} \quad \text{for all } m \in \mathbb{N}^{(\mathbb{N})}.$$

By (3) and (4) we see that

$$\Big\| f - \sum_{m\in J} P_m \Big\|_{K+W} \le M \sum_{m\in\mathbb{N}^{(\mathbb{N})}\setminus J} \delta^{-m}$$

for any finite subset J of $\mathbb{N}^{(\mathbb{N})}$.

The above, together with further details from [5, Propositions 5.24 and 5.25], now shows the following.

PROPOSITION 2. *If $\lambda(A)$ is a Fréchet nuclear space, $\{E_n\}_n$ is a sequence of Banach spaces and $E = \lambda(\{E_n\}_n)$ then $\{\mathcal{P}(^m E)\}_{m\in\mathbb{N}^{(\mathbb{N})}}$ is an absolute Schauder decomposition for $(\mathcal{H}(U), \tau)$, $\tau = \tau_0$ or τ_ω and U a balanced open subset of polydisc type in E. Moreover, $\{\mathcal{P}(^m E)\}_{m\in\mathbb{N}^{(\mathbb{N})}}$ is an absolute Schauder decomposition for $(\mathcal{H}(E), \tau_\delta)$ and an unconditional equicontinuous Schauder decomposition for $(\mathcal{H}(U), \tau_\delta)$ for U a balanced open subset of polydisc type in E.*

In proving our main result we shall need to consider polynomials which are homogeneous in some but not in all variables. To simplify the notation we introduce the collection of polynomials which are homogeneous in each even variable (and consequently in the combined odd variables).

Let $\{E_n\}_n$ denote an unconditional Schauder decomposition of the Fréchet space E. If $m = (m_1, \ldots, m_n) \in \mathbb{N}^{(\mathbb{N})}$ we let $\mathcal{P}_e(^m E)$ denote the subspace of $\mathcal{P}(^{|m|}E)$ consisting of all polynomials P satisfying

$$P\Big(\sum_{i=1}^{\infty} x_{2i-1} + \sum_{\substack{i=1\\ i\ne j}}^{\infty} x_{2i} + \lambda x_{2j}\Big) = \lambda^{m_{2j}} P\Big(\sum_{i=1}^{\infty} x_i\Big)$$

for $\sum_{i=1}^{\infty} x_i \in E$ and $j \in \mathbb{N}$. If $m_0 = \sum_{i=1}^{\infty} m_{2i-1}$, $m_e = (m_2, m_4, \ldots)$ and A is the $|m|$-linear symmetric form on E associated with P then

$$\begin{aligned} P\Big(\lambda_0 \sum_{i=1}^{\infty} x_{2i-1} + \sum_{i=1}^{\infty} \lambda_i x_{2i}\Big) &= \lambda^m P\Big(\sum_{i=1}^{\infty} x_i\Big) \\ &= \lambda^m \frac{|m|!}{m_0! m_e!} A\Big(\sum_{i=1}^{\infty} x_{2i-1}\Big)^{m_0} \Big(\sum_{i=1}^{\infty} x_{2i}\Big)^{m_e} \end{aligned}$$

where $\lambda^m = \prod_{i=0}^{\infty} \lambda_i^{m_{2i}}$.

Now let E denote a Banach space with an unconditional Schauder decomposition $\{E_n\}_n$. We may and shall always suppose that the given norm on E satisfies

$$\Big\| \sum_{n=1}^{\infty} x_n \Big\| = \sup_{|\lambda_n|\le 1} \Big\| \sum_{n=1}^{\infty} \lambda_n x_n \Big\| \tag{5}$$

for all $\sum_{n=1}^{\infty} x_n \in E$, where $x_n \in E_n$ for all n. If $(\beta_n)_n$ is a sequence of positive real numbers, $\|\cdot\|$ is a continuous norm satisfying (5) and j is a positive integer we let

$$\Big\| \sum_{n=1}^{\infty} x_n \Big\|_{\beta,j} = \Big\| \sum_{n=1}^{j} x_n + \sum_{n=j+1}^{\infty} \beta_n x_n \Big\| \tag{6}$$

and

$$\Big\| \sum_{n=1}^{\infty} x_n \Big\|_{\beta} = \Big\| \sum_{n=1}^{\infty} \beta_n x_n \Big\|$$

for all $\sum_{n=1}^{\infty} x_n \in E$. If $\beta_n = 2$ for all n we write $\|x\|_j$ in place of $\|x\|_{\beta,j}$. For any strictly increasing sequence $(j_k)_{k=1}^{\infty}$ of positive integers we define $\|\cdot\|_{j_1,\ldots,j_k}$ inductively, using (6), as

$$\|\cdot\|_{j_1,\ldots,j_k} = (\|\cdot\|_{j_1,\ldots,j_{k-1}})_{j_k}. \tag{7}$$

If $\beta_n \ge 1$ for all n and $\sum_{n=1}^{\infty} \beta_n x_n \in E$ for all $\sum_{n=1}^{\infty} x_n \in E$, then $(\|\cdot\|_{\beta,j})_{j=1}^{\infty}$ is a decreasing sequence of continuous norms on E, $\|\cdot\|_\beta$ is a continuous norm on E and

$$\{x : \|x\|_\beta \le 1\} \subset \bigcap_j \{x : \|x\|_{\beta,j} \le 1\} \subset \{x : \|x\| \le 1\}. \tag{8}$$

If $\dim(E_n) < \infty$ for each n and $(j_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers then

$$K_{(j_k)_k} := \bigcap_k \{x : \|x\|_{j_1,\ldots,j_k} \le 1\}$$

is a compact subset of E. Moreover, sets of the form $cK_{(j_k)_k}, c \in \mathbb{R}$, form a fundamental system of compact subsets of E and if $\delta_k > 1$ for all k and $\delta_k \to 1$ as $k \to \infty$ then

$$\{x : \|x\|_{j_1,\ldots,j_k} \le \delta_k\}, \quad k = 1, 2, \ldots,$$

forms a fundamental system of neighbourhoods of $K_{(j_k)_k}$ (1).

Now let $E = \lambda(\{E_n\}_n)$ where each E_n is a Banach space with an unconditional Schauder decomposition and $\lambda(A)$ is nuclear. Let $(k_l)_{l=0}^{\infty}$ denote a strictly increasing sequence of positive integers with $k_0 = 1$. For each $j \in \mathbb{N}$ there exists a unique s_j such that $k_{s_j} \le j < k_{s_j+1}$. Note that $s_1 = 0$ and $s_{k_l-1} = l - 1$. Suppose for each j we have a strictly increasing sequence of positive integers $(n_{i,j})_{i\ge1}$. Let $\aleph_{l,j} = \{n_{1,j}, \ldots, n_{l,j}\}$ and let

$$\Big\|\sum_{j=1}^{\infty} x_j\Big\|_{\aleph_l} = \sum_{j=1}^{k_l-1} \alpha_{j,s_j+1}\|x_j\|_{\aleph_{l-s_j,j}} + \sum_{j=k_l}^{\infty} \alpha_{j,l+1}\|x_j\|$$

where $\|x_j\|_{\aleph_{l-s_j,j}}$ is defined using (6) and the unconditional Schauder decomposition of E_j for each j. Let $B_{\aleph_l} := \{x : \|x\|_{\aleph_l} \le 1\}$. If the unconditional Schauder decomposition of each E_j is finite-dimensional then $K_{\aleph} := \bigcap_l B_{\aleph_l}$ is a compact subset of E and sets of the form $cK_{\aleph}$, $c \in \mathbb{R}$, form a fundamental system of compact subsets of E. Moreover, if $\delta_l > 1$ and $\delta_l \to 1$ as $l \to \infty$ then $\delta_l B_{\aleph_l}$, $l = 1, 2, \ldots$, form a fundamental system of neighbourhoods of $K_{\aleph}$.

Proposition 3. *Let $\{E_n\}_n$ denote an unconditional Schauder decomposition of the Fréchet space E, let F denote a Banach space and let $T : \mathcal{H}(E) \to F$ denote a τ_δ-continuous linear function. Let $\beta_i \ge 1$ for all i, $\beta_{2i-1} = 2$ for all i and suppose $\sum_{i=1}^{\infty} \beta_i^p x_i \in E$ for any $\sum_{i=1}^{\infty} x_i \in E$ and any $p \ge 0$. Let $\|\cdot\|$ denote a continuous seminorm on E satisfying* (5) *and suppose $c > 0$ is such that*

$$\text{(9)}\qquad \|T(P)\| \le c\|P\| \quad \textit{for all } P \in \mathcal{P}_e(^mE) \textit{ and all } m \in \mathbb{N}^{(\mathbb{N})}$$

where $\|P\| = \sup\{|P(x)| : \|x\| \le 1\}$. Then for any $\delta > 1$ there exists a positive integer j and $c_1 > 0$ such that

$$\|T(P)\| \le c_1\|P\|_{\beta,j} \quad \textit{for all } P \in \mathcal{P}_e(^mE) \textit{ and all } m \in \mathbb{N}^{(\mathbb{N})}$$

where $\|P\|_{\beta,j} = \delta^{|m|}\sup\{|P(x)| : \|x\|_{\beta,j} \le 1\} = \sup\{|P(x)| : \|x\|_{\beta,j} \le \delta\}$.

(1) Any sequence $(\beta_n)_n$ for which $(\beta_1 \ldots \beta_n)^{-1} \to 0$ as $n \to \infty$ could be used to obtain a similar characterization.

Proof. Suppose the result is not true. Then for every positive integer j there exists $m_j \in \mathbb{N}^{(\mathbb{N})}$ and $P_j \in \mathcal{P}_e(^{m_j}E)$ such that

$$\text{(10)}\qquad \|T(P_j)\| > j\|P_j\|_{\beta,2j}.$$

If $|m_j| = l$ for an infinite number of j then, by (8), $\{P_j/\|P_j\|_{\beta,2j}\}_{|m_j|=l}$ is a locally bounded, and hence τ_δ-bounded, subset of $\mathcal{P}(^lE)$ on which T is unbounded. This is impossible and so we may assume that $|m_j|$ is strictly increasing. Let $m_j = (m_1^j, \ldots, m_{n_j}^j)$, $m_0^j = \sum_{i=1}^{\infty} m_{2i-1}^j$, $m_j^e = (m_2^j, m_4^j, \ldots)$ and $s_j = (0, \ldots, 0, m_{2j}^j, m_{2j+1}^j, \ldots)$. If A_j is the symmetric $|m_j|$-linear form associated with P_j then for $0 \le l \le m_0^j$ let

$$P_{j,l}\Big(\sum_{i=1}^{\infty} x_i\Big) = \binom{|m_j|}{m_j^e\ l} A_j\Big(\sum_{i=1}^{j} x_{2i-1}\Big)^l\Big(\sum_{i=j+1}^{\infty} x_{2i-1}\Big)^{m_0^j-l}\Big(\sum_{i=1}^{\infty} x_{2i}\Big)^{m_j^e}.$$

Then $P_{j,l} \in \mathcal{P}_e(^{m_j}E)$ and $\sum_{l=0}^{m_0^j} P_{j,l} = P_j$.

Hence, by (10) and Lemma 1, for each j there exists l_j, $0 \le l_j \le m_0^j$, such that

$$\text{(11)}\qquad \|T(P_{j,l_j})\| > \frac{j}{|m_j|+1}\|P_j\|_{\beta,2j} \ge \frac{j}{|m_j|+1}\|P_{j,l_j}\|_{\beta,2j}.$$

Let $\beta = (\beta_j)_{j=1}^{\infty}$. Since $\beta_j \ge 1$ for all j the following two cases cover all possibilities. We may suppose, by taking a subsequence if necessary, that the limit exists in case 2.

Case 1: $\lim_{j\to\infty}(\beta^{s_j})^{1/|m_j|} = 1$.

Case 2: $\limsup_{j\to\infty}(\beta^{s_j})^{1/|m_j|} = w > 1$.

We consider case 1. By (9),

$$\limsup_{j\to\infty}\Big(\frac{\|T(P_{j,l_j})\|}{\|P_{j,l_j}\|}\Big)^{1/|m_j|} \le 1.$$

By (11),

$$\begin{aligned}\liminf_{j\to\infty}\Big(\frac{\|T(P_{j,l_j})\|}{\|P_{j,l_j}\|}\Big)^{1/|m_j|} &\ge \liminf_{j\to\infty}\Big(\frac{j}{|m_j|+1}\,\frac{\|P_{j,l_j}\|_{\beta,2j}}{\|P_{j,l_j}\|}\Big)^{1/|m_j|}\\ &\ge \delta\liminf_{j\to\infty}(\beta^{s_j})^{1/|m_j|} \quad \text{(by Lemma 1)}\\ &= \delta.\end{aligned}$$

Since $\delta > 1$ this is impossible and so case 1 leads to a contradiction.

We now consider case 2. Let $x = \sum_{n=1}^{\infty} x_n \in E$ be arbitrary and let p be a fixed positive integer. Since $\sum_{i=1}^{\infty} \beta_i^p x_i \in E$ and $\|\cdot\|_{\beta,j}$ is decreasing in j

we can choose n_1 such that

$$\Big\|\sum_{i=n_1+1}^{\infty}\beta_i^p x_i\Big\|_{\beta,2j}\le 1 \qquad \text{for all } j\ge n_1.$$

Since β_{2i-1} is independent of i, we have

$$P_{j,l_j}(x)=\frac{(\|x\|_{\beta,2j})^{l_j}}{(\beta^{s_j})^p}P_{j,l_j}\Big(\frac{\sum_{i=1}^{2j-1}x_i}{\|x\|_{\beta,2j}}+\sum_{i=2j+1}^{\infty}\beta_i^p x_i\Big).$$

Hence

$$\limsup_{j\to\infty}\Big(\frac{|P_{j,l_j}(x)|}{\|P_{j,l_j}\|_{\beta,2j}}\Big)^{1/|m_j|}\le\frac{2}{w^p}(1+\|x\|_\beta)\frac{1}{\delta}$$

since

$$\Big\|\frac{\sum_{i=1}^{2j-1}x_i}{\|x\|_{\beta,2j}}+\sum_{i=2j+1}^{\infty}\beta_i^p x_i\Big\|_{\beta,2j}\le 2.$$

Since $w>1$ and p was arbitrary this implies that

$$\sum_{j=1}^{\infty}\frac{P_{j,l_j}}{\|P_{j,l_j}\|_{\beta,2j}}\in\mathcal{H}(E).$$

By (11) we have

$$\limsup_{j\to\infty}\Big\|\frac{T(P_{j,l_j})}{\|P_{j,l_j}\|_{\beta,2j}}\Big\|^{1/|m_j|}\ge\limsup_{j\to\infty}\Big(\frac{j}{|m_j|+1}\Big)^{1/|m_j|}=1.$$

This is impossible and hence case 2 leads to a contradiction. This completes the proof.

We note that the integer j in Proposition 3 may be taken arbitrarily large. We shall refer to the seminorm $\|\cdot\|$ and to the seminorm $\delta^{-1}\|\cdot\|_{\beta,j}$, which occur in Proposition 3, as the *initial* and *final seminorms* of Proposition 3, respectively. In Theorem 4 the Fréchet nuclear space $\lambda(A)$ has DN and hence it admits a continuous norm ([5, 6, 11]).

THEOREM 4. *Let $\lambda(A)$ denote a Fréchet nuclear space with DN and let $\{E_n\}_n$ denote a sequence of Banach spaces each of which has an unconditional finite-dimensional Schauder decomposition. Then $\tau_\omega=\tau_\delta$ on $\mathcal{H}(\lambda(\{E_n\}_n))$.*

Proof. Let $\{E_{n,m}\}_m$ denote the unconditional finite-dimensional decomposition of E_n. By unconditionality we have $E:=\lambda(\{E_n\}_n)=\bigoplus_{n,m}E_{n,m}=\bigoplus_n E_{J_n}$ where $\{J_n\}_n$ is any partition of $\mathbb{N}\times\mathbb{N}$. We apply Proposition 3 to a decomposition $\{F_n\}_n$ of E in two different situations which we refer to as

(A_l) and (B_s):

$$(A_l):\quad F_{2n-1}=\{0\},\ F_{2n}=E_n,\ \beta_{2n}=a_{n,l+2}/a_{n,l+1},$$
$$(B_s):\quad F_{2n-1}=E_{n,s},\ F_{2n}=E_{\phi(n)}$$

where $\phi:N\to N\backslash\{s\}$ is bijective, $\beta_{2n}=1$.

Let $\mathcal{P}_{e,l}({}^mE)$ and $\mathcal{P}_{e,k}({}^mE)$, $m\in\mathbb{N}^{(\mathbb{N})}$, denote the subspaces of $\mathcal{P}({}^{|m|}E)$ consisting of those polynomials which are homogeneous with respect to the even variables for the decompositions given for (A_l) and (B_s) respectively and let $\mathcal{P}({}^mE)$, $m\in\mathbb{N}^{(\mathbb{N})}$, denote the elements of $\mathcal{P}({}^{|m|}E)$ which are homogeneous with respect to all the variables of the decomposition $\{E_n\}_n$ of $\lambda(\{E_n\}_n)$. For each (A_l) the odd variables are all zero and the even variables are the variables of the decomposition $\{E_n\}_n$ of $\lambda(\{E_n\}_n)$. Hence $\mathcal{P}_{e,l}({}^mE)=\mathcal{P}({}^mE)$ for all $m\in\mathbb{N}^{(\mathbb{N})}$ and all k. For each (B_s) the odd variables are the variables of the unconditional decomposition $\{E_{n,s}\}_n$ of E_s and so the combined odd variables give the sth variable of the decomposition $\{E_n\}_n$ of $\lambda(\{E_n\}_n)$ and the even variables are the other variables of the same decomposition. Hence $\mathcal{P}_{e,s}({}^mE)=\mathcal{P}({}^mE)$ for all $m\in\mathbb{N}^{(\mathbb{N})}$ and all s. The DN hypothesis on $\lambda(A)$ will guarantee that the hypothesis of Proposition 3 holds for (A_l). Hence in both cases we are applying Proposition 3 to $\{\mathcal{P}({}^mE)\}_{m\in\mathbb{N}^{(\mathbb{N})}}$.

Let F denote a Banach space and suppose $T:\mathcal{H}(E)\to F$ is a τ_δ-continuous linear function. Since T is τ_δ-continuous there exist $c>0$ and $\delta>0$ such that

$$\|T(P)\|\le c\|P\| \quad \text{for all } P\in\mathcal{P}({}^mE) \text{ and all } m\in\mathbb{N}^{(\mathbb{N})},$$

where $\|P\|=\sup\{\|P(x)\|:\|x\|\le\delta\}$ and

(12) $$\Big\|\sum_{n=1}^{\infty}x_n\Big\|=\sum_{n=1}^{\infty}a_{n,1}\|x_n\|.$$

Let $\delta'>\delta$ be arbitrary. We claim that for each non-negative integer l,

$(13)_l$ there exists δ_l, $\delta\le\delta_l<\delta'$, a strictly increasing sequence of non-negative integers $\{k_0=0,k_1,\ldots,k_l\}$ and strictly increasing sequences $\aleph_{l-s_j,j}:=\{n_{1,j},\ldots,n_{l-s_j,j}\}$, $1\le j<k_l$, where s_j is defined by $k_{s_j}\le j<k_{s_j+1}$, and $c_l>0$ such that

$$\|T(P)\|\le c_l\|P\| \quad \text{for all } P\in\mathcal{P}({}^mE) \text{ and } m\in\mathbb{N}^{(\mathbb{N})},$$

where $\|P\|=\sup\{|P(x)|:\|x\|_{\aleph_l}\le\delta_l\}$.

We establish this claim by induction. Let $\delta_0=\delta$, $c_0=c$ and $k_0=0$ (the set $\aleph_{l-s_0,j}$ is empty for all j). By (12), $(13)_0$ is satisfied.

We now suppose that we have found δ_l, k_l and $n_{i,j}$ such that $(13)_l$ is satisfied. Choose $\delta_{l,0}>1$ such that $\delta_l\delta_{l,0}<\delta'$. Now apply Proposition 3 with decomposition (A_l), $\delta_{l,0}$ and initial norm $\delta_l^{-1}\|\cdot\|_{\aleph_l}$. This gives a positive integer k_{l+1}, which we may suppose to be strictly larger than k_l, and the final norm

$$(14)\qquad \Big\|\sum_{j=1}^{\infty} x_j\Big\|_{\aleph_{l,0}} := (\delta_l\delta_{l,0})^{-1}\Big(\Big\|\sum_{j=1}^{k_{l+1}-1} x_j\Big\|_{\aleph_l} + \sum_{j=k_{l+1}}^{\infty} \alpha_{j,l+2}\|x_j\|\Big).$$

Now choose $\delta_{l,i}$ for $1 \le i < k_{l+1}$ such that $\delta_{l,i} > 1$ for all i and $\delta_{l+1} := \delta_l \prod_{i=0}^{k_{l+1}-1}\delta_{l,i} < \delta'$.

We apply Proposition 3 with the decomposition (B_1), $\delta_{l,1}$ and the initial norm given by (14). This gives a positive integer $n_{l+1-s_1,1}$, which we may suppose to be strictly larger than $n_{l-s_1,1}$, and the final norm

$$(15)\qquad \Big\|\sum_{j=1}^{\infty} x_j\Big\|_{\aleph_{l,1}} := (\delta_l\delta_{l,0}\delta_{l,1})^{-1}\Big((\|x_1\|_{\aleph_l})_{n_{l+1}-s_1,1} + \Big\|\sum_{j=2}^{\infty} x_j\Big\|_{\aleph_{l,0}}\Big).$$

We now apply Proposition 3 in succession to $(B_2),\dots,(B_{k_{l+1}-1})$ and $\delta_{l,2},\dots,\delta_{l,k_{l+1}-1}$ using as initial norm the final norm of the previous application. In particular, (15) gives the initial norm for the application using (B_2) and $\delta_{l,2}$. The final norm obtained after these applications of Proposition 3 is

$$\Big(\delta_l\prod_{i=0}^{k_{l+1}-1}\delta_{l,i}\Big)^{-1}\Big(\sum_{j=1}^{k_{l+1}-1}(\|x_j\|_{\aleph_l})_{n_{l+1}-s_j,j} + \Big\|\sum_{j=k_{l+1}}^{\infty} x_j\Big\|_{\aleph_{l,0}}\Big)$$
$$= \delta_{l+1}^{-1}\Big(\sum_{j=1}^{k_{l+1}-1}\alpha_{j,s_j+1}\|x_j\|_{\aleph_{l+1-s_j,j}} + \sum_{j=k_{l+1}}^{\infty}\alpha_{j,l+2}\|x_j\|\Big)$$
$$= \delta_{l+1}^{-1}\Big\|\sum_{j=1}^{\infty} x_j\Big\|_{\aleph_{l+1}}.$$

Hence $(13)_{l+1}$ is satisfied and by induction $(13)_l$ holds for all l. Hence there exist sequences $(c_l)_{l\ge 0}$, $(k_l)_{l\ge 0}$ and $(n_{i,j})_{i\ge 1, j\ge 1}$ such that

$$(16)\qquad \|T(P)\| \le c_l\|P\|_l \quad \text{for all } P \in \mathcal{P}(^mE) \text{ and all } m \in \mathbb{N}^{(\mathbb{N})}$$

where $\|P\|_l = \sup\{|P(x)| : \|x\|_{\aleph_l} \le \delta_{l+1}\}$.

Let $K = \delta' K_\aleph$. The set K is a compact subset of E and if V is a neighbourhood of K then (16) implies that there exists $C(V) > 0$ such that $\|T(P)\| \le C(V)\|P\|_V$ for all $P \in \mathcal{P}(^mE)$ and all m_0.

By $(13)_l$ and Proposition 2 this implies that T is τ_ω-continuous and completes the proof.

The above methods together with modifications given in [5, Theorem 4.38] show that we also have $\tau_\omega = \tau_\delta$ for balanced open subsets of $\lambda(\{E_n\}_n)$ which are of polydisc type.

The τ_b topology on $\mathcal{H}(U)$, U a balanced open subset of a locally convex space, is generated by all seminorms of the form

$$p\Big(\sum_{n=0}^{\infty}\frac{\widehat{d}^n f(0)}{n!}\Big) = \sum_{n=0}^{\infty}\Big\|\frac{\widehat{d}^n f(0)}{n!}\Big\|_{B_n}$$

where $(B_n)_n$ is a sequence of bounded subsets of E converging to some compact subset of U.

In [7] we proved that $\tau_b = \tau_\omega$ for T-invariant convex balanced domains in a Fréchet space with T-Schauder decomposition. It is easily shown (see for instance the proof that every Fréchet–Schwartz space with an unconditional finite-dimensional decomposition is T-decomposable in [1]) that $\lambda(\{E_n\}_n)$ has a T-Schauder decomposition when $\lambda(A)$ is a Fréchet nuclear space and each E_n is a Banach space. Thus, by Theorem 4 and [7] we have the following result.

THEOREM 5. *If $\lambda(A)$ is a Fréchet nuclear space with DN and $\{E_n\}_n$ is a sequence of Banach spaces each of which has an unconditional finite-dimensional Schauder decomposition then $\lambda(\{E_n\}_n)$ has a fundamental neighbourhood basis at the origin $(V_j)_j$ such that $\tau_\delta = \tau_\omega = \tau_b$ on $\mathcal{H}(V_j)$ for all j.*

Theorem 5 applies, in particular, to Banach spaces with an unconditional finite-dimensional decomposition. Examples of Banach spaces of this kind and which do not have unconditional basis are given in [8] and [9]. Examples of twisted quojections with unconditional finite-dimensional decompositions but without an unconditional basis are given in [10] (such spaces do have a basis). It is also easy to show, using Theorem 5, that $\tau_b = \tau_\delta$ on $\mathcal{H}(E)$, where E is a complemented subspace of $\lambda(\{E_n\})$. Hence Theorem 5 applies to reflexive subspaces with the approximation property of Banach spaces with an unconditional finite-dimensional decomposition and gives examples which do not have a finite-dimensional decomposition.

Since E_n and $\lambda(A)$ are closed complemented subspaces of $\lambda(\{E_n\}_n)$ it follows that some restrictions are necessary on both $\lambda(A)$ and the Banach spaces $\{E_n\}_n$ in order to obtain $\tau_\omega = \tau_\delta$ on $\mathcal{H}(E)$. For instance, if $E_n \cong \ell_\infty$ for some n then $\tau_\omega \ne \tau_\delta$ on $\mathcal{H}(E_n)$ [3, 5] and if $\lambda(A)$ does not have DN then $\tau_\omega \ne \tau_\delta$ on $\mathcal{H}(\lambda(A))$ [5, 6] and hence either of these implies that $\tau_\omega \ne \tau_\delta$ on $\mathcal{H}(\lambda(\{E_n\}))$.

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Polynomial asymptotics and approximation of Sobolev functions

by

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Abstract. We prove several results concerning density of C_0^∞, behaviour at infinity and integral representations for elements of the space $L^{m,p} = \{f \mid \nabla^m f \in L^p\}$.

1. Introduction. It was O. Nikodym who first introduced Sobolev type spaces. They appeared in [9] under the name of Beppo Levi spaces. Today this name is reserved for spaces of the type $L^{m,p}(\mathbb{R}^n) = \{f \in \mathcal{D}'(\mathbb{R}^n) \mid \nabla^m f \in L^p\}$, also denoted by $BL_m(L^p(\mathbb{R}^n))$. However, an interest in spaces of this type really begun with the paper of Deny and Lions [4].

The space $L^{m,p}$ is equipped with a quasinorm $\|\nabla^m f\|_{L^p}$. It is well known that elements of $L^{m,p}$ are locally integrable with exponent p. However, they need not be p-integrable in the entire space $\mathbb{R}^n$. As an example, take any polynomial of degree less than m.

In this paper we prove several results concerning behaviour at infinity, approximation by C_0^∞ and integral representations for functions from the space $L^{m,p}$. We also deal with the space $W_{r,p}^m = L^r \cap L^{m,p}$.

The general framework of the subject and the problems discussed here are certainly not new. They have been developed in many directions (cf. [1]–[3], [6], [8], [11], [13]). The most comprehensive source is [3]. However, the approach presented in these papers is very technical, based upon complicated integral representations and singular integrals. For this reason the authors deal *only* with $1 < p < \infty$.

Our approach is more elementary, because it depends only on a Poincaré type inequality. We also cover the missing case $p = 1$. The Poincaré

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