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Topological conditions for bound-2 isomorphisms of $C(X)$

by

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Abstract. We establish the topological relationship between compact Hausdorff spaces X and Y equivalent to the existence of a bound-2 isomorphism of the sup norm Banach spaces $C(X)$ and $C(Y)$.

1. Introduction. The Banach–Stone Theorem, which states that the Banach spaces $C(X)$ and $C(Y)$ are linearly isometric iff X and Y are homeomorphic, has been generalized in a number of directions by Amir [1], Behrends [2], Cambern [4, 5, 7], Jarosz [16], and others [11, 12]. See Jarosz [16] for discussion and references. In this paper we consider the near homeomorphic relationship between X and Y when $C(X)$ and $C(Y)$ admit an onto isomorphism φ whose bound $b = \|\varphi\|\,\|\varphi^{-1}\|$ is sufficiently small. From the seminal work of Amir [1] and Cambern [5], we know X and Y are homeomorphic if the bound is less than 2. And from [10], homeomorphism does not follow from a bound of 2 or more. However, [11] showed the existence of a relation on $X \times Y$ carrying topological properties when $b < 3$. The structure of this relation is unknown when $2 < b < 3$. This paper determines its structure when $b = 2$ and gives several applications. In particular, we are indebted to Chris Lennard for suggesting our extension of the result in [3; Theorem 2] and [6].

2. Examples. Throughout, given a compact Hausdorff space T, $C(T)$ is the Banach space of real-valued continuous functions on T with the supremum norm and $M(T) = C(T)^*$ the Banach space of regular (signed) real-valued Borel measures on T with total variation norm. A continuous linear operator φ from a Banach space B into $C(T)$ is determined by the action of φ^* on the set δT of all unit point mass measures δ_t on T; indeed, given any weak* continuous function $\Phi : T \to B^*$, the transformation $\varphi : B \to C(T)$ given by $\varphi(b)(t) = \Phi(t)(b)$ is linear with norm equal to $\sup\{\|\Phi(t)\| : t \in T\}$; furthermore, $\varphi^*(\delta_t) = \Phi(t)$ for all t in T. The map Φ is called the *dual*

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representation of φ and we will construct isomorphisms $\varphi : C(X) \to C(Y)$ by defining dual representations $\Phi : Y \to M(X)$ and $\Psi : X \to M(Y)$ such that $\varphi^*\Psi(x) = \delta_x$ $(x \in X)$ and $\psi^*\Phi$ $(y) = \delta_y$ $(y \in Y)$. Given a matrix $M = (c_{ij})$ of scalars, $\|M\|$ denotes $\max_i \sum_j |c_{ij}|$, the largest ℓ_1 norm of a row.

EXAMPLE 1. Any continuous $h : Y \to X$ gives rise to the dual representation $\Phi(y) = \delta_{h(y)}$ from Y into $M(X)$. If $w \in C(Y)$, then $\Phi(y) = w(y)\delta_{h(y)}$ is a *point-mass* representation with function h and continuous weight w. In particular, if h is a homeomorphism (onto) and w has no zeros, Φ represents an isomorphism whose inverse is represented by

$$\Psi(x) = \frac{1}{w \circ h^{-1}(x)} \delta_{h^{-1}(x)}$$

and whose bound is $\max |w| / \min |w|$.

EXAMPLE 2. Construct each of X and Y as pictured from four pairwise disjoint connected compact Hausdorff spaces, A, B, C, D, two curves x_o and x_e for X, and two curves y_o and y_e for Y. The four curves, homeomorphisms defined on $I = [0,1]$, are disjoint from one another and from A, B, C, and D except at their endpoints where $y_o(0) = x_o(0)$ in C, $y_e(0) = x_e(0)$ in A, $y_o(1) = x_e(1)$ in D, and $y_e(1) = x_o(1)$ in B.

There is no continuous extension $H : Y \to X$ of the identity map on $A \cup B \cup C \cup D$ since, for example, $H \circ y_e$ would join A to B in X. Consequently, a continuously weighted identity map $y \to w(y)\delta_y \in M(X)$ defined on $A \cup B \cup C \cup D$, with $w(y) \neq 0$ for all y in Y, cannot be extended to a *point-mass* representation $\Phi : Y \to M(X)$.

There is an extension $\Phi : Y \to M(X)$, however, if the measures $\Phi(y)$, $y \in y_o[I] \cup y_e[I]$, are permitted to have two-point supports. For any $w \neq 0$, we look for a matrix

$$\mathcal{C} = \begin{bmatrix} c_{ee} & c_{eo} \\ c_{oe} & c_{oo} \end{bmatrix}$$

of continuous functions on I for which

$$\mathcal{C}(0) = \begin{bmatrix} w(y_e(0)) & 0 \\ 0 & w(y_o(0)) \end{bmatrix} \quad \text{and} \quad \mathcal{C}(1) = \begin{bmatrix} 0 & w(y_e(1)) \\ w(y_o(1)) & 0 \end{bmatrix},$$

then define

$$\begin{bmatrix} \Phi(y_e(t)) \\ \Phi(y_o(t)) \end{bmatrix} \equiv \begin{bmatrix} c_{ee}(t)\delta_{x_e(t)} + c_{eo}(t)\delta_{x_o(t)} \\ c_{oe}(t)\delta_{x_e(t)} + c_{oo}(t)\delta_{x_o(t)} \end{bmatrix} = \begin{bmatrix} c_{ee}(t) & c_{eo}(t) \\ c_{oe}(t) & c_{oo}(t) \end{bmatrix} \begin{bmatrix} \delta_{x_e(t)} \\ \delta_{x_o(t)} \end{bmatrix} = \mathcal{C}(t) \begin{bmatrix} \delta_{x_e(t)} \\ \delta_{x_o(t)} \end{bmatrix}$$

With $w \equiv 1$, the $\mathcal{C}$ with linear entries satisfying the conditions at 0 and 1 is

$$\begin{bmatrix} 1-t & t \\ t & 1-t \end{bmatrix},$$

but the continuous linear transformation represented is not an isomorphism since $\Phi(y_o(1/2)) = \Phi(y_e(1/2))$ but $y_o(1/2) \neq y_e(1/2)$. Setting $w \equiv 1$ on $A \cup B \cup D$ and $w \equiv -1$ on C,

$$\mathcal{C}(t) \equiv \begin{bmatrix} 1-t & t \\ t & -(1-t) \end{bmatrix}$$

is invertible for all t and provides the representation of an isomorphism whose inverse is represented by the (same) weight $1/w = w$ on $A \cup B \cup C \cup D$ and by

$$\begin{bmatrix} \Psi(x_o(t)) \\ \Psi(x_e(t)) \end{bmatrix} = \mathcal{C}^{-1}(t) \begin{bmatrix} \delta_{y_o(t)} \\ \delta_{y_e(t)} \end{bmatrix} = \frac{1}{(1-t)^2 + t^2} \begin{bmatrix} 1-t & t \\ t & -(1-t) \end{bmatrix} \begin{bmatrix} \delta_{y_o(t)} \\ \delta_{y_e(t)} \end{bmatrix} \quad \text{for all } t \text{ in } I.$$

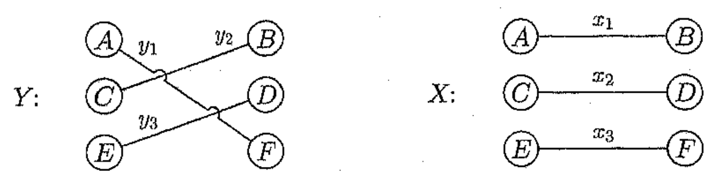
Since $|w(y)| = 1$ for all y,

$$\|\Phi\| = \max_{0 \le t \le 1} \|\mathcal{C}(t)\| = \max_{0 \le t \le 1} (|1-t| + |t|) = 1 \quad \text{and}$$

$$\|\Psi\| = \max_{0 \le t \le 1} \|\mathcal{C}^{-1}(t)\| = \max_{0 \le t \le 1} \frac{1}{(1-t)^2 + t^2} = 2.$$

The isomorphism has bound 2. In this example, a reworking of [10], the coefficients avoid discontinuity of the transformation by shifting weights as t varies from 0 to 1. In the following we give one more example to illustrate this idea.

EXAMPLE 3. Construct X and Y from six compact Hausdorff spaces A, B, C, D, E, F, and the curves y_1, y_2, y_3, x_1, x_2, x_3 as shown:



with all the component parts disjoint except at the endpoints of the curves where $y_i(0) = x_i(0)$ for $i = 1,2,3$ and $y_1(1) = x_3(1)$, $y_2(1) = x_1(1)$, and $y_3(1) = x_2(1)$. Suppose w is any non-zero continuous weight for $A \cup B \cup C \cup D \cup E \cup F$ and $C(t)$ any 3×3 matrix of real-valued continuous functions c_{ij} on I, $i,j = 1,2,3$, for which

$$C(0) = \begin{bmatrix} w(y_1(0)) & 0 & 0 \\ 0 & w(y_2(0)) & 0 \\ 0 & 0 & w(y_3(0)) \end{bmatrix}$$

and

$$C(1) = \begin{bmatrix} 0 & 0 & w(y_1(1)) \\ w(y_2(1)) & 0 & 0 \\ 0 & w(y_3(1)) & 0 \end{bmatrix}.$$

Then $\Phi(y) \equiv w(y)\delta_y$ for all $y \in A \cup B \cup C \cup D \cup E \cup F$ and

$$\begin{bmatrix} \Phi(y_1(t)) \\ \Phi(y_2(t)) \\ \Phi(y_3(t)) \end{bmatrix} \equiv C(t) \begin{bmatrix} \delta_{x_1(t)} \\ \delta_{x_2(t)} \\ \delta_{x_3(t)} \end{bmatrix}$$

agree at the endpoints of the curves y_i, $i = 1,2,3$, and define a representation $Y \to M(X)$. If $C(t)$ is non-singular for each t, then Φ is an isomorphism and Φ^{-1} is represented by $\Psi(x) \equiv (1/w(x))\delta_x$ for all $x \in A \cup B \cup C \cup D \cup E \cup F$, and

$$\begin{bmatrix} \Psi(x_1(t)) \\ \Psi(x_2(t)) \\ \Psi(x_3(t)) \end{bmatrix} \equiv C^{-1}(t) \begin{bmatrix} \delta_{y_1(t)} \\ \delta_{y_2(t)} \\ \delta_{y_3(t)} \end{bmatrix}.$$

One such coefficient matrix is

$$C(t) = \begin{bmatrix} (1-t)^p & 0 & t^p \\ t^p & (1-t)^p & 0 \\ 0 & t^p & (1-t)^p \end{bmatrix}$$

for the weight function $w = 1$ on $A \cup B \cup C \cup D \cup E \cup F$. Here $p \geq 1$, and is fixed. We find $\det C(t) = (1-t)^{3p} + t^{3p} \neq 0$ on I and

$$C^{-1}(t) = \frac{1}{\det C(t)} \begin{bmatrix} (1-t)^{2p} & t^{2p} & -t^p(1-t)^p \\ -t^p(1-t)^p & (1-t)^{2p} & t^{2p} \\ t^{2p} & -t^p(1-t)^p & (1-t)^{2p} \end{bmatrix}.$$

Consequently,

$$\|\Phi\| = \max_{0 \leq t \leq 1} \|C(t)\| = \max_{0 \leq t \leq 1} ((1-t)^p + t^p) = 2^{1-p}$$

and

$$\|\Psi\| = \max_{0 \leq t \leq 1} \|C^{-1}(t)\| = \max_{0 \leq t \leq 1} \frac{(1-t)^{2p} + t^{2p} + t^p(1-t)^p}{(1-t)^{3p} + t^{3p}} = 3 \cdot 2^{p-1};$$

so the bound is 3.

For the X and Y of this example, Lisa Koch has found an isomorphism of bound $2 + 7/9$ between $C(X)$ and $C(Y)$ and we wonder how small a bound can be achieved.

3. Statement of main results

DEFINITION. A *bridge* in a topological space T is a triple (F, W, S) where W is an open set and F, S two disjoint closed sets such that $\mathrm{bdry}(W) = \partial W = F \cup S$ (hence $W \cap (F \cup S) = \emptyset$). We will speak of the bridge W with *first endset* $F(W)$ and *second endset* $S(W)$, and we will say that W *joins* a set T_1 *to* a set T_2 in T provided $W \cap (T_1 \cup T_2) = \emptyset$, $F \subseteq T_1$, and $S \subseteq T_2$. Here T_1 may intersect T_2; indeed, a bridge may join a set to itself. In general, $F \cup W \cup S$ is a regular closed set whose interior contains the dense open set W.

The notation in the following main theorem is motivated by later constructions. In particular, eight bridges are distinguished by means of three binary indices: $T \in \{X, Y\}$ indicating which space contains the bridge, the sign $s \in \{+, -\}$ indicating whether the bridge is "positive" or "negative", and the parity $p \in \{e, o\}$ indicating whether the bridge is "even" or "odd". For a particular value of T, s or p, the other value of the variable will be denoted by $\widehat{T}$, $\widehat{s}$, $\widehat{p}$, respectively. When s occurs as a factor in an expression, it stands for $+1$ if s is $+$ and -1 if s is $-$.

THEOREM. *Let X and Y be compact Hausdorff spaces. Then there is an isomorphism from $C(X)$ onto $C(Y)$ of bound 2 or less if and only if*

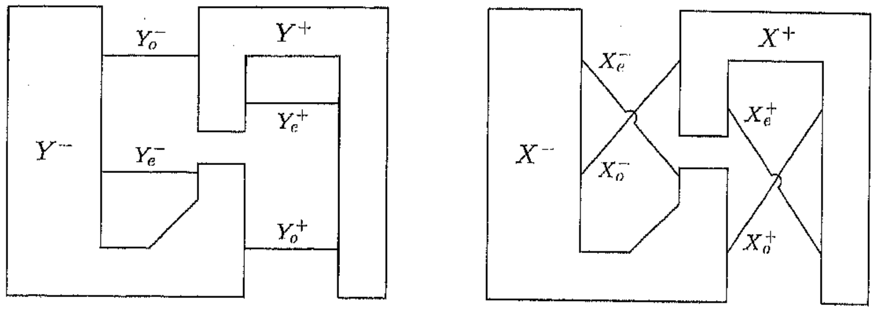
(1) *Each $T \in \{X, Y\}$ is the disjoint union of six sets: two closed sets T^s ($s = +, -$) and four bridges T^s_p ($s = +, -$ and $p = e, o$) such that T^s_e joins T^s to T^s and T^s_o joins $T^{\widehat{s}}$ to T^s and all the endsets are pairwise disjoint.*

(2) *There is a homeomorphism H carrying $X^+ \cup X^-$ onto $Y^+ \cup Y^-$ such that $H[X^s] = Y^s$, $H[F(X^s_p)] = F(Y^s_p)$ and $H[S(X^s_p)] = S(Y^s_{\widehat{p}})$ for all s and p.*

(3) *For each s, there is a family of four homeomorphisms h_{spq}, $p, q \in \{e, o\}$, carrying $\mathrm{cl}\, X^s_p$ onto $\mathrm{cl}\, Y^s_q$ such that $h_{spq} = h_{s\widehat{p}q} \circ h^{-1}_{s\widehat{p}\widehat{q}} \circ h_{sp\widehat{q}}$, $h_{spp} = H$ on $F(X^s_p)$, and $h_{sp\widehat{p}} = H$ on $S(X^s_p)$.*

The Theorem is illustrated by the diagram on the next page.

The third condition in the theorem amounts to the existence of a family of homeomorphisms, one between each pair of the four positive (negative) bridges in X and Y, that preserve the orientation of endsets, left endset to left and right to right, and that are compatible with one another in the following sense, which we state in general terms. Given four homeomorphic spaces T_i ($i = 1, \ldots 4$), and a homeomorphism $T_{ij} : T_i \to T_j$ for each i, j, we say these homeomorphisms are *compatible* if T_{ii} is the identity on T_i,



$T_{ij} = T_{ji}^{-1}$, and $T_{jk} \circ T_{ij} = T_{ik}$. Such a family is obviously constructible from any three homeomorphisms joining the four spaces or, in our condition, from the four homeomorphisms postulated for a given s.

We see that Example 2 is a special case of the conditions in the Theorem by setting $Y^- = X^- = A$, $Y^+ = X^+ = B \cup C \cup D$, and $Y_e^- = Y_o^- = X_e^- = X_o^- = \emptyset$.

Corollary 1. *If $C(X)$ and $C(Y)$ are bound-2 isomorphic, and if X is totally disconnected, then Y is homeomorphic to X.*

Proof. We can disconnect each $\mathrm{cl}\, X_p^s$ in its own topology into two clopen sets: $S(X_p^s)$ containing its second endset and $F(X_p^s)$ containing its first. This shows that X is the disjoint union of the two clopen subsets

$$C^+ = X^+ \cup S(X_e^+) \cup S(X_o^+) \cup F(X_e^+) \cup F(X_o^-) \quad \text{and}$$
$$C^- = X^- \cup S(X_e^-) \cup S(X_o^-) \cup F(X_e^-) \cup F(X_o^+).$$

Define $h^+ : C^+ \to Y$ to be H on X^+, h_{+eo} on $S(X_e^+)$, h_{+oo} on $S(X_o^+)$, h_{+ee} on $F(X_e^+)$, and h_{-oo} on $F(X_o^-)$, and $h^- : C^- \to Y$ to be H on X^-, h_{+oo} on $F(X_o^+)$, h_{-ee} on $F(X_e^-)$, h_{-oe} on $S(X_o^-)$, and h_{-eo} on $S(X_e^-)$. Then h^s is continuous and 1-1 on the compact C^s, hence a homeomorphism, for each s; so the two maps carry $X = C^+ \cup C^{-1}$ homeomorphically onto Y.

The following answers a question of Cambern [6].

Corollary 2. *For σ-finite measure spaces (S_i, Σ_i, μ_i), $i = 1, 2$, if the $L_1(S_i, \Sigma_i, \mu_i)$ are bound-2 isomorphic, then they are linearly isometric.*

Proof. Note that $L_1(S_i, \Sigma_i, \mu_i)^*$ $(i = 1, 2)$ is linearly isometric to $C(X_i)$ with X_i hyperstonean [13]. By Corollary 1, X_1 is homeomorphic with X_2, implying that the Σ_i are regular set-isomorphic in the sense of [19]. So the $L_1(S_i, \Sigma_i, \mu_i)$ are linearly isometric.

Corollary 3. *If $C(X)$ is bound-2 isomorphic with a P_1 space, it is a P_1 space.*

Proof. The P_1 space must be of the form $C(Y)$ with Y extremally disconnected [15, 18]. By Corollary 1, X is homeomorphic with Y. Then X is also extremally disconnected, and so $C(X)$ is P_1.

Conjecture. The three corollaries hold for bound $2 < b < 3$.

4. The topological conditions are sufficient. Suppose the topological objects of our main theorem above are given. To mimic the construction in Example 2, we index the bridges of sign s with the compact Hausdorff space $I(s)$. Indeed, we can suppose the existence of closed disjoint subsets $F(s)$ and $S(s)$ of $I(s)$ and homeomorphisms $h(T_p^s)$ carrying $I(s)$ onto $\mathrm{cl}\, T_p^s$, $F(s)$ onto $F(T_p^s)$, $S(s)$ onto $S(T_p^s)$, and such that $H \circ h(X_p^s) = h(Y_p^s)$ on $F(s)$ and $H \circ h(X_p^s) = h(Y_p^s)$ on $S(s)$. Let $W(s) = I(s) \setminus (F(s) \cup S(s))$ for each s so that $I(s) = \mathrm{cl}(W(s))$. If each $W(s)$ is disconnected or void, then X and Y are homeomorphic, hence $C(X)$ and $C(Y)$ are linearly isometric. Assume then that at least one $W(s)$ is non-void and connected, so $I(s)$ is also connected, and let f_s be a continuous function on $I(s)$ with values in $[0,1]$ which is 0 on $F(s)$ and 1 on $S(s)$. Define $\Phi : Y \to M(X)$ by

$$\Phi(z) \equiv s\delta_{H^{-1}z} \quad \text{if } z \in Y^s, s = + \text{ or } -,$$
$$\Phi \begin{bmatrix} h(Y_e^s)(t) \\ h(Y_o^s)(t) \end{bmatrix} = s \begin{bmatrix} 1 - f_s(t) & f_s(t) \\ f_s(t) & -(1 - f_s(t)) \end{bmatrix} \begin{bmatrix} \delta_{h(X_e^s)(t)} \\ \delta_{h(X_o^s)(t)} \end{bmatrix}.$$

The weak* continuity of Φ on each of the six closed sets Y^s and $\mathrm{cl}\, Y_p^s$ is evident, so Φ will be continuous on Y provided the defining formulas agree where these closed sets overlap on the endsets. Consider, for instance, $F(Y_e^s)$, which is contained in Y^s because Y_e^s joins Y^s to itself. Let $z \in F(Y_e^s)$. There is a $t \in F(s)$ such that $z = h(Y_e^s)(t)$. Then $f_s(t) = o$ and the matrix equation yields $\Phi(z) = \Phi(h(Y_e^s)(t)) = s\delta_{h(X_e^s)(t)} = s\delta_{H^{-1} \circ h(Y_e^s)(t)} = s\delta_{H^{-1}z}$, which is the value for Φ at $z \in Y^s$.

Similarly, let $z \in F(Y^s)$ be a subset of $Y^{\hat{s}}$. Again, there is a $t \in F(S)$ such that $z = h(Y_o^s)(t)$ and $f(s(t)) = 0$. From the matrix equation, $\Phi(z) = \Phi(H(Y_o^s)(t)) = s(-1)\delta_{h(X_o^s)(t)} = \hat{s}\delta_{H^{-1} \circ h(Y_o^s)(t)} = \hat{s}\delta_{H^{-1}z}$ is the value of Φ at $z \in Y^{\hat{s}}$. The calculation is similar for $S(Y_e^s)$ and $S(Y_o^s)$.

Thus Φ is the dual representation of a linear transformation $\varphi : C(X) \to C(Y)$ whose norm, $\sup\{\|\Phi(y)\| : y \in Y\}$, computes to 1 since $|s(1 - f_s(t))| + |sf_s(t))| = 1 - f_s(t) + f_s(t) = 1$. For each $t \in I(s)$, set $D(t) = (1 - f_s(t))^2 + f_s(t)^2$ and define $\Psi : X \to M(Y)$ by

$$\Psi(w) = s\delta_{Hw} \quad \text{if } w \in X^s, s = +, -,$$
$$\Psi \begin{bmatrix} h(X_e^s)(t) \\ h(X_o^s)(t) \end{bmatrix} = \frac{s}{D(t)} \begin{bmatrix} 1 - f_s(t) & f_s(t) \\ f_s(t) & -(1 - f_s(t)) \end{bmatrix} \begin{bmatrix} \delta_{h(Y_e^s)(t)} \\ \delta_{h(Y_o^s)(t)} \end{bmatrix}$$

for all $t \in I(s)$. Arguments similar to those for Φ establish Ψ as the dual representation of a linear operator $\psi : C(Y) \to C(X)$. Moreover, $\psi = \varphi^{-1}$. We have $\|\psi\| \le 2$ because, e.g.,

$$\frac{1}{D(t)}(|1 - f_s(t)| + |f_s(t)|) = \frac{1}{D(t)} = \frac{1}{1 - f_s(t)^2 + f_s(t)^2} \le 2.$$

For at least one $s, I(s)$ is connected and so f_s assumes the value 1/2 providing an x in X for which $|\Psi(x)| = 2$. Thus $\|\psi\| = 2$.

5. The topological conditions are necessary. We are given an isomorphism $\varphi : C(X) \to C(Y)$ of bound 2 and inverse ψ. Following [11], suppose $\|\varphi\| = 2$ and $\|\psi\| = 1$. To smoothly expose the argument for necessity we refer the reader forward to the Appendix and to [11] for all results that depend upon isolated points in compact Hausdorff spaces T, in particular in the spaces kT for which $C(kT)$ represents $C(T)^{**}$. We make immediate use here of Lemma A1 in the Appendix which states that the points of T are in 1-1 correspondence, $t \to \widehat{t}$, with the isolated points of kT.

DEFINITION. For x in X and y in Y, write $x \backsim y$ when $|\psi^{**}\chi_{\widehat{y}}|$ assumes its norm at $\widehat{x}$, where $\chi_{\widehat{y}}$ denotes the characteristic function of $\widehat{y}$, and write $y \backsim x$ when $|\varphi^{**}\chi_{\widehat{x}}|$ assumes its norm at $\widehat{y}$. By [11; Lemma 2], $\backsim(x) = \{y \in Y \mid x \backsim y\}$ has one or two points and x is called *simple* or *compound* accordingly; and similarly, y is *simple* or *compound* according to whether $\backsim(y)$ has one or two points.

From [11] and the Appendix, $\varphi^*\delta_y(\{x\}) = \varphi^{**}\chi_{\widehat{x}}(\widehat{y})$ and $\psi^*\delta_x(\{y\}) = \psi^{**}\chi_{\widehat{y}}(\widehat{x})$. Moreover, the following are equivalent:

$$\begin{array}{lll} x \backsim y & |\psi^{**}\chi_{\widehat{y}}(\widehat{x})| \ge 1/2 & |\psi^*\delta_x(\{y\})| \ge 1/2 \\ y \backsim x & |\varphi^{**}\chi_{\widehat{x}}(\widehat{y})| \ge 1 & |\varphi^*\delta_y(\{x\})| \ge 1. \end{array}$$

We also know that for $x \backsim y$, x is simple if and only if y is simple and x is compound iff y is compound. Moreover, if x_1 (resp. y_1) is compound, there are unique points x_2, y_1, y_2 (resp. x_1, x_2, y_2) such that $x_1 \ne x_2$, $y_1 \ne y_2$ and $\backsim(x_1) = \backsim(x_2) = \{y_1, y_2\}$ and $\backsim(y_1) = \backsim(y_2) = \{x_1, x_2\}$. Let us refer to x_1, x_2, y_1, y_2 so related as a *quadruple of compound points*. By Theorem 11 of [11], $\backsim$ is a homeomorphism between the open sets of simple points $\mathrm{simp}(X), \mathrm{simp}(Y)$ in X and Y respectively. Hence the compound points, $\mathrm{comp}(X)$ and $\mathrm{comp}(Y)$, are closed. Moreover, if there are no compound points, X and Y consist of simple points and so are homeomorphic and we are finished. In the remainder of this proof of necessity, we assume that compound points exist.

From Lemma 7 and its proof in [11], we have

LEMMA 5.1. *A quadruple of compound points can be ordered and then relabeled* (x^e, x^o, y^e, y^o) *so that for each sign* s,

$$\varphi^*\begin{bmatrix}\delta_{y^e}\\ \delta_{y^o}\end{bmatrix} = s\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\delta_{x^e}\\ \delta_{x^o}\end{bmatrix} \quad \textit{and} \quad \psi^*\begin{bmatrix}\delta_{x^e}\\ \delta_{x^o}\end{bmatrix} = s\begin{bmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{bmatrix}\begin{bmatrix}\delta_{y^e}\\ \delta_{y^o}\end{bmatrix}.$$

DEFINITION. If $x \in \mathrm{comp}(X)$, then $\psi^*\delta_x = (\alpha/2)\delta_{y_1} + (\beta/2)\delta_{y_2}$ where $|\alpha| = |\beta| = 1$. We label $x = x^e$ and say x is *even* if $\alpha = \beta$, and label $x = x^o$ and say x is *odd* if $\alpha \ne \beta$. Similarly, if $y \in \mathrm{comp}(Y)$, then $\varphi^*\delta_y = \alpha\delta_{x_1} + \beta\delta_{x_2}$ where $|\alpha| = |\beta| = 1$; we then label $y = y^e$ and say y is even if $\alpha = \beta$, and label $y = y^o$ and say y is odd if $\alpha \ne \beta$. A quadruple of compound points and its entries are called *positive* if s is $+$ in the matrix equations above, *negative* if s is $-$.

For each choice of $T = X$ or Y, $p = e$ or o, and $s = +$ or $-$, we let $\mathrm{comp}(T, s, p)$ denote the set of compound points in T with sign s and parity p, and $\mathrm{comp}(T, s) = \mathrm{comp}(T, s, p) \cup \mathrm{comp}(T, s, \widehat{p})$. We let Z denote the set of all quadruples in the order (x^e, x^o, y^e, y^o), Z_+ the subset of positive quadruples, and Z_- the subset of negative quadruples in Z. Note that the three pairs of indices T, s, p of Section 3 have now been defined in this proof of necessity. The eight bridges we seek will emerge as neighborhoods of the eight sets $\mathrm{comp}(T, s, p)$.

LEMMA 5.2. *The sets* $\mathrm{comp}(T, s, p)$ *are pairwise disjoint, and each is closed in* T. *Moreover, for each* s, Z_s *defines a homeomorphism between each pair of the four sets* $\mathrm{comp}(T, s, p)$.

Proof. We assume X and Y are disjoint from the outset. As $t \in \mathrm{comp}(T)$ is either positive or negative and not both, and even or odd and not both, $\mathrm{comp}(T)$ is the disjoint union of the four sets $\mathrm{comp}(T, s, p)$. Lemma 9 of [11] states that Z is closed in $X \times X \times Y \times Y$. The proof shows that if a net $(x^e_\alpha, x^o_\alpha, y^e_\alpha, y^o_\alpha)$ in Z converges to (x^e, x^o, y^e, y^o), then this point is in Z and

$$\varphi^*\delta_{y^{p_1}}(\{x^{p_2}\}) = \lim_\alpha \varphi^*\delta_{y^{p_1}_\alpha}(\{x^{p_2}_\alpha\})$$

for each of the four choices of (p_1, p_2). So if the net is in Z_+ (resp. in Z_-), so is the limit. This shows both Z_+ and Z_- are closed sets; hence, compact. The four sets $\mathrm{comp}(T, +, p)$ are the images of Z_+ under the coordinate projections; hence, each is closed. Similarly, the sets $\mathrm{comp}(T, -, p)$ are closed. For each s, Z_s is compact, so the 1-1 map between $\mathrm{comp}(T_1, s, p_1)$ and $\mathrm{comp}(T_2, s, p_2)$ provided by Z_s has a closed graph and is, therefore, a homeomorphism.

Example 2 has only one quadruple (x^e, x^o, y^e, y^o) of compound points, at $t = 1/2$, and since $\{x^e\}$, $\{x^o\}$, $\{y^e\}$, $\{y^o\}$ have void interiors, they cannot serve as bridges. To obtain bridges, we now generalize our notion of compound point.

DEFINITION. Let $\gamma : C(S) \to C(T)$ be a bound-2 isomorphism (onto), $\|\gamma\|/4 < c \le \|\gamma\|/2$, $s_1, s_2 \in S$, and $t_1, t_2 \in T$. We say $\{s_1, s_2, t_1, t_2\}$ is a *(γ, c)-set* if $s_1 \ne s_2$, $t_1 \ne t_2$, and $|\gamma^*\delta_{t_i}(\{s_j\})| \ge c$ for all $i, j = 1, 2$. For $i \in \{1, 2\}, \hat{\imath}$ denotes the other number. Note that since a $(\gamma, \|\gamma\|/2)$-set is a quadruple of compound points, we will be interested in $c < \|\gamma\|/2$.

LEMMA 5.3. *Let $\{s_1, s_2, t_1, t_2\}$ be a (γ, c)-set.*

(1) *For each i there is a j such that $\|\gamma\| - c \ge |\gamma^*\delta_{t_i}(\{s_j\})| \ge \|\gamma\|/2$; i.e., $t_i \backsim s_j$. Moreover, if $s \in S\backslash\{s_1, s_2\}$, then $|\gamma^*\delta_{t_i}(\{s\})| < c$.*

(2) *Either s_1, s_2, t_1, t_2 form a quadruple of compound points, or they are all simple.*

(3) *If s_1, s_2, t_1, t_2 are all simple and $t_i \backsim s_j$, then $s_{\hat\jmath}$ is the unique element of S at which the function $s \to |\gamma^*\delta_{t_i}(\{s\})|$ assumes its second largest value.*

(4) *If $\{s_1, s_2, t_1, t_2\}$ and $\{\overline{s}_1, \overline{s}_2, \overline{t}_1, \overline{t}_2\}$ are (γ, c)-sets and either $\{s_1, s_2\} \cap \{\overline{s}_1, \overline{s}_2\} \ne \emptyset$ or $\{t_1, t_2\} \cap \{\overline{t}_1, \overline{t}_2\} \ne \emptyset$, then $\{s_1, s_2\} = \{\overline{s}_1, \overline{s}_2\}$ and $\{t_1, t_2\} = \{\overline{t}_1, \overline{t}_2\}$.*

Proof. For (1), what if $|\gamma^*\delta_{t_i}(\{s_j\})| < \|\gamma\|/2$ for both j?! Then since there is an $s \backsim t_i$, we have $|\gamma^*\delta_{t_i}(\{s\})| \ge \|\gamma\|/2$ and s_1, s_2, and s are distinct. But then $\|\gamma\| \ge |\gamma^*\delta_{t_i}| \ge |\gamma^*\delta_{t_i}|(\{s_1, s_2, s\}) \ge 2c + \|\gamma\|/2 > \|\gamma\|$, impossible. This, with $|\gamma^*\delta_{t_i}(\{s_j\})| + c \le |\gamma^*\delta_{t_i}(\{s_j\})| + |\gamma^*\delta_{t_i}(\{s_{\hat\jmath}\})| \le \|\gamma\|$, proves the first part of (1). If, in the second part of (1), $|\gamma^*\delta_{t_i}(\{s\})| \ge c$, then $\|\gamma\| > \|\gamma\|$ as above, a contradiction.

To prove (2), first suppose t_1 and t_2 are simple. Then $t_1 \backsim s_j$ for some j and $t_2 \backsim s_{\hat\jmath}$. The other possibility is that one of $\{t_1, t_2\}$, say t_i, is compound. Then since $|\gamma^*\delta_{t_i}(\{s\})| < c \le \|\gamma\|/2$ for all $s \in S\backslash\{s_1, s_2\}$, we must have $\backsim(t_i) = \{s_1, s_2\}$. Then s_1 and s_2 are compound and $t_{\hat\imath} \backsim s_j$ for one of $j = 1, 2$. Consequently, $t_{\hat\imath}$ is compound, so by the same reasoning used for t_i, $\backsim(t_{\hat\imath}) = \{s_1, s_2\}$. Thus also $\backsim(s_1) = \{t_1, t_2\} = \backsim(s_2)$ and s_1, s_2, t_1, t_2 is a quadruple of compound points.

The result (3) follows immediately from (1).

To prove (4), suppose first that all the points are compound. If, say, $s_1 = \widehat{s}_1$, then $\{t_1, t_2\} = \backsim(s_1) = \backsim(\widehat{s}_1) = \{\widehat{t}_1, \widehat{t}_2\}$. Then, say, $t_1 = \widehat{t}_1$ so $\{s_1, s_2\} = \backsim(t_1) = \backsim(\widehat{t}_1) = \{\widehat{s}_1, \widehat{s}_2\}$. The other possibility is that all the points are simple. Suppose first that two of the t's are equal, say $t \equiv t_1 = \widehat{t}_1$. Then $\{s_1, s_2\} = \{s \in S : |\gamma^*\delta_t(\{s\})| \ge c\} = \{\widehat{s}_1, \widehat{s}_2\}$. Say $t \backsim s \equiv s_1 = \widehat{s}_1$ (the argument is the same if $s = s_1 = \widehat{s}_2$, $s = s_2 = \widehat{s}_2$ or $s = s_2 = \widehat{s}_1$). Then since s is simple, neither $t_2 \backsim s$ nor $\widehat{t}_2 \backsim s$. Consequently, $t_2, \widehat{t}_2 \backsim s_2 = \widehat{s}_2$. Since s_2 is simple, $t_2 = \widehat{t}_2$ as desired. If, second, two of the s's are equal, say $s \equiv s_1 = \widehat{s}_1$, then for some i and j, $s \backsim t_i, t_j$. Since s is simple, $t_i = \widehat{t}_j$ and our prior argument applies.

LEMMA 5.4. *If $4/5 < c < 1$, and if $\{x_1, x_2, y_1, y_2\}$ is a (φ, c)-set, it is also a $(\psi, \frac{1}{2}(5 - 4/c))$-set. Furthermore, for each i, j, $\varphi^*\delta_{y_i}(\{x_j\})$ and $\psi^*\delta_{x_j}(\{y_i\})$ have the same sign.*

Proof. For each i there is a j such that $x_i \backsim y_j$ and consequently, $|\psi^*\delta_{x_i}(\{y_j\})| \ge 1/2 > \frac{1}{2}(5 - 4/c)$ as well as

$$(\star) \qquad |\varphi^{**}\chi_{\hat{x}_i}(\widehat{y}_j)| = \|\varphi^{**}\chi_{\hat{x}_i}\|.$$

It remains to show $|\psi^*\delta_{x_i}(\{y_{\hat\jmath}\})| \ge \frac{1}{2}(5 - 4/c)$. By Lemma 5.3(1) and $(\star)$,

$$|\varphi^{**}\chi_{x_i}(y_{\hat\jmath})| \ge c = \frac{c}{2-c}(2-c) \ge \frac{c}{2-c}|\varphi^*\delta_{y_i}(\{x_i\})|$$
$$= \frac{c}{2-c}|\varphi^{**}\chi_{\hat{x}_i}(\widehat{y}_i)| = \frac{c}{2-c}\|\varphi^{**}\chi_{\hat{x}_i}\|.$$

By Lemma A2, since $3/(2-c) > 2/3$,

$$|\psi^*\delta_{x_i}(\{y_{\hat\jmath}\})| = |\psi^{**}\chi_{y_{\hat\jmath}}(\widehat{x}_i)| \ge \frac{\|\psi\|}{2}\left(3 - \frac{2}{\frac{c}{2-c}}\right) = \frac{1}{2}\left(5 - \frac{4}{c}\right).$$

The second assertion follows from Lemma A7.

LEMMA 5.5. *If $68/71 < c < 1$, a (φ, c)-set $\{x_1, x_2, y_1, y_2\}$ can be labelled x^e, x^o, y^e, y^o so that either*

(1) *$\varphi^*\delta_{y^e}(\{x^e\}), \varphi^*\delta_{y^e}(\{x^o\}), \varphi^*\delta_{y^o}(\{x^e\}) > 0$ and $\varphi^*\delta_{y^o}(\{x^o\}) < 0$, whereupon we call the value of the foursome* positive, *or*

(2) *$\varphi^*\delta_{y^e}(\{x^e\}), \varphi^*\delta_{y^e}(\{x^o\}), \varphi^*\delta_{y^o}(\{x^e\}) < 0$ and $\varphi^*\delta_{y^o}(\{x^o\}) > 0$, whereupon we call the value of the foursome* negative.

Proof. Since $68/71 < c < 1$, $\frac{1}{2}(5 - 4/c) > 7/17$ so $\{x_1, x_2, x_3, x_4\}$ is a $(\psi, 7/14)$-set and, by Lemma 5.4, for each i, j, $\varphi^*\delta_{y_i}(\{x_j\})$ and $\psi^*\delta_{x_j}(\{y_i\})$ have the same sign. For each $i = 1, 2$, $\psi^*\delta_{x_i} = \psi^*\delta_{x_i}(\{y_1\})\delta_{y_1} + \psi^*\delta_{x_i}(\{y_2\})\delta_{y_2} + \nu_i$ where $|\nu_i|(\{x_1, x_2\}) = 0$ and $|\nu_i| \le 1 - 2 \cdot 7/17 < 1/8$. Then $|\varphi^*\nu_i| < 1/4$. Since

$$\begin{bmatrix} \psi^*\delta_{x_1}(\{y_1\}) & \psi^*\delta_{x_1}(\{y_2\}) \\ \psi^*\delta_{x_2}(\{y_1\}) & \psi^*\delta_{x_2}(\{y_2\}) \end{bmatrix} \begin{bmatrix} \varphi^*\delta_{y_1}(\{x_1\}) & \varphi^*\delta_{y_1}(\{x_2\}) \\ \varphi^*\delta_{y_2}(\{x_1\}) & \varphi^*\delta_{y_2}(\{x_1\}) \end{bmatrix}$$
$$= \begin{bmatrix} 1 - \varphi^*\nu_1(\{x_1\}) & -\varphi^*\nu_2(\{x_2\}) \\ -\varphi^*\nu_1(\{x_1\}) & 1 - \varphi^*\nu_2(\{x_2\}) \end{bmatrix},$$

the hypotheses of the following lemma are fulfilled, which completes the proof.

DEFINITION. If the four numbers $\varphi^*\delta_{y^e}(\{x^e\}), \varphi^*\delta_{y^e}(\{x^o\}), \varphi^*\delta_{y^o}(\{x^e\})$, $\varphi^*\delta_{y^o}(\{x^o\})$ consist of three positives (negatives) and one negative (positive), we will say the *majority sign* is positive (negative) and the minority sign negative (positive) and call $\{x^e, x^o, y^e, y^o\}$ a *positive* (*negative*) (φ, c)-

set. Note that x^o and y^o are characterized by $\varphi^*\delta_{y^o}(\{x^o\})$ having the minority sign.

LEMMA. *Let δ_{ij} denote 0 if $i \neq j$ and 1 if $i = j$. Suppose A and B are 2×2 matrices of real numbers and $C = AB$ with entries a_{ij}, b_{ij}, c_{ij} respectively that satisfy the conditions $||a_{ij}|-1/2|, ||b_{ij}|-1|, |c_{ij}-\delta_{ij}| < 1/4$ and $\operatorname{sign} a_{ij} = \operatorname{sign} b_{ji}$ for all $i,j = 1,2$. Then either A and B each have three positive and one negative entry or three negative and one positive.*

Proof. If not, then several cases are possible:

Case 1: All entries are positive. Then

$$\tfrac14 > c_{12} = a_{11}b_{12} + a_{12}b_{22} > \tfrac14\cdot\tfrac34 + \tfrac14\cdot\tfrac34 = \tfrac38,$$

impossible.

Case 2: All entries are negative. Then $1/4 > c_{12} = |a_{11}||b_{12}| + |a_{12}||b_{22}| > 3/8$ again.

Case 3: $a_{11}, a_{12} > 0$ and $a_{21}, a_{22} < 0$. Then $b_{11}, b_{21} > 0$ and $b_{12}, b_{22} < 0$. Hence

$$-\tfrac14 < c_{12} = a_{11}b_{12} + a_{12}b_{22} < \tfrac14\cdot\tfrac{-3}{4} + \tfrac14\cdot\tfrac{-3}{4} = \tfrac{-6}{16},$$

impossible. (The other cases are similar.)

Case 4: $a_{11}, a_{21} > 0$ and $a_{12}, a_{22} < 0$. Then $b_{11}, b_{12} > 0$ and $b_{21}, b_{22} < 0$. Hence

$$\tfrac14 > c_{12} = a_{11}b_{12} + a_{12}b_{22} = a_{11}b_{12} + |a_{12}||b_{22}| > \tfrac14\cdot\tfrac34 + \tfrac14\cdot\tfrac34 = \tfrac38,$$

impossible.

Case 5: $a_{11}, a_{22} > 0$ and $a_{12}, a_{21} < 0$. Then $b_{11}, b_{22} > 0$ and $b_{21}, b_{12} < 0$. Hence

$$\tfrac{-1}{4} < c_{12} = a_{11}b_{12} + a_{12}b_{22} < \tfrac14\cdot\tfrac{-3}{4} + \tfrac14\cdot\tfrac{-3}{4} = \tfrac{-6}{16}.$$

DEFINITION. Let $68/71 < c < 1$ so that any (φ, c)-set is also a (ψ, d)-set for $d = \frac12(5 - 4/c) > 7/17$, hence the conclusions of Lemmas 5.4 and 5.5 hold and we may use the notation x^e, x^o, y^e, y^o of Lemma 5.5. A *(φ, c)-quadruple* (a *(ψ, d)-quadruple*) is an ordered quadruple (x^e, x^o, y^e, y^o) such that $\{x^e, x^o, y^e, y^o\}$ is a (φ, c)-set (a (ψ, d)-set)); Φ^c (resp. Ψ^d) shall denote the set of all (φ, c)-quadruples (resp. (ψ, d)-quadruples), which is not empty since it contains all quadruples of compound points. The ranges of the canonical projections of Φ^c onto its four coordinates are denoted by $\mathcal{X}_e$, $\mathcal{X}_o$, $\mathcal{Y}_e$, $\mathcal{Y}_o$ respectively. Let $\Phi^{c,s}$ denote the set of quadruples of sign s in Φ^c and let $\mathcal{X}_e^s$, $\mathcal{X}_o^s$, $\mathcal{Y}_e^s$, $\mathcal{Y}_o^s$ denote the ranges of the coordinate projections of $\Phi^{c,s}$. We shall say that a set of ordered quadruples is 1-1 if two quadruples in the set are equal whenever they agree at one coordinate.

LEMMA 5.6. *Φ^c is 1-1 whenever $68/71 < c < 1$.*

Proof. If (x^e, x^o, y^e, y^o) and $(\widehat{x}^e, \widehat{x}^o, \widehat{y}^e, \widehat{y}^o)$ in Φ^c agree at one coordinate, then by Lemma 5.3(4), $\{x^e, x^o\} = \{\widehat{x}^e, \widehat{x}^o\}$ and $\{y^e, y^o\} = \{\widehat{y}^e, \widehat{y}^o\}$. If they agree at an x-coordinate, then $x^e = \widehat{x}^e$ and $x^o = \widehat{x}^o$. If $y^e = \widehat{y}^o$ and $y^o = \widehat{y}^e$, then $\varphi^*\delta_{y^o}(\{x^o\})$, which has the minority sign of the (φ, c)-set, is equal to $\varphi^*\delta_{\widehat{y}^e}(\{\widehat{x}^o\})$ which has the majority sign, a contradiction. Thus, $y^e = \widehat{y}^e$ and $y^o = \widehat{y}^o$. If they agree at a y-coordinate, we reach the same conclusion because (x^e, x^o, y^e, y^o) is a (ψ, d)-quadruple.

Remark. If Φ^c were closed in $X\times X\times Y\times Y$, the range of the natural projection of Φ^c onto $\mathcal{X}_e \times \mathcal{X}_o$ would be a homeomorphism of $\mathcal{X}_e$ with $\mathcal{X}_o$; indeed, the range of the coordinate projection onto any two of $\mathcal{X}_e$, $\mathcal{X}_o$, $\mathcal{Y}_e$, $\mathcal{Y}_o$ would be a homeomorphism. We have not shown Φ^c is closed, but we can prove $\operatorname{cl}\Phi^c \subseteq \Phi^{2c-1}$ for $2c - 1 > 68/71$. Since Φ^{2c-1} is then 1-1, $\operatorname{cl}\Phi^c$ provides a compatible family of homeomorphisms. See the remarks after the main theorem in Section 3 above. We will need the following lemma, whose proof, a slight generalization of the proof of Lemma 4 in [11], is omitted.

DEFINITION. In the sequel, c is a fixed number satisfying $139/142 < c < 1$, hence $68/71 < 2c - 1 < 1$.

LEMMA 5.7. *Let $1/2 < r \le 1$ and suppose T is compact Hausdorff, $t_\alpha \to x \neq t$ in T, μ_α a net in $M(T)$ weak* convergent to μ, $|\mu_\alpha| \le 2$ for all α, and $\mu_\alpha(\{t_\alpha\}) \ge r$ (resp. $\le -r$) for all α. Then $\mu(\{x\}) \ge 2r - 1$ (resp. $\le -2r+1$).*

LEMMA 5.8. $\operatorname{cl}\Phi^c \subseteq \Phi^{2c-1}$; *indeed, for each sign s, $\operatorname{cl}\Phi^{c,s} \subseteq \Phi^{2c-1,s}$.*

Proof. Suppose $(x^e_\alpha, x^o_\alpha, y^e_\alpha, y^o_\alpha)$ is a net in $\Phi^{c,s}$ coordinatewise convergent to $(x^e, x^o, y^e, y^o) \in X\times X\times Y\times Y$. What if $x \equiv x^e = x^o$?! To reach a contradiction, let $\delta > 0$ be given and select an open neighborhood U of x such that $|\varphi^*\delta_{y^o}|(U\setminus\{x\}) < \delta$. Let $f : X \to [0,1]$ be continuous with $f = 0$ on $X\setminus U$ and $f = 1$ near x. For every α, $a_\alpha \equiv \varphi^*\delta_{y^o_\alpha}(\{x^o_\alpha\})$ and $b_\alpha \equiv \varphi^*\delta_{y^o_\alpha}(\{x^e_\alpha\})$ have opposite signs and by Lemma 5.3(1), one of them, say a_α, satisfies $1 \le |a_\alpha| \le 2 - c$. Moreover, for large α, $f(x^o_\alpha) = 1 = f(x^e_\alpha)$. Consequently,

$$\begin{aligned}
&|\varphi^*\delta_{y^o}(\{x\})| - \delta \\
&\quad\le |\varphi^*\delta_{y^o}(\{x\})| - \Big|\int_{U\setminus\{x\}} f\,d(\varphi^*\delta_{y^o})\Big| \\
&\quad\le \Big|\varphi^*\delta_{y^o}(\{x\}) + \int_{U\setminus\{x\}} f\,d(\varphi^*\delta_{y^o})\Big| \\
&\quad= \Big|\int f\,d(\varphi^*\delta_{y^o})\Big| = \lim_\alpha \Big|\int f\,(d\varphi^*\delta_{y^o_\alpha})\Big|
\end{aligned}$$

$$= \lim_\alpha \Big| f(x^o_\alpha)(\varphi^*\delta_{y^o_\alpha})(\{x^o_\alpha\}) + f(x^e_\alpha)(\varphi^*\delta_{y^o_\alpha})(\{x^e_\alpha\}) + \int\limits_{U\setminus\{x^o_\alpha, x^e_\alpha\}} f\,d(\varphi^*\delta_{y^o_\alpha})\Big|$$

$$= \lim_\alpha \Big| a_\alpha + b_\alpha + \int\limits_{U\setminus\{x^o_\alpha, x^e_\alpha\}} f\,d(\varphi^*\delta_{y^o_\alpha})\Big|.$$

Also, for every α,

$$\Big| a_\alpha + b_\alpha + \int\limits_{U\setminus\{x^o_\alpha, x^e_\alpha\}} f\,d(\varphi^*\delta_{y^o_\alpha})\Big| \le |a_\alpha + b_\alpha| + |\varphi^*\delta_{y^o_\alpha}|(U\setminus\{x^o_\alpha, x^e_\alpha\})$$
$$= ||a_\alpha| - |b_\alpha|| + |\varphi^o\delta_{y^o_\alpha}|(U\setminus\{x^o_\alpha, x^e_\alpha\})$$
$$\le |(2-c) - c| + 2 - (1+c) = 3(1-c).$$

Hence $|\varphi^*\delta_{y^o}(\{x\})| - \delta \le 3(1-c)$ and since δ is arbitrary, $|\varphi^*\delta_{y^o}(\{x\})| \le 3(1-c) < 1$. So if $x \backsim y^o$, then $|\varphi^*\delta_{y^o}(\{x\})| \ge 1$, a contradiction. If not $x \backsim y^o$, then $y^o \backsim t \ne x$ for some t in X. For each α, let $t_\alpha \in \{x^e_\alpha, x^o_\alpha\}$ be such that $y^o_\alpha \backsim t_\alpha$, hence $t_\alpha \to x$ and $|\varphi^*\delta_{y^o_\alpha}(\{t_\alpha\})| \ge 1$. By Lemma 4 of [11], $|\varphi^*\delta_{y^o}(\{x\})| \ge 1$, impossible. Thus $x^e \ne x^o$. A similar argument applied to $\Psi^{d,s}$ yields $y^e \ne y^o$. By Lemma 5.7, $(x^e, x^o, y^e, y^o) \in \Phi^{2c-1,s}$.

For each sign s, $\mathrm{cl}\,\Phi^{c,s}$ is 1-1 and therefore embodies a compatible set of homeomorphisms between any pair of $\mathrm{cl}\,\mathcal{X}^{c,s}_e$, $\mathrm{cl}\,\mathcal{X}^{c,s}_o$, $\mathrm{cl}\,\mathcal{Y}^{c,s}_e$, $\mathrm{cl}\,\mathcal{Y}^{c,s}_o$. Therefore $\Phi^{c,s}$ also embodies a compatible set of homeomorphisms between any two of $\mathcal{X}^{c,s}_e$, $\mathcal{X}^{c,s}_o$, $\mathcal{Y}^{c,s}_e$, $\mathcal{Y}^{c,s}_o$. The key remaining step to our result is

PROPOSITION. *A compound point in any one of the sets* $\mathcal{X}^{c,s}_e$, $\mathcal{X}^{c,s}_o$, $\mathcal{Y}^{c,s}_e$, $\mathcal{Y}^{c,s}_o$ *is interior to it.*

The proof of this proposition requires several auxiliary lemmas.

DEFINITION. Let $H : X \to Y$ denote the 1-1 onto map that associates with simple $x \in X$ the unique $y \in Y$ such that $x \backsim y$, and with the even (odd) compound point x^e (resp. x^o) in X the unique compound point y^e (resp. y^o) in Y such that $x^e \backsim y^e$ ($x^o \backsim y^o$). Recall that by Theorem 11 of [11], H is a homeomorphism of the simple points of X with the simple points of Y. We say a net *converges to a set* C iff it is eventually in every neighborhood of C.

LEMMA 5.9. *Let* x_α *be a net of simple points in* X, $y_\alpha = H(x_\alpha)$ *the corresponding net of simple points in* Y, *and* (x^e, x^o, y^e, y^o) *a quadruple of compound points. Then* $x_\alpha \to \{x^e, x^o\}$ *iff* $y_\alpha \to \{y^e, y^o\}$.

Proof. Assume $y_\alpha \to \{y^e, y^o\}$. If x_α does not converge to $\{x^e, x^o\}$, then passing to subnets by compactness, we can suppose $y_\alpha \to y \in \{y^e, y^o\}$ and $x_\alpha \to x \notin \{x^e, x^o\}$. Let $h \in C(X)$ have values in $[-1,1]$, be 0 at x, and $\operatorname{sign}\varphi^*\delta_y(\{x\})$ at x^p for each $p = e$ or o. For each α, let $\mu_\alpha = \varphi^*\delta_{y_\alpha}(\{x_\alpha\})\delta_{x_\alpha}$ and $\nu_\alpha = \varphi^*\delta_{y_\alpha} - \mu_\alpha =$ the restriction of $\varphi^*\delta_{y_\alpha}$ to $X\setminus\{x_\alpha\}$, and note that $\varphi^*\delta_{y_\alpha}$ is weak* convergent to $\varphi^*\delta_y = h(x^e)\delta_{x^e} + h(x^o)\delta_{x^o}$, $\varphi^*\delta_{y_\alpha}(\{x_\alpha\})$ is a bounded net of scalars, and $h(x_\alpha)$ converges to 0. Then $\nu_\alpha(h)$ converges to 2, so for large α, $2 < 1 + \nu_\alpha(h) \le |\varphi^*\delta_{y_\alpha}(\{x_\alpha\})| + \nu_\alpha(h) \le |\mu_\alpha| + |\nu_\alpha| = |\varphi^*\delta_{y_\alpha}| \le 2$, a contradiction. The converse argument is similar.

LEMMA 5.10. *If* x_o *is compound in* X *and* $\varepsilon > 0$, *there is a neighborhood* U *of* x_o *such that* $1/2 \le |\psi^*\delta_x(\{Hx\})| \le 1/2 + \varepsilon$ *for all* $x \in U$. *If* y_o *is compound in* Y *and* $\varepsilon > 0$, *there is a neighborhood* V *of* y_o *such that* $1 \le |\varphi^*\delta_y(\{H^{-1}y\})| \le 1 + \varepsilon$ *for all* $y \in V$.

Proof. Let $\varepsilon > 0$ be given and y_o compound in Y. We know $1 \le |\varphi^*\delta_y(\{H^{-1}y\})|$ for all $y \in Y$. What if every open neighborhood V of y_o contains an element y for which $|\varphi^*\delta_y(\{H^{-1}y\})| > 1 + \varepsilon$? Then there is a net y_α in Y such that $y_\alpha \to y_o$ and $|\varphi^*\delta_{y_\alpha}(\{H^{-1}y_\alpha\})| > 1 + \varepsilon$ for all α. By Lemma 5.9, $x_\alpha \equiv H^{-1}y_\alpha \to \{x^e, x^o\}$ where $y^o \backsim x^e, x^o$, and by passing to a subnet, we can suppose x_α converges to one of x^e, x^o, say x^e.

Let W be an open neighborhood of x^o whose closure does not contain x^e and $h : X \to [0,1]$ continuous with $h(x^o) = 1$ and $h = 0$ on $X\setminus W$. Let γ_α denote $\varphi^*\delta_{y_\alpha}$ restricted to $X\setminus\{x_\alpha\}$. Then for large α, $x_\alpha \notin \mathrm{cl}\,W$ so $|\gamma_\alpha|(W) = |\varphi^*\delta_{y_\alpha}|(W) \ge |\varphi^*\delta_{y_\alpha}|(h)$ and since $\lim_\alpha |\varphi^*\delta_{y_\alpha}(h)| = |\varphi^*\delta_{y_o}(h)| = |\pm h(x^e) \pm h(x^o)| = |h(x^o)| = 1$, we have $|\gamma_\alpha| \ge 1$ for all large α. But then $2 \ge |\varphi^*\delta_{y_\alpha}| = |\varphi^*\delta_{y_\alpha}(\{x_\alpha\})| + |\gamma_\alpha| > (1+\varepsilon) + 1 > 2$, impossible. The proof for x_o is similar.

LEMMA 5.11. *There is an* $\varepsilon > 0$ *such that if* $H(x_1) = y_1$ *with*

(a) $1 < |\varphi^*\delta_{y_1}(\{x_1\})| \le 1 + \varepsilon$ *and*
(b) $1/2 \le |\psi^*\delta_{x_1}(\{y_1\})| \le 1/2 + \varepsilon$,

then there exist unique x_2 *and* y_2 *such that* $x_2 \ne x_1, y_2 \ne y_1$, *and*

(c) $1 \ge |\varphi^*\delta_{y_2}(\{x_1\})| \ge c$, (f) $1/2 \ge |\psi^*\delta_{x_2}(\{y_1\})| \ge c/2$,
(d) $1/2 \ge |\psi^*\delta_{x_1}(\{y_2\}) \ge c/2$, (g) $1 \le |\varphi^*\delta_{y_2}(\{x_2\})| \le 2 - c$,
(e) $1 \ge |\varphi^*\delta_{y_1}(\{x_2\})| \ge c$, (h) $1/2 \le |\psi^*\delta_{x_2}(\{y_2\})| \le 1 - c/2$.

Proof. An immediate consequence of Lemma A8 (see Appendix) and the relationships $\varphi^*\delta_y(\{x\}) = \varphi^{**}\chi_{\hat{x}}(\hat{y})$ and $\psi^*\delta_x(\{y\}) = \psi^{**}\chi_{\hat{y}}(\hat{x})$.

We are now ready to prove the above Proposition. Let $\varepsilon > 0$ have the properties of Lemma 5.11. From Lemma 5.10, we obtain the eight open subsets $U(T,s,p)$, each containing the set of compound points $\mathrm{comp}(T,s,p)$, and such that $1/2 \le |\psi^*\delta_x(\{Hx\})| \le 1/2 + \varepsilon$ if $x \in U(X,s,p)$ and $1 \le |\varphi^*\delta_y(\{H^{-1}y\})| \le 1 + \varepsilon$ if $y \in U(Y,s,p)$. Since the sets $\mathrm{comp}(T,s,p)$ are closed and pairwise disjoint, we can and do suppose the eight open sets $U(T,s,p)$ have pairwise disjoint closures. From Example 2, we see that $H[U(X,s,p)]$ may not be open. Nevertheless, setting $U(T,s) = U(T,s,e) \cup U(T,s,o)$, each $H[U(X,s)]$ is open. For $H[U(X,s)]$ is the disjoint union of the closed set $\mathrm{comp}(Y,s) = H[\mathrm{comp}(X,s)]$ and the open set $H[U(X,s)]\setminus\mathrm{comp}(Y,s) = H[U(X,s)\setminus\mathrm{comp}(X,s)]$. If $H[U(X,s)]$ were not open, then some point of $\mathrm{comp}(Y,s)$ would not be in the interior of $H[U(X,s)]$ and consequently a net of simple points $y_\alpha \notin H[U(X,s)]$ would exist such that $y_\alpha \to \mathrm{comp}(Y,s)$. But then $H^{-1}y_\alpha \notin U(X,s)$ and by Lemma 5.9, $H^{-1}y_\alpha \to \mathrm{comp}(X,s)$, impossible.

Set $W(Y,s,p) \equiv U(Y,s,p) \cap H[U(X,s)]$ for each s and p, an open set containing $\mathrm{comp}(Y,s,p)$, and set $W(X,s,p) \equiv U(X,s,p)\cap H^{-1}[W(Y,s)]$, an open set containing $\mathrm{comp}(X,s,p)$. Given $y_1 \in W(Y,s,p)$, set $x_1 = H^{-1}y_1$ and note that both $y_1 \in U(Y,s,p)$ and $x_1 \in U(X,s,e) \cup U(X,s,o)$. Consequently, both $1/2 \le |\psi^*\delta_x(\{y_1\})| \le 1/2+\varepsilon$ and $1 \le |\varphi^*\delta_y(\{x_1\})| \le 1+\varepsilon$. Let x_2 and y_2 be the points given by Lemma 5.11. Then $\{x_1,x_2,y_1,y_2\}$ is a (φ,c)-set and consequently $y_1 \in \mathcal{Y}^{c,s}$. Thus $W(Y,s,p) \subseteq \mathcal{Y}^{c,s}$. Given $x_1 \in W(X,s,p), y_1 \equiv H(x_1)$ is in $W(Y,s)$ so that x_1 belongs to a (φ,c)-set, hence $x_1 \in \mathcal{X}^{c,s}$. Thus $W(X,s,p) \subseteq \mathcal{X}^{c,s}$.

Next, consider the directed set $\Gamma(T,s,p)$ of open sets V in $T = X$ or Y satisfying $\mathrm{comp}(T,s,p) \subseteq V \subseteq W(T,s,p)$. To be definite, set $T = X$, the discussion for Y being similar. If no $V \in \Gamma(X,s,p)$ is a subset of $\mathcal{X}^{c,s}_{\hat p}$ then there is an element $x_V \in V\setminus\mathcal{X}^{c,s}_p = V \cap \mathcal{X}^{c,s}_{\hat p}$ for each V in $\Gamma(X,s,p)$. Then the net x_V converges to $\mathrm{comp}(X,s,p)$ and therefore has a subnet x_α converging to some $x \in \mathrm{comp}(X,s,p)$. Let $\{x^e_\alpha, x^o_\alpha, y^e_\alpha, y^o_\alpha\}$ be the (φ,c)-set to which x_α belongs for each α. Then

$$\bigstar \qquad \text{either} \quad x_\alpha = x^e_\alpha \to x^o = x \quad \text{or} \quad x_\alpha = x^o_\alpha \to x^e = x.$$

Passing to subnets once more, we assume $(x^e_\alpha, x^o_\alpha, y^e_\alpha, y^o_\alpha)$ converges to a member of $X\times X\times Y\times Y$ which, by Lemma 5.8, is a quadruple (x^e,x^o,y^e,y^o) of compound points, incidentally, because x is compound. But then either $x_\alpha = x^e_\alpha \to x^e = x$ or $x_\alpha = x^o_\alpha \to x^o = x$, contradicting $\bigstar$. Thus we have shown that $\mathrm{comp}(X,s,p)$ is interior to $\mathcal{X}^{c,s}_p$ and $\mathrm{comp}(Y,s,p)$ is interior to $\mathcal{Y}^{c,s}_p$.

We will now complete the argument for the necessity of the three conditions of our main theorem. For each sign s and pair of parity values $p,q \in \{e,o\}$, let h_{spq} denote the homeomorphism $\mathcal{X}^{c,s}_p \to \mathcal{Y}^{c,s}_q$ deriving from $\Phi^{c,s}$; i.e. the homeomorphism whose graph is the range of the natural projection of $\Phi^{c,s}$ into $X\times Y$. It is a simple exercise in topology to show the existence of open sets X^s_p in X and Y^s_q in Y satisfying $\mathrm{comp}(X,s,p) \subseteq X^s_p \subseteq \mathcal{X}^{c,s}_p$ and $\mathrm{comp}(Y,s,q) \subseteq Y^s_q$ with $h_{spq}[X^s_q] = Y^s_q$. We next show that these four open sets $X^s_e, X^s_o, Y^s_e, Y^s_o$ are bridges. Their boundaries ∂X^s_p and ∂Y^s_p are closed sets disjoint from them which we partition into two closed subsets:

$$F(X^s_p) \equiv \{x \in \partial X^s_p : H(x) \in \mathcal{Y}^s_p\}, \qquad S(X^s_p) \equiv \{x \in \partial X^s_p : H(x) \in \mathcal{Y}^s_{\hat p}\},$$
$$F(Y^s_p) \equiv \{y \in \partial Y^s_p : H^{-1}(y) \in \mathcal{X}^s_p\}, \qquad S(Y^s_p) \equiv \{y \in \partial Y^s_p : H^{-1}(y) \in \mathcal{X}^s_{\hat p}\}.$$

The definition makes sense because if $x \in \mathcal{X}^{c,s}_p$ for instance, then $x \in \{x^e, x^o\}$ for some $(x^e,x^o,y^e,y^o) \in \Phi^{c,s}$ and $H(x)$ is one of y^e, y^o and therefore $H(x) \in \mathcal{Y}^s_e \cup \mathcal{Y}^s_o$. Since $\mathcal{X}^{c,s}$ is closed and contains X^s_p, $\partial X^s_p \subseteq \mathcal{X}^{c,s}$. Hence ∂X^s_p is the disjoint union of $F(X^s_p)$ and $S(X^s_p)$. Since H is continuous on $X\setminus\mathrm{comp}(X)$, $F(X^s_p) = \partial X^s_p \cap H^{-1}[\mathcal{Y}^s_p\setminus Y^s_p]$ and $S(X^s_p) = \partial X^s_p \cap H^{-1}[\mathcal{Y}^s_{\hat p}\setminus Y^s_{\hat p}]$ are closed as well. We have $h_{spq}[\mathrm{cl}\, X^s_p] = \mathrm{cl}\, h_{spq}[X^s_p] = \mathrm{cl}\, Y^s_p$ and by their construction from $\Phi^{c,s}$, the homeomorphisms h_{spq} are compatible as required by condition (3) of the main theorem. Also, if $x \in F(X^s_p)$ then $x = x^p$ where $p = e$ or o and $(x^e,x^o,y^e,y^o) \in \Phi^{c,s}$. Then $H(x) = H(x^p) = y^p = h_{spp}(x)$. And if $x \in S(X^s_p)$, then $H(x) = H(x^p) = y^{\hat p} = h_{sp\hat p}(x)$. Thus condition (3) holds.

For the second assertion of (2), let $x \in F(X^s_p)$ so that $H(x) = h_{spp}(x)$. Since h_{spp} is a homeomorphism carrying $\mathcal{X}^s_p$ onto $\mathcal{Y}^s_p$ and X^s_p onto Y^s_p, we have $H(x) \in \partial Y^s_p$, and of course $H^{-1}(Hx) = x \in \mathcal{X}^s_p$. Hence, by its definition, $F(Y^s_p)$ contains the element $H(x)$. Thus $H[F(X^s_p)] \subseteq F(Y^s_p)$ and, since a similar argument yields $H^{-1}[F(Y^s_p)] \subseteq F(X^s_p)$, we conclude $H[F(X^s_p)] = F(Y^s_p)$. Similarly, $H[S(X^s_p)] = S(Y^s_p)$.

For each T, let $W(T)$ denote the union of the four sets T^s_p, and set

$$X^+ = \{x \in X\setminus W(X) : \psi^*\delta_x(\{Hx\}) \ge 1/2\},$$
$$X^- = \{x \in X\setminus W(X) : \psi^*\delta_x(\{Hx\}) \le -1/2\},$$
$$Y^+ = \{y \in Y\setminus W(Y) : \varphi^*\delta_y(\{H^{-1}y\}) \ge 1\},$$
$$Y^- = \{y \in Y\setminus W(Y) : \varphi^*\delta_y(\{H^{-1}y\}) \le -1\}.$$

Evidently, each T is the disjoint union of the four open sets T^s_p and the two sets T^s, hence $T^+ \cup T^-$ is closed.

LEMMA 5.12. *T^s is closed and $H[X^s] = Y^s$.*

Proof. We give the argument for X^+. Suppose $x_\alpha \in X^+$ and $x_\alpha \to x$. Then $x \in X^+ \cup X^-$. What if $x \in X^-$?! Let $0 < \varepsilon < 1/4$. By regularity of $\psi^*\delta_x$, let V be an open neighborhood of Hx such that $|\psi^*\delta_x|(V\setminus\{Hx\}) < \varepsilon$ and W a closed neighborhood of Hx contained in V. There is an $f \in C(Y)$ with values in $[0,1]$ that is 1 on W and 0 on $Y\setminus V$. Then $|\psi^*\delta_{x_\alpha}| \le 1$ and $\psi^*\delta_{x_\alpha}(\{Hx_\alpha\}) \ge 1/2$ for all α, so

$$\int_{V\setminus\{Hx_\alpha\}} f\, d(\psi^*\delta_{x_\alpha}) \ge -1/2;$$

hence $\psi^*\delta_{x_\alpha}(f) \ge 0$ for all α. From this,

$$0 \le \lim_\alpha \psi^*\delta_{x_\alpha}(f) = \psi^*\delta_x(f) = \int_{V\setminus\{Hx\}} f\, d(\psi^*\delta_x) + \psi^*\delta_x(\{Hx\})$$
$$\le \varepsilon - 1/2 < 1/4 - 1/2 < 0,$$

impossible, so $x \in X^+$ as desired.

From Lemma A7, $\psi^*\delta_x(\{Hx\})$ and $\varphi^*\delta_{Hx}(\{x\})$ have the same sign. Also, $H[W(X)] = W(Y)$ so $H[X^+ \cup X^-] = Y^+ \cup Y^-$. But $H[X^+] \subseteq Y^+$ since if $x \in X^+$, then $\psi^*\delta_x(\{Hx\}) \ge 1/2$ and so $\varphi^*\delta_{Hx}(\{H^{-1}(Hx)\}) = \varphi^*\delta_{Hx}(\{x\}) \ge 1$. Similarly, $H[X^-] \subseteq Y^-$. Then $H[X^s] = Y^s$ for each s.

It remains to prove that for each T and s, T^s_e joins T^s to itself and T^s_o joins $T^{\hat{s}}$ to T^s; that is, $F(T^s_e), S(T^s_e), S(T^s_o) \subseteq T^s$ and $F(T^s_o) \subseteq T^{\hat{s}}$. We will demonstrate only one of these eight inclusions since the arguments are similar. To prove $F(X^+_o) \subseteq X^-$, for instance, let $x \in F(X^+_o)$. Then $x \in \partial X^+_o$ and $Hx \in \mathcal{Y}^+_o$. The four sets X^s_p are pairwise disjoint and open so $x \in \partial X^+_o$ gives us $x \in X\backslash W(X)$, indeed, $x \in \mathcal{X}^{c,+}_o\backslash W(X)$. To show $\psi^*\delta_x(\{Hx\}) \le -1/2$, let (x^e, x^o, y^e, y^o) be the quadruple in $\Phi^{c,+}$ for which $x = x^o$. Since $Hx \in \mathcal{Y}^+_o$, $Hx = y^o$. Then $\psi^*\delta_x(\{Hx\}) = \psi^*\delta_{x^o}(\{y^o\})$ has the minority sign, $-$, of the quadruple, and since $x \backsim y$, $|\psi^*\delta_x(\{Hx\})| \ge 1/2$. Thus $\psi^*\delta_x(\{Hx\}) \le -1/2$ as desired. Thus $x \in X^-$.

Appendix on isolated points. We denote by T_0 the set of isolated points of a topological space T.

LEMMA A1. *Let T be compact Hausdorff and kT denote the compact Hausdorff space for which $C(kT)$ is linearly isometric with $C(T)^{**}$. Then there is a 1-1 function $t \to \hat{t}$ from T onto $(kT)_0$ for which*

(i) *given $t \in T$, the action of $\chi_{\hat{t}} \in C(kT)$ upon a given $\eta \in M(T)$ is $\eta(\{t\})$,*

(ii) *given $h \in C(kT)$ and $t \in T$, the action of h upon $\delta_t \in M(T)$ is $h(\hat{t})$.*

Proof. These results can be deduced from the construction of H. Gordon [14] or the Kakutani abstract L-space construction [17] applied to $M(T)$. Moreover, Cambern and Greim [8], [9] have developed these representations and extended them to spaces of vector-valued measures.

Hence if $\varphi : C(X) \to C(Y)$ is continuous and linear and if $x \in X$ and $y \in Y$, then $\varphi^*\delta_y(\{x\}) = \varphi^{**}\chi_{\hat{x}}(\hat{y})$, by (i) and (ii) above. For convenient reference, we record in Lemma A2 the results of Lemmas 1 and 2 of [11] specialized to bound-2 isomorphisms.

LEMMA A2. (1) *Suppose $\varphi : C(X) \to C(Y)$ is an onto isomorphism with $\|\varphi^{-1}\| = 1$ and $\|\varphi\| = 2$. Let $y \in Y_0$ and $F \equiv \varphi^{-1}(\chi_y)/\|\varphi^{-1}(\chi_y)\|$ and let $2/3 \le \Theta \le 1$. Suppose $f, g \in C(X)$ such that $F = f + g$ and $\Theta|\mu| \le \|f + \mu g\| \le |\mu|$ if $|\mu| \ge 1$. Then $\varphi(g)(y) \ge 3 - 2/\Theta$.*

(2) *If $\gamma : C(X) \to C(Y)$ is bound-2 and onto, $y \in Y_0$, and $2/3 < \Theta \le 1$, then $X_0(y, \Theta) \equiv \{x \in X : |(\gamma^{-1}\chi_y)(x)| \ge \|\gamma^{-1}\chi_y\|\Theta\}$ consists of m isolated points where $1 \le m \le 2/(3 - 2/\Theta)$.*

Moreover, if $x \in X_0(y, \Theta)$, then $|(\gamma\chi_x)(y)| \ge (\|\gamma\|/2)(3 - 2/\Theta)$. By changing the roles of X and Y, we have similar results for $Y_0(x, \Theta)$.

In the following Lemmas A3 and A5, and their corollaries, we refine our approach to Lemma 5 of [11] in order to repair the proof given there. In the sequel, $\gamma : C(X) \to C(Y)$ is always an onto bound-2 isomorphism. Given $y \in Y_0$, $X_0(y)$ denotes $X_0(y, 1)$, a single isolated point or a pair by Lemma A2. For $x \in X_0$ and $y \in Y_0$, let

$$\overline{\gamma\chi}^y_x = (\gamma\chi_x)\chi_{Y\backslash\{y\}} = \gamma\chi_x - (\gamma\chi_x)(y)\chi_y$$

so that $\gamma\chi_x = (\gamma\chi_x)(y)\chi_y + \overline{\gamma\chi}^y_x$, $\overline{\gamma\chi}^y_x(y) = 0$ and $\|\gamma\chi_x\| = \max(|(\gamma\chi_x)(y)|, \|\overline{\gamma\chi}^y_x\|)$. Similarly, $\|\gamma^{-1}\chi_y\| = \max(|(\gamma^{-1}\chi_y)(x)|, \|\overline{\gamma^{-1}\chi_y}^x\|)$.

LEMMA A3. *If $y \in Y_0$ and $x \in X_0(y)$, then*

$$\|\overline{\gamma\chi}^y_x\| \le \frac{\|\gamma\|}{1 + \|\gamma^{-1}\chi_y\|/\|\overline{\gamma^{-1}\chi_y}^x\|}.$$

Proof. If $\|\overline{\gamma^{-1}\chi_y}^x\| = 0$, the right side of the inequality is 0 and $\gamma^{-1}\chi_y = (\gamma^{-1}\chi_y)(x)\chi_x$; hence $\chi_y = (\gamma^{-1}\chi_y)(x)\gamma\chi_x$, from which $\overline{\gamma\chi}^y_x = 0$.

Now suppose $\|\overline{\gamma^{-1}\chi_y}^x\| > 0$, and let $\nu = \|\gamma^{-1}\chi_y\|/\|\overline{\gamma^{-1}\chi_y}^x\|$. Let $w \in Y$, $w \ne y$. Then

$$0 = \chi_y(w) = \gamma[(\gamma^{-1}\chi_y)(x)\chi_x + \overline{\gamma^{-1}\chi_y}^x](w)$$

implies

$$|\gamma^{-1}\chi_y(x)| \cdot |(\gamma\chi_x)(w)| = |\gamma(\overline{\gamma^{-1}\chi_y}^x)(w)|.$$

Using this and one of $\alpha = \pm 1$ gives us

$$\begin{aligned}(1 + \nu)|(\gamma^{-1}\chi_y)(x)| \cdot |(\gamma\chi_x)(w)| &= |(\gamma^{-1}\chi_y)(x)| \cdot |(\gamma\chi_x)(w)| \\ &\quad + \nu|\gamma(\overline{\gamma^{-1}\chi_y}^x)(w)| \\ &= |(\gamma^{-1}\chi_y)(x)(\gamma\chi_x)(w) + \alpha\nu\gamma(\overline{\gamma^{-1}\chi_y}^x)(w)| \\ &\le \|\gamma\| \cdot \|(\gamma^{-1}\chi_y)(x)\chi_x + \alpha\nu\overline{\gamma^{-1}\chi_y}^x\| \\ &= \|\gamma\| \cdot \|\gamma^{-1}\chi_y\|.\end{aligned}$$

Strike $|(\gamma^{-1}\chi_y)(x)| = \|\gamma^{-1}\chi_y\|$ from the inequality to obtain $|(\gamma\chi_x)(w)| \le \|\gamma\|/(1 + \nu)$.

COROLLARY A4. *If $y \in Y_0$ and $x \in X_0(y)$, then $\|\overline{\gamma\chi_x}^y\| \le \|\gamma\|/2 \le \|(\gamma\chi_x)\| = |(\gamma\chi_x)(y)|$. Consequently, $y \in Y_0(x)$. Indeed, for isolated x and y, $x \in X_0(y)$ iff $y \in Y_0(x)$.*

Proof. Lemma A3 yields the first inequality since $\|\gamma^{-1}\chi_y\| \ge \|\overline{\gamma^{-1}\chi_y}^x\|$. The middle inequality is just

$$1 = \|\chi_x\| = \|\gamma^{-1}(\gamma\chi_x)\| \le \|\gamma^{-1}\| \cdot \|\gamma\chi_x\| = \frac{2}{\|\gamma\|}\|\gamma\chi_x\|.$$

Lemma A2 implies $|(\gamma\chi_x)(y)| \ge \|\gamma\|/2$ so the last equality follows from the first inequality.

DEFINITION. We say $x \in X_0$ is *simple* if $Y_0(x)$ is a singleton, *compound* if $Y_0(x)$ is a doubleton. These are the only possibilities according to Lemma A2.

LEMMA A5. *Suppose $y \in Y_0$ and $x \in X_0(y)$.*

(a) *These are equivalent: y is simple; $\|\overline{\gamma\chi_x}^y\| < \|\gamma\|/2$; x is simple; $\|\overline{\gamma^{-1}\chi_y}^x\| < \|\gamma^{-1}\|/2$.*

(b) *These are also equivalent: y is compound; $\|\overline{\gamma\chi_x}^y\| = \|\gamma\|/2$; x is compound; $\|\overline{\gamma^{-1}\chi_y}^x\| = \|\gamma^{-1}\|/2$, and when they hold, then $|(\gamma\chi_x)(y)| = \|\gamma\|/2$ and $|(\gamma^{-1}\chi_y)(x)| = \|\gamma^{-1}\|/2$.*

Proof. (a) If y is simple, this means $\|\overline{\gamma^{-1}\chi_y}^x\| < \|\gamma^{-1}\chi_y\|$, which, by Lemma A3, yields $\|\overline{\gamma\chi_x}^y\| < \|\gamma\|/2$, hence $\|\overline{\gamma\chi_x}^y\| < \|\gamma\chi_x\|$ by Corollary A4, which means x is simple. Similarly, x simple yields $\|\overline{\gamma\chi_x}^y\| < \|\gamma\chi_x\|$, whence $\|\overline{\gamma^{-1}\chi_y}^x\| < \|\gamma^{-1}\|/2$ by Lemma A3, so $\|\overline{\gamma^{-1}\chi_y}^x\| < \|\gamma^{-1}\chi_y\|$ by Corollary A4, hence y simple.

The string of equivalences in (b) is a simple consequence of (a). Assuming these, $X_0(y) = \{x_1, x_2\}$ where $x = x_1 \neq x_2$. By Corollary A4,

$$\|\gamma\|/2 \le \|\gamma\chi_{x_i}\| = |(\gamma\chi_{x_i})(y)| = |(\gamma^*\delta_y)(\{x_i\})| \quad (i = 1, 2);$$

hence, $\|\gamma\| \le |(\gamma^*\delta_y)(\{x_1\})| + |(\gamma^*\delta_y)(\{x_2\})| \le \|\gamma\|$. Therefore $|(\gamma\chi_{x_i})(y)| = \|\gamma\|/2$ $(i = 1, 2)$ and, similarly, $|(\gamma^{-1}\chi_y)(x)| = \|\gamma^{-1}\|/2$.

COROLLARY A6. *If $x \in X_0$ and $|(\gamma\chi_x)(y)| \ge \|\gamma\|/2$, then $y \in Y_0(x)$.*

Proof. If x is simple, then $|\gamma\chi_x| \ge \|\gamma\|/2$ at a unique point t in Y and $Y_0(x) = \{t\}$. By uniqueness, $t = y$. If x is compound, by Corollary A4 and Lemma A5, $\|\gamma\chi_x\| = \|\gamma\|/2$, hence $|(\gamma\chi_x)(y)| = \|\gamma\chi_x\|$; i.e., $y \in Y_0(x)$.

In what follows, $\varphi : C(X) \to C(Y)$ is a bound-2 isomorphism with $\|\varphi\| = 2$ and $\psi = \varphi^{-1}$.

LEMMA A7. *Suppose $y \in Y_0$, $\|\psi\chi_y\| = |(\psi\chi_y)(x_1)|$, and suppose the second largest value r of $|\psi\chi_y|$ is such that $\Theta \equiv r/\|\psi\chi_y\| > 2/3$ and is assumed at $x_2 \neq x_1$. Then $(\psi\chi_y)(x_1)(\varphi\chi_{x_1})(y) \ge 1/2$ and $(\psi\chi_y)(x_2)(\varphi\chi_2)(y) > 0$.*

Proof. First, set $F \equiv \psi\chi_y/\|\psi\chi_y\|$, $g \equiv F(x_1)\chi_{x_1}$, and $f \equiv F - g = F\chi_{X\setminus\{x_1\}}$ so that $1 = \|F\| = \|g\| \ge \|f\|$. Then $\|f + \mu g\| = |\mu|$ whenever $|\mu| \ge 1$. By Lemma A2, with $\Theta = 1$,

$$1 = 3 - \frac{2}{\Theta} \le \varphi(g)(y) = \varphi(F(x_1)\chi_{x_1})(y) = \frac{(\psi\chi_y)(x_1)}{\|\psi\chi_y\|}(\varphi\chi_{x_1})(y),$$

hence $(\psi\chi_y)(x_1)(\varphi\chi_{x_1})(y) \ge \|\psi\chi_y\| \ge 1/2$. Secondly, set $g' \equiv F(x_2)\chi_{x_2}$ and $f' \equiv F - g' = F\chi_{X\setminus\{x_2\}}$ so $\|f'\| = \|F\| = 1$ and $\|g'\| = |F(x_2)| \ge r/\|\psi\chi_y\| = \Theta$. For $|\mu| \ge 1$ we calculate $|\mu|\Theta \le \|f' + \mu g'\| \le |\mu|$ and hence, by Lemma A2,

$$\frac{(\psi\chi_y)(x_2)}{\|\psi\chi_y\|}(\varphi\chi_{x_2})(y) \ge 3 - \frac{2}{\Theta} > 0.$$

LEMMA A8. *Let $\|\varphi\| = 2$, $\|\varphi^{-1}\| = 1$, and $3/4 < \sigma < 1$. Then there is an $\varepsilon > 0$ such that if $x_1 \in X_0(y_1)$ and $y_1 \in Y_0(x_1)$ are simple with*

(a) $1 \le |(\varphi\chi_{x_1})(y_1)| \le 1 + \varepsilon$ *and*
(b) $1/2 \le |(\psi\chi_{y_1})(x_1)| \le 1/2 + \varepsilon$,

then there exist unique isolated points x_2 and y_2 such that $x_2 \neq x_1$, $y_2 \neq y_1$ and

(c) $1 \ge \|(\varphi\chi_{x_1})(y_2)\| \ge \sigma$,	(f) $1/2 \ge \|(\psi\chi_{y_1})(x_2)\| \ge \sigma/2$,
(d) $1/2 \ge \|(\psi\chi_{y_2})(x_1)\| \ge \sigma/2$,	(g) $1 \le \|(\varphi\chi_{x_2})(y_2)\| \le 2 - \sigma$,
(e) $1 \ge \|(\varphi\chi_{x_2})(y_1)\| \ge \sigma$,	(h) $1/2 \le \|(\psi\chi_{y_2})(x_2)\| \le 1 - \sigma/2$.

Proof. If y_2 and z_2 exist in Y with property (d) and $y_1, y_2 \neq z_2$, then

$$1 \ge |\psi^*\delta_{x_1}| \ge |(\psi^*\delta_{x_1})(\{y_1\})| + |(\psi^*\delta_{x_1})(\{y_2\})| + |(\psi^*\delta_{x_1})(\{z_2\})|$$
$$= |(\psi\chi_{y_1})(x_1)| + |(\psi\chi_{y_2})(x_1)| + |(\psi\chi_{z_2})(x_1)| \ge \frac{1}{2} + \frac{\sigma}{2} + \frac{\sigma}{2} = \frac{1}{2} + \sigma > 1,$$

impossible. Thus y_2 is unique and similarly x_2 is unique.

For existence, first fix $2/3 < \sigma < 1$ and pick ε such that both

$$0 < \varepsilon < \frac{3\sigma}{2} - 1 \quad\text{and}\quad \frac{-2}{1+2\varepsilon} + \frac{1+\varepsilon}{1-\sigma} < -1 + \frac{1}{1-\sigma}.$$

We now produce $y_2 \neq y_1$ satisfying (c). Set $M = (\psi\chi_{y_1})(x_1)$ and $N = (\varphi\chi_{x_1})(y_1)$ and let F, f, g be as in Lemma A7, which implies $MN \ge 1/2$. Setting $\mu = 1 - 1/(1-\sigma)$, simple calculations yield $\mu < -1 \le 1 - 1/(MN)$, hence $1 + (\mu - 1)MN \le 0$. There exists $y_2 \in Y$ such that $|\varphi(f + \mu g)(y_2)| = \|\varphi(f + \mu g)\| \ge \|f + \mu g\| = -\mu$. What if $y_2 = y_1$?! Then

$$\begin{aligned}-\mu \le |\varphi(f+\mu g)(y_2)| &= |\varphi(F)(y_1) + (\mu - 1)\varphi(g)(y_1)| \\ &= \frac{1}{\|\psi\chi_{y_1}\|}|1 + (\mu - 1)MN| \\ &= \frac{1}{|M|}(-1 + (1-\mu)|M|\cdot|N|) = -\frac{1}{|M|} + (1-\mu)|N| \\ &\le \frac{-2}{1+2\varepsilon} + \frac{1+\varepsilon}{1-\sigma} < -1 + \frac{1}{1-\sigma} = -\mu,\end{aligned}$$

impossible. So, $y_2 \neq y_1$, whence

$$-\mu \le |\varphi(F)(y_2) + (\mu - 1)\varphi(g)(y_2)| = (1-\mu)|(\varphi\chi_{x_1})(y_2)|,$$

from which $|(\varphi\chi_{x_1})(y_2)| \ge \sigma$. To see that y_2 is isolated, note that $\sigma/(1+\varepsilon) > 2/3$ and choose $\sigma/(1+\varepsilon) > \sigma_1 > 2/3$. Then $|(\varphi\chi_{x_1})(y_2)| \ge \sigma > \sigma_1(1+\varepsilon) \ge$

$\sigma_1\|\varphi\chi_{x_1}\|$, from which $y_2 \in Y_0(x_1, \sigma_1)$, a set of isolated points by Lemma A2. Note that $1 > |(\varphi\chi_{x_1})(y_2)|$ because x_1 is simple, by Lemma A5.

We now start over to find y_2 satisfying both (c) and (d). Given $2/3 < \sigma < 1$, choose $\sigma < \sigma_1 < \sigma_2 < 1$ such that $3 - 2/\sigma_1 > \sigma$. By the above paragraph, there is an $\varepsilon > 0$, which we may take so small that $(1+\varepsilon)\sigma_1 < \sigma_2$, such that if x_1, y_1 are simple isolated points satisfying (a) and (b), there is an isolated $y_2 \neq y_1$ for which $1 \geq |(\varphi\chi_{x_1})(y_2)| \geq \sigma_2$. With such ε, y_1, y_2, x_1, we have $|(\varphi\chi_{x_1})(y_2)| > (1+\varepsilon)\sigma_1 \geq \sigma_1|(\varphi\chi_{x_1})(y_1)| = \sigma_1\|\varphi\chi_{x_1}\|$; hence, $y_2 \in Y_0(x_1, \sigma_1)$. By Lemma A2,

$$|(\psi\chi_{y_2})(x_1)| \geq \frac{\|\psi\|}{2}\left(3 - \frac{2}{\sigma_1}\right) = \frac{1}{2}\left(3 - \frac{2}{\sigma_1}\right) > \frac{\sigma}{2}.$$

If $|(\psi\chi_{y_2})(x_1)| \geq 1/2$, then $x_1 \in X_0(y_2)$ so $y_1, y_2 \in Y_0(x_1)$, violating x_1 simple. Thus y_2 satisfies both (c) and (d).

Given $2/3 < \sigma < 1$, there is an $\varepsilon_1 > 0$ such that if x_1 and y_1 are corresponding simple isolated points satisfying (a) and (b), then there exists $y_2 \neq y_1$ such that (c) and (d) hold. For 2ψ and $\frac{1}{2}\varphi$, there is an $\varepsilon_2 > 0$ such that if y_1 and x_1 are corresponding simple isolated points satisfying $1 \leq |(2\psi\chi_{y_1})(x_1)| \leq 1 + \varepsilon_2$ and $1/2 \leq |(\frac{1}{2}\varphi\chi_{x_1})(y_1)| \leq 1/2 + \varepsilon_2$, then there exists $x_2 \neq x_1$ such that $1 > |(2\psi\chi_{y_1})(x_2)| \geq \sigma$ and $1/2 > |(\frac{1}{2}\varphi\chi_{x_2})(y_1)| \geq \sigma/2$. Let $\varepsilon = \min(\varepsilon_1/2, \varepsilon_2/2)$. Let x_1 and y_1 be corresponding (for φ and ψ) simple isolated points satisfying (a) and (b):

$$(\star)\qquad 1 \leq |(\varphi\chi_{x_1})(y_1)| \leq 1 + \varepsilon \quad\text{and}\quad 1/2 \leq |(\psi\chi_{y_1})(x_1)| \leq 1/2 + \varepsilon.$$

Then $y_2 \neq y_1$ isolated exists satisfying (c) and (d). From $(\star)$, $1 \leq |(2\psi\chi_{y_1})(x_1)| \leq 1 + 2\varepsilon \leq 1 + \varepsilon_2$ and $\frac{1}{2}|(\frac{1}{2}\varphi\chi_{x_1})(y_1)| \leq 1/2 + \varepsilon/2 \leq 1/2 + \varepsilon_2$. This implies that x_1 and y_1 are corresponding isolated points for the maps 2ψ and $\frac{1}{2}\varphi$, and if they were compound for 2ψ and $\frac{1}{2}\varphi$ they would be compound for φ and ψ, a contradiction. So they are corresponding simple points for 2ψ and $\frac{1}{2}\varphi$. By definition of ε_2, there exists $x_2 \neq x_1$ such that $1 > |(2\psi\chi_{y_1})(x_2)| \geq \sigma$ and $1/2 > |(\frac{1}{2}\varphi\chi_{x_2})(y_1)| \geq \sigma/2$. Then we have (f) $1/2 > |(\psi\chi_{y_1})(x_2)| \geq \sigma/2$ and (e) $1 > |(\varphi\chi_{x_2})(y_1)| \geq \sigma$.

To complete the argument, suppose $3/4 < \sigma < 1$ and $\varepsilon > 0$ has the property that if x_1 and y_1 are corresponding simple isolated points satisfying (a) and (b) for φ and ψ, there exist isolated $x_2 \neq x_1$ and $y_2 \neq y_1$ satisfying (c)–(f). We will show that (g) and (h) also hold. Given such x_1, x_2, y_1, y_2, it suffices to prove $|(\varphi\chi_{x_2})(y_2)| \geq 1$. For then $y_2 \in Y_0(x_2)$, so $x_2 \in X_0(y_2)$, from which $|(\psi\chi_{y_2})(x_2)| \geq 1/2$. Moreover,

$$\begin{aligned}|(\varphi\chi_{x_2})(y_2)| + \sigma &\leq |(\varphi\chi_{x_2})(y_2)| + |(\varphi\chi_{x_1})(y_2)| \\ &= |(\varphi^*\delta_{y_2})(\{x_2\})| + |(\varphi^*\delta_{y_2})(\{x_1\})| \leq |\varphi^*\delta_{y_2}| \leq 2,\end{aligned}$$

whence $|(\varphi\chi_{x_2})(y_2)| \leq 2 - \sigma$ and, similarly, $|(\psi\chi_{y_2})(x_2)| \leq 1 - \sigma/2$.

What, then, if $|(\varphi\chi_{x_2})(y_2)| < 1$?! Then $y_2 \notin Y_0(x_2)$ so $x_2 \notin X_0(y_2)$, and since $|(\psi\chi_{y_2})(x_1)| < 1/2$, $x_1 \notin X_0(y_2)$. Then there is an $x_3 \in X_0(y_2)$ and x_1, x_2, and x_3 are distinct. We obtain

$$\begin{aligned}|(\varphi\chi_{x_2})(y_2)| &= |(\varphi^*\delta_{y_2})(\{x_2\})| \leq 2 - |(\varphi^*\delta_{y_2})(\{x_3\})| - |(\varphi^*\delta_{y_2})(\{x_1\})| \\ &\leq 2 - 1 - \sigma = 1 - \sigma < 3\sigma - 2\end{aligned}$$

because $\sigma > 3/4$. We will reach the contradiction $3\sigma - 2 \leq |(\varphi\chi_{x_2})(y_2)|$. To this end, define $h \in C(Y)$ by

$$h \equiv (\varphi\chi_{x_2})\chi_{Y\setminus\{y_1,y_2\}} = \varphi\chi_{x_2} - (\varphi\chi_{x_2})(y_2)\chi_{y_2} - (\varphi\chi_{x_2})(y_1)\chi_{y_1}.$$

For suitable $\alpha, \beta = \pm 1$,

$$\begin{aligned}2 &\geq \|2\alpha\chi_{y_1} + 2\beta\chi_{y_2} + h\| \geq \|\psi(2\alpha\chi_{y_1} + 2\beta\chi_{y_2} + h)\| \\ &\geq |2\alpha(\psi\chi_{y_1})(x_1) + 2\beta(\psi\chi_{y_2})(x_1) + \psi(h)(x_1)| \\ &= 2[|(\psi\chi_{y_1})(x_1)| + |(\psi\chi_{y_2})(x_1)|] + |\psi(h)(x_1)| \\ &\geq 2[1/2 + \sigma/2] + |\psi(h)(x_1)|;\end{aligned}$$

thus

$$(\star)\qquad |\psi(h)(x_1)| \leq 1 - \sigma.$$

Applying ψ to the defining equation of h, evaluating at x_1, and using $(\star)$ we obtain

$$\begin{aligned}|(\varphi\chi_{x_2})(y_2)| &\geq \frac{1}{|(\psi\chi_{y_2})(x_1)|}[|(\varphi\chi_{x_2})(y_1)||(\psi\chi_{y_1})(x_1)| - |\psi(h)(x_1)|] \\ &\geq 2[\sigma(1/2) - |\psi(h)(x_2)|] = \sigma - 2|\psi(h)(x_1)| \\ &\geq \sigma - 2(1 - \sigma) = 3\sigma - 2,\end{aligned}$$

which completes the argument.

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Asymptotic expansion of solutions of Laplace–Beltrami type singular operators

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Abstract. The theory of Mellin analytic functionals with unbounded carrier is developed. The generalized Mellin transform for such functionals is defined and applied to solve the Laplace–Beltrami type singular equations on a hyperbolic space. Then the asymptotic expansion of solutions is found.

0. Introduction. This paper may be regarded as a sequel and correction of [1], and we use similar notations.

In Section 2 we define directly the space of Mellin analytic functionals with not necessarily bounded carrier, without using the notion of Fourier analytic functionals.

Section 3 contains the definition of the generalized Mellin transform of a Mellin analytic functional as some Fourier analytic functional. It is shown in Theorem 2 that if the carrier of a Mellin analytic functional is compact then its generalized Mellin transform is the boundary value (in some special sense) of its ordinary Mellin transform.

In the next section we prove two Paley–Wiener type theorems for the Mellin transform of Mellin analytic functionals. In the proof of Lemma 5 we use estimates similar to those used in the proof of Theorem 3 in [1] but in a corrected form.

In Section 6 we apply the theory of the Mellin transform of Mellin analytic functionals to solve the equation $Pu = f$, where P is a Laplace–Beltrami type operator. We find a solution in the space of Mellin analytic functionals, by a method similar to that used in Section 7 in [1]. The estimate of F_j in [1], p. 274, is incorrect, because the "constant" A is not constant (depends on $\operatorname{Re} z_2$).

Here we find a correct but slightly worse estimate; thus the conclusion on the Laplace–Beltrami operator in Section 8 of [1] is not true. The main result

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