to know whether such an operator A has the property similar to the one stated in Theorem 3.3.

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The dual of Besov spaces on fractals

by

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Abstract. For certain classes of fractal subsets F of $\mathbb{R}^n$, the Besov spaces $B_\alpha^{p,q}(F)$ have been studied for $\alpha > 0$ and $1 \le p,q \le \infty$. In this paper the Besov spaces $B_\alpha^{p,q}(F)$ are introduced for $\alpha < 0$, and it is shown that the dual of $B_\alpha^{p,q}(F)$ is $B_{-\alpha}^{p',q'}(F)$, $\alpha \ne 0$, $1 < p,q < \infty$, where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.

1. Introduction and notation. The Besov spaces $B_\alpha^{p,q}(F)$ consisting of functions defined on a fractal subset F of $\mathbb{R}^n$ have been studied for $\alpha > 0$ and $1 \le p,q \le \infty$ by the present authors, e.g. in [5]. In this paper we introduce Besov spaces $B_\alpha^{p,q}(F)$ for negative indices α, and show that the dual of $B_\alpha^{p,q}(F)$, $\alpha > 0$, $1 \le p,q < \infty$ or $\alpha < 0$, $1 < p,q < \infty$, is $B_{-\alpha}^{p',q'}(F)$, where p' and q' are the dual indices of p and q. This is well known if $F = \mathbb{R}^n$ (see e.g. [6, p. 178]). Duality theory involving the Lipschitz spaces $B_\alpha^{\infty,\infty}(F)$, $\alpha > 0$, where F is a fractal set (and in fact an arbitrary closed set if $\alpha < 1$) was given in [3].

Our definition of $B_\alpha^{p,q}(F), \alpha < 0$, is in terms of atomic decompositions and is inspired by the atomic decomposition of distributions in $B_\alpha^{p,q}(\mathbb{R}^n)$ given in [1]. If $F = \mathbb{R}^n$ and $\alpha < 0$, our decomposition reduces to one given in [1], except that we use atoms normed in L^p rather than smooth atoms normed in L^∞. Atomic decompositions of functions in $B_\alpha^{p,q}(F), \alpha > 0$, F a fractal set, were given in [4].

For the definition of $B_\alpha^{p,q}(F)$ we refer to Section 2 ($\alpha > 0$) and Section 3 ($\alpha < 0$). The duality results are given in Sections 4 and 5. In particular, Theorems 4.3 and 5.1 give precise statements describing the duality. Throughout the paper, the assumption on F is that F is a d-set preserving Markov's inequality. We now define these concepts, referring to [5, Chapter II] for the general theory.

Let F be a closed subset of $\mathbb{R}^n$ and $0 < d \le n$, and denote by $B(x,r)$ the closed ball with center x and radius r. A positive Borel measure μ with

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support F is called a *d-measure* on F if, for some constants $c_1, c_2 > 0$,

$$c_1 r^d \le \mu(B(x,r)) \le c_2 r^d, \quad x \in F,\ 0 < r \le 1.$$

The set F is a *d-set* if there exists a d-measure on F. Any two d-measures on F are equivalent [5, p. 30]. A closed set $F \subset \mathbb{R}^n$ (not necessarily a d-set) *preserves Markov's inequality* if the following holds for all positive integers k: For all polynomials P in n variables of degree $\le k$ and all balls $B(x,r)$, $x \in F$, $0 < r \le 1$, we have

$$\max_{F\cap B} |\operatorname{grad} P| \le c r^{-1} \max_{F\cap B} |P|$$

with a constant c depending on F, n, and k only. Any d-set with $d > n-1$ preserves Markov's inequality. Examples of d-sets preserving Markov's inequality are classes of self-similar fractals [7], the closure of Lipschitz domains and, more generally, (ε,δ)-domains.

In Theorems 4.3 and 5.1 as well as in the propositions and lemmas we get constants depending only on n, F, μ, and α. However, the d-measure μ can be chosen in a canonical way as the restriction to F of the d-dimensional Hausdorff measure [5, p. 32], and the dimension n of $\mathbb{R}^n$ depends uniquely on F when F preserves Markov's inequality [7, Section 3]. Because of this we can make the constants depend only on F and α.

As already mentioned, we assume throughout the paper that F is a d-set preserving Markov's inequality. Whenever we deal with the Besov spaces $B_\alpha^{p,q}(F)$ with $|\alpha| < 1$, the results remain true, however, without the assumption that F preserves Markov's inequality.

NOTATION.

F	a closed subset of $\mathbb{R}^n$ which is a d-set preserving Markov's inequality, $0 < d \le n$
μ	a fixed d-measure on F
$[\alpha]$	the integer part of α
$f, g, \ldots$	real-valued functions
$\Vert f\Vert_p$	the $L^p(\mu)$-norm of f
$\Vert f\Vert_{p,E}$	the $L^p(\mu, E)$-norm of f
$A, B, \ldots$	real-valued sequences $A = (A_\nu)_{\nu=0}^\infty$, $B = (B_\nu)_{\nu=0}^\infty, \ldots$
$\Vert A\Vert_q$	the ℓ^q-norm of A
supp f	the support of f
Q	halfopen cubes of the form $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : a_i \le x_i < a_i + r,\ i = 1, \ldots, n\}$
$2Q$	the cube obtained by expanding Q twice around its center
$B(x,r)$	the closed ball with center x and radius r
$\mathcal{N}$	a net with mesh r, i.e. a division of $\mathbb{R}^n$ into equally big cubes Q with edges of length r, obtained by intersecting $\mathbb{R}^n$ with hyperplanes orthogonal to the axes
$\mathcal{N}_\nu$	the net with mesh $2^{-\nu}$ such that the origin is a corner of some cube in the net
$\mathcal{N}_\nu(F)$	$\{Q \in \mathcal{N}_\nu : Q \cap F \ne \emptyset\}$
$\mathcal{P}_k$	the set of polynomials of total degree at most k
$\mathcal{P}_k(\mathcal{N})$	the set of functions which on each cube Q in the net $\mathcal{N}$ coincide with a polynomial of degree at most k
$s(\mathcal{N})$	a non-smooth spline, element of $\mathcal{P}_k(\mathcal{N})$
B'	the topological dual of the Banach space B
p', q'	the dual indices of p and q, i.e. $1/p + 1/p' = 1, 1/q + 1/q' = 1$
$j = (j_1, \ldots, j_n)$	a multi-index with length $\vert j\vert = j_1 + \ldots + j_n$
x^j	$x_1^{j_1} \ldots x_n^{j_n}$, j a multi-index, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$
c	a non-negative constant, not necessarily the same each time it appears.

2. The space $B_\alpha^{p,q}(F)$, $\alpha > 0$. There are different, equivalent definitions of the Besov space $B_\alpha^{p,q}(F)$ for $\alpha > 0$. One of these is as follows (see [5, p. 135]).

DEFINITION 2.1. $f \in B_\alpha^{p,q}(F)$, $\alpha > 0, 1 \le p, q \le \infty$, if $f \in L^p(\mu)$ and there is a sequence $B = (B_\nu)_{\nu=0}^\infty \in \ell^q$ such that for every net $\mathcal{N}$ with mesh $2^{-\nu}$, $\nu = 0, 1, 2, \ldots$, there exists a function $s(\mathcal{N}) \in \mathcal{P}_{[\alpha]}(\mathcal{N})$ satisfying

$$\|f - s(\mathcal{N})\|_p \le 2^{-\nu\alpha} B_\nu. \tag{1}$$

The $B_\alpha^{p,q}(F)$-norm of f is

$$\|f\|_p + \inf \|B\|_q,$$

where the infimum is over all such sequences B.

Next we shall give an alternative definition of $B_\alpha^{p,q}(F)$, $\alpha > 0$, where the functions in $\mathcal{P}_{[\alpha]}(\mathcal{N})$ are defined in a constructive way. Let $\mathcal{N}_\nu, \nu = 0, 1, 2, \ldots$, be the fixed net with mesh $2^{-\nu}$ introduced in the notation in Section 1. For $Q \in \mathcal{N}_\nu(F) := \{Q \in \mathcal{N}_\nu : Q \cap F \ne \emptyset\}$, let $(\pi_j)_{|j| \le [\alpha]}$ be an orthonormal basis in the subspace $\mathcal{P}_{[\alpha]}$ of $L^2(\mu, 2Q)$ and define the projection on $\mathcal{P}_{[\alpha]}$ for every $f \in L^1(\mu, 2Q)$:

$$P_Q = P_Q(f) := \begin{cases} \sum_{|j| \le [\alpha]} \pi_j \int_{2Q} f \pi_j \, d\mu & \text{for } \nu > 0, \\ 0 & \text{for } \nu = 0. \end{cases} \tag{2}$$

We shall need the following lemma. For its validity it is important that the integration in (2) is over a cube which is an expansion of Q by a factor larger than 1, for instance the factor 2 which we have chosen.

LEMMA 2.2 [2, p. 310]. *For $Q \in \mathcal{N}_\nu(F)$ with $\nu > 0$ we have*

$$\|f - P_Q(f)\|_{p,2Q} \le c\|f - P\|_{p,2Q}$$

for $1 \le p \le \infty$, $P \in \mathcal{P}_{[\alpha]}$, and $f \in L^1(\mu, 2Q)$, where c depends only on n, F, μ, and α.

The alternative definition of $B^{p,q}_\alpha(F)$, $\alpha > 0$, which we have in mind is given by the following proposition.

PROPOSITION 2.3. *Let $\alpha > 0$ and $1 \le p, q \le \infty$, and let f be a μ-measurable function defined on F. Introduce the sequence $A = (A_\nu)_{\nu=0}^\infty$ by*

$$(3) \qquad \Big(\sum_{Q \in \mathcal{N}_\nu(F)} \int_{2Q} |f - P_Q|^p \, d\mu\Big)^{1/p} = 2^{-\nu\alpha} A_\nu, \qquad \nu = 0, 1, \ldots,$$

where $P_Q = P_Q(f)$ is given by (2). Then $f \in B^{p,q}_\alpha(F)$ if and only if $A \in \ell^q$ and the norm of f in $B^{p,q}_\alpha(F)$ is equivalent to $\|A\|_q$. The constants in the inequalities giving the equivalence of the norms depend on n, F, μ, and α only.

Before we turn to the proof of the proposition we shall write (3) in a different form. The fact that in (2) and (3) we work with polynomials in $2Q$, and in (1) with polynomials in Q, causes some technical difficulties. One way to overcome this is as follows. The cubes $2Q$, $Q \in \mathcal{N}_\nu$, belong to 2^n different nets, $\mathcal{N}_{\nu 1}, \ldots, \mathcal{N}_{\nu K}$, $K = 2^n$, with mesh $2^{-(\nu-1)}$. Put

$$I_{\nu k} := \{Q \in \mathcal{N}_\nu : 2Q \in \mathcal{N}_{\nu k}\}, \qquad 1 \le k \le K,$$

and define functions in $2Q$ by

$$m_{\nu k}(x) := P_Q(x) \quad \text{if } x \in 2Q,\ Q \in I_{\nu k},\ Q \cap F \ne \emptyset,$$

and, for $Q \in I_{\nu k}$ with $Q \cap F = \emptyset$,

$$m_{\nu k}(x) = \begin{cases} f(x) & \text{if } x \in 2Q \cap F, \\ 0 & \text{if } x \in 2Q \setminus F. \end{cases}$$

Then $m_{\nu k}$ is defined in $\mathbb{R}^n$ and (3) may be written

$$(3') \qquad \Big(\sum_{k=1}^K \|f - m_{\nu k}\|_p^p\Big)^{1/p} = 2^{-\nu\alpha} A_\nu.$$

Proof of Proposition 2.3. 1) Assume that $f \in B^{p,q}_\alpha(F)$. Take a sequence $B = (B_\nu)_{\nu=0}^\infty \in \ell^q$ such that for every net $\mathcal{N}$ with mesh $2^{-\nu}$, $\nu \ge 0$, there exists $s(\mathcal{N}) \in \mathcal{P}_{[\alpha]}(\mathcal{N})$ satisfying (1) in Definition 2.1. By Lemma 2.2, if $Q \in I_{\nu k}$, $Q \cap F \ne \emptyset$, $\nu \ge 1$, and $P_Q = P_Q(f)$ is given by (2) and $s(\mathcal{N}_{\nu k}) \in \mathcal{P}_{[\alpha]}(\mathcal{N}_{\nu k})$ by (1), we obtain

$$\|f - P_Q\|_{p,2Q} \le c\|f - s(\mathcal{N}_{\nu k})\|_{p,2Q}.$$

By summing over $Q \in I_{\nu k}$ with $Q \cap F \ne \emptyset$, we obtain, for some constant c,

$$\|f - m_{\nu k}\|_p \le c 2^{-\nu\alpha} B_{\nu-1},$$

and, consequently, by (3′),

$$2^{-\nu\alpha} A_\nu = \Big(\sum_{k=1}^K \|f - m_{\nu k}\|_p^p\Big)^{1/p} \le K^{1/p} c 2^{-\nu\alpha} B_{\nu-1} \quad \text{for } \nu \ge 1.$$

For $\nu = 0$ we get $A_0 \le c\|f\|_p$. The last two inequalities prove that $A = (A_\nu)_{\nu=0}^\infty \in \ell^q$ and that the ℓ^q-norm of A is at most a constant times the norm of f in $B^{p,q}_\alpha(F)$.

2) Conversely, assume that $A = (A_\nu)_{\nu=0}^\infty \in \ell^q$, where A_ν is defined by (3). By taking $\nu = 0$ in (3) we see that $f \in L^p(\mu)$. Let $\mathcal{N}$ be an arbitrary net with mesh $2^{-\nu}$ and let $Q \in \mathcal{N}$ with $Q \cap F \ne \emptyset$. For $\nu > 0$, take $Q' \in \mathcal{N}_{\nu-1}$ such that $Q' \cap Q \cap F \ne \emptyset$ and $P_{Q'} = P_{Q'}(f)$ according to (2). Then $Q \subset 2Q'$ and the function $s(\mathcal{N}) \in \mathcal{P}_{[\alpha]}(\mathcal{N})$ defined by $s(\mathcal{N})(x) = P_{Q'}(x)$ on Q will do in (1), and taking $s(\mathcal{N})$ identically zero if $\nu = 0$, we see that the $B^{p,q}_\alpha(F)$-norm of f is at most a constant times the ℓ^q-norm of A.

3. The space $B^{p,q}_\alpha(F)$, $\alpha < 0$.

The definition of $B^{p,q}_\alpha(F)$ for $\alpha < 0$ will be made by means of an atomic decomposition. We start by defining the atoms.

DEFINITION 3.1. Let $\alpha < 0$, $1 \le p \le \infty$, and let Q with $Q \cap F \ne \emptyset$ be a cube with edge length $2^{-\nu}$, where ν is a non-negative integer. A function $a = a_Q \in L^p(\mu)$ is an (α, p)-*atom* associated with Q if

(i) $\operatorname{supp} a \subset 2Q$,
(ii) $\int x^\gamma a(x)\, d\mu(x) = 0$ for $|\gamma| \le [-\alpha]$ if $\nu > 0$,
(iii) $\|a\|_p \le 2^{-\nu\alpha}$.

We shall work with the fixed dyadic net $\mathcal{N}_\nu$, $\nu = 0, 1, \ldots$, and cubes $Q \in \mathcal{N}_\nu(F)$. The following lemma is basic; notice that $\alpha > 0$ in this lemma and that p' and q' are the conjugate indices to p and q, respectively.

LEMMA 3.2. *Assume that $\alpha > 0$, $1 \le p, q \le \infty$, and let $f \in B^{p,q}_\alpha(F)$. For every $Q \in \mathcal{N}_\nu(F)$, let s_Q be a number and a_Q a $(-\alpha, p')$-atom. If $S' = (S'_\nu)_{\nu=0}^\infty \in \ell^{q'}$ where*

$$(4) \qquad S'_\nu := \Big(\sum_{Q \in \mathcal{N}_\nu(F)} |s_Q|^{p'}\Big)^{1/p'},$$

and $A = (A_\nu)_{\nu=0}^\infty$ where A_ν is defined by (3), then

$$(5) \qquad \sum_{\nu=0}^\infty \sum_{Q \in \mathcal{N}_\nu(F)} \Big| s_Q \int a_Q f \, d\mu \Big| \le \|S'\|_{q'} \|A\|_q < \infty.$$

Proof. By using the moment condition in Definition 3.1(ii), we get, for $Q \in \mathcal{N}_\nu(F)$, $\nu \geq 0$,

$$\int a_Q f\,d\mu = \int_{2Q} a_Q(f - P_Q)\,d\mu,$$

where P_Q is the polynomial in (2) which has degree at most $[\alpha] = [-(-\alpha)]$; notice that $P_Q = 0$ if $\nu = 0$. By combining this with Hölder's inequality and Definition 3.1(iii) we obtain

$$\Big|\int a_Q f\,d\mu\Big|^p \leq \int_{2Q} |f - P_Q|^p\,d\mu \cdot 2^{\nu\alpha p}.$$

Putting

$$M_\nu := \sum_{Q\in\mathcal{N}_\nu(F)} \Big|s_Q \int a_Q f\,d\mu\Big|,$$

by Hölder's inequality, the previous estimate, and (3) we get

$$M_\nu \leq S'_\nu 2^{\nu\alpha}\Big(\sum_{Q\in\mathcal{N}_\nu(F)} \int_{2Q} |f - P_Q|^p\,d\mu\Big)^{1/p} = S'_\nu A_\nu.$$

By Hölder's inequality we finally get

$$\sum_{\nu=0}^{\infty} M_\nu \leq \|S'\|_{q'}\|A\|_q,$$

which is (5). ■

By means of Lemma 3.2 (with α, p, q changed to $-\alpha, p', q'$) we shall now define $B^{p,q}_\alpha(F)$ for $\alpha < 0$ as a subspace of Schwartz's distributions $\mathcal{D}'(\mathbb{R}^n)$. So we now take $\alpha < 0$. For $Q \in \mathcal{N}_\nu(F)$, let a_Q be an (α, p)-atom associated with Q and let s_Q be numbers such that $S = (S_\nu)_{\nu=0}^\infty \in \ell^q$ where S_ν is given by

$$S_\nu := \Big(\sum_{Q\in\mathcal{N}_\nu(F)} |s_Q|^p\Big)^{1/p}. \tag{6}$$

Then the function

$$g_\nu := \sum_{Q\in\mathcal{N}_\nu(F)} s_Q a_Q$$

is locally in $L^1(\mu)$ since the sum defining g_ν is a finite sum on any compact subset of $\mathbb{R}^n$. We identify g_ν with the distribution

$$\langle g_\nu, \varphi\rangle := \sum_{Q\in\mathcal{N}_\nu(F)} s_Q \int a_Q\varphi\,d\mu, \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

Then $f_m := \sum_{\nu=0}^m g_\nu$ is the distribution given by

$$\langle f_m, \varphi\rangle := \sum_{\nu=0}^{m} \sum_{Q\in\mathcal{N}_\nu(F)} s_Q \int a_Q\varphi\,d\mu, \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

We claim that by Lemma 3.2, f_m converges to f in the distribution sense, i.e. that

$$\langle f_m, \varphi\rangle \to \langle f, \varphi\rangle \quad \text{as } m \to \infty, \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^n),$$

where the distribution f is given by

$$\langle f, \varphi\rangle := \sum_{\nu=0}^{\infty} \sum_{Q\in\mathcal{N}_\nu(F)} s_Q \int a_Q\varphi\,d\mu, \quad \varphi \in C_0^\infty(\mathbb{R}^n). \tag{7}$$

In fact, since $\varphi \in C_0^\infty(\mathbb{R}^n)$ the trace theorem for Besov spaces [5, p. 141] gives $\varphi|F \in B^{p',q'}_{-\alpha}(F)$, where $\varphi|F$ denotes the restriction of φ to F. Hence our claim follows from Lemma 3.2.

When (7) holds we write

$$f = \sum_{\nu=0}^{\infty} \sum_{Q\in\mathcal{N}_\nu(F)} s_Q a_Q, \tag{8}$$

and we refer to (8) as an *atomic decomposition* of f. Observe that the atomic decomposition (8) is not necessarily unique, i.e. different (α, p)-atoms a_Q and numbers $s_Q, S = (S_\nu)_{\nu=0}^\infty \in \ell^q$, S_ν given by (6), may give the same distribution f in (7).

Definition 3.3. We define $B^{p,q}_\alpha(F), \alpha < 0, 1 \leq p, q \leq \infty$, to consist of those $f \in \mathcal{D}'(\mathbb{R}^n)$ which are given by (7) where we assume that a_Q are (α, p)-atoms and s_Q are numbers such that $S = (S_\nu)_{\nu=0}^\infty \in \ell^q$ and S_ν is defined by (6). We define the norm of f by

$$\|f\|_{B^{p,q}_\alpha(F)} := \inf \|S\|_q,$$

where the infimum is taken over all possible atomic decompositions (8) of f.

With this definition $B^{p,q}_\alpha(F)$ is a normed linear space.

4. The dual of $B^{p,q}_\alpha(F), \alpha > 0$.

The object of this section is to prove that

$$(B^{p,q}_\alpha(F))' = B^{p',q'}_{-\alpha}(F) \quad \text{for } \alpha > 0,\ 1 \leq p, q < \infty. \tag{9}$$

One half of (9) essentially follows from the work already done in Section 3. In fact, if $f \in B^{p,q}_\alpha(F)$, $\alpha > 0$, $1 \leq p, q < \infty$, $g \in B^{p',q'}_{-\alpha}(F)$, and g is given by the atomic decomposition

$$g=\sum_{\nu=0}^{\infty}\sum_{Q\in\mathcal{N}_\nu(F)} s_Q a_Q,$$

then the duality is given by

$$\langle g,f\rangle := \sum_{\nu=0}^{\infty}\sum_{Q\in\mathcal{N}_\nu(F)} s_Q \int a_Q f\,d\mu. \tag{10}$$

We need to prove that the value of the double sum in (10) is independent of the particular atomic decomposition used for g. This follows from

LEMMA 4.1. *Let $f\in B_\alpha^{p,q}(F)$ and $g\in B_{-\alpha}^{p',q'}(F)$, $\alpha>0$, $1\le p,q<\infty$, be given. Then the double sum in* (10) *has the same value for all atomic decompositions of g.*

Proof. By the trace theorem [5, p. 141], $f=f_0|F$ where $f_0\in B_\beta^{p,q}(\mathbb{R}^n)$ with $\beta=\alpha+(n-d)/p$. For every $\varepsilon>0$ there exists $\varphi\in C_0^\infty(\mathbb{R}^n)$ such that

$$\|f_0-\varphi\|_{B_\beta^{p,q}(\mathbb{R}^n)}<\varepsilon,$$

since $C_0^\infty(\mathbb{R}^n)$ is dense in $B_\beta^{p,q}(\mathbb{R}^n)$ if $p,q<\infty$. By using again the trace theorem we get, for some constant c,

$$\|f-\varphi\|_{B_\alpha^{p,q}(F)}<c\varepsilon. \tag{11}$$

Let

$$g=\sum_{\nu=0}^{\infty}\sum_{Q\in\mathcal{N}_\nu(F)} s_Q a_Q$$

be an atomic decomposition of g, i.e. a_Q are $(-\alpha,p')$-atoms and s_Q are numbers such that $S'=(S'_\nu)_{\nu=0}^\infty\in\ell^{q'}$, where S'_ν is given by (4). From Lemma 3.2, Proposition 2.3 and (11) we conclude

$$\sum_{\nu=0}^{\infty}\sum_{Q\in\mathcal{N}_\nu(F)}\Big|s_Q\int a_Q(f-\varphi)\,d\mu\Big|\le c\varepsilon\|S'\|_{q'}.$$

This means that the value of the double sum in (10) is independent of the atomic decomposition of g. ■

From (10), Lemma 3.2, Proposition 2.3, and Lemma 4.1 we get

$$|\langle g,f\rangle|\le c\|g\|_{B_{-\alpha}^{p',q'}(F)}\|f\|_{B_\alpha^{p,q}(F)}, \tag{12}$$

where c depends only on n,F,μ, and α. This means that every $g\in B_{-\alpha}^{p',q'}(F)$ determines a bounded linear functional L on $B_\alpha^{p,q}(F)$ by means of (10), $L(f)=\langle g,f\rangle$, and that

$$\|L\|\le c\|g\|_{B_{-\alpha}^{p',q'}(F)}.$$

We also note that different functions g in $B_{-\alpha}^{p',q'}(F)$ give different bounded linear functionals L on $B_\alpha^{p,q}(F)$, since the restriction to F of a function in $C_0^\infty(\mathbb{R}^n)$ is in $B_\alpha^{p,q}(F)$.

In the proof of the second half of (9) we shall use the spaces $\ell^q(\overrightarrow{L_p})$ of sequences $v=(v_\nu)_{\nu=0}^\infty$ of vectors $v_\nu=(v_{\nu1},\dots,v_{\nu K})$, $K=2^n$, of $L^p(\mu)$ functions on F with norm $\|v\|_{q,p}$ equal to the ℓ^q-norm of the sequence

$$\|v_\nu\|_{\overrightarrow{L_p}}:=\Big(\sum_{k=1}^{K}\int|v_{\nu k}|^p\,d\mu\Big)^{1/p},\quad \nu=0,1,\dots$$

We shall need the following lemma and since we have not found any reference for it, it is proved in the Appendix.

LEMMA 4.2. $(\ell^q(\overrightarrow{L_p}))'=\ell^{q'}(\overrightarrow{L_{p'}})$ *for $1\le p<\infty$ and $1\le q<\infty$, and if $v\in\ell^q(\overrightarrow{L_p})$, $v=(v_\nu)_{\nu=0}^\infty$, $v_\nu=(v_{\nu1},\dots,v_{\nu K})$, and $g\in\ell^{q'}(\overrightarrow{L_{p'}})$, $g=(g_\nu)_{\nu=0}^\infty$, $g_\nu=(g_{\nu1},\dots,g_{\nu K})$, then the duality is given by*

$$L_1(v):=\sum_{\nu=0}^{\infty}\sum_{k=1}^{K}\int v_{\nu k}g_{\nu k}\,d\mu, \tag{13}$$

with $\|L_1\|=\|g\|_{q',p'}$. Here $\|L_1\|$ denotes the norm of the element in $(\ell^q(\overrightarrow{L_p}))'$ determined by (13).

We are now ready to prove the second half of (9) but first we state the duality result (9) in more detail.

THEOREM 4.3. *Assume that $\alpha>0$ and $1\le p,q<\infty$.*

(i) *If*

$$g=\sum_{\nu=0}^{\infty}\sum_{Q\in\mathcal{N}_\nu(F)} s_Q a_Q\in B_{-\alpha}^{p',q'}(F) \tag{14}$$

and L is defined by

$$L(f)=\sum_{\nu=0}^{\infty}\sum_{Q\in\mathcal{N}_\nu(F)} s_Q\int a_Q f\,d\mu, \tag{15}$$

then $L\in(B_\alpha^{p,q}(F))'$ and

$$\|L\|\le c\|g\|_{B_{-\alpha}^{p',q'}(F)}, \tag{16}$$

where c is a constant depending only on n,F,μ, and α.

(ii) *If* $L \in (B^{p,q}_\alpha(F))'$, *then there exists a unique* g *as in* (14) *such that* (15) *and* (16) *hold and*

(17) $$\|g\|_{B^{p',q'}_{-\alpha}(F)} \le c\|L\|,$$

where c *is a constant depending only on* n, F, μ, *and* α.

Proof. We have already proved (i); observe that the value of $L(f)$ given by (15) depends on g but not on the particular atomic decomposition of g.

To prove (ii) we first return to the functions $m_{\nu k}, 1 \le k \le K$, $K = 2^n$, introduced before (3'). We define an operator T on $B^{p,q}_\alpha(F)$ by $Tf = (T_\nu f)_{\nu=0}^\infty$, where $T_\nu f = (T_{\nu 1} f, \ldots, T_{\nu K} f)$ and

$$T_{\nu k} f := 2^{\nu\alpha}(f - m_{\nu k}), \quad \nu = 0, 1, \ldots, \ 1 \le k \le K.$$

Since (3') is another way to write (3) we can use Proposition 2.3 to conclude that $f \to Tf$ is an embedding of $B^{p,q}_\alpha(F)$ in $\ell^q(\overrightarrow{L_p})$ such that the $B^{p,q}_\alpha(F)$-norm of f is equivalent to the $\ell^q(\overrightarrow{L_p})$-norm of Tf; in particular, $Tf = 0$ implies $f = 0$.

Now, for a given $L \in (B^{p,q}_\alpha(F))'$ we use the operator T to define a functional L_1 on $V := T(B^{p,q}_\alpha(F))$ by

$$L_1(v) := L(f) \quad \text{if } v = Tf, \ v \in V.$$

Then L_1 is a bounded linear functional on $V \subset \ell^q(\overrightarrow{L_p})$ with $\|L_1\| \le c\|L\|$ and by Hahn–Banach's theorem we can extend L_1 to a bounded linear functional on $\ell^q(\overrightarrow{L_p})$ without increasing its norm. By Lemma 4.2 this functional is given by $g \in \ell^{q'}(\overrightarrow{L_{p'}})$, $g = (g_\nu)_{\nu=0}^\infty$, $g_\nu = (g_{\nu 1}, \ldots, g_{\nu K})$ and (13). Hence, for $f \in B^{p,q}_\alpha(F)$ we get

$$L(f) = L_1(Tf) = \sum_{\nu=0}^{\infty}\sum_{k=1}^{K} \int 2^{\nu\alpha}(f - m_{\nu k}) g_{\nu k}\, d\mu.$$

By using the polynomials $P_Q = P_Q(f)$ defined by (2) instead of $m_{\nu k}$, we find

$$L(f) = \sum_{\nu=0}^{\infty} \sum_{Q \in \mathcal{N}_\nu(F)} \int_{2Q} 2^{\nu\alpha}(f - P_Q(f)) g_{\nu k}\, d\mu.$$

(In this formula the index k in $g_{\nu k}$ depends on Q and is determined by the condition $2Q \in \mathcal{N}_{\nu k}$.) But by (2) and Fubini's theorem,

$$\int_{2Q} P_Q(f) g_{\nu k}\, d\mu = \int_{2Q} \Big(\sum_{|j| \le [\alpha]} \pi_j(x) \int_{2Q} f\pi_j\, d\mu(y)\Big) g_{\nu k}(x)\, d\mu(x) = \int_{2Q} f P_Q(g_{\nu k})\, d\mu,$$

and consequently,

$$L(f) = \sum_{\nu=0}^{\infty} \sum_{Q \in \mathcal{N}_\nu(F)} \int_{2Q} f(g_{\nu k} - P_Q(g_{\nu k})) 2^{\nu\alpha}\, d\mu(x).$$

We now introduce s_Q and a_Q for $Q \in \mathcal{N}_\nu(F)$:

$$s_Q := \|g_{\nu k} - P_Q(g_{\nu k})\|_{p',2Q}$$

and, for $x \in (2Q) \cap F$,

$$a_Q(x) := \begin{cases} 2^{\nu\alpha}(g_{\nu k} - P_Q(g_{\nu k}))/s_Q & \text{if } s_Q \ne 0, \\ 0 & \text{if } s_Q = 0; \end{cases}$$

for $x \notin (2Q) \cap F$ we put $a_Q(x) = 0$.

Since $P_Q(g_{\nu k})$ is the projection of $g_{\nu k}$ on $\mathcal{P}_{[\alpha]}$, it is easy to check that a_Q are $(-\alpha, p')$-atoms. We also have the desired representation (15) for $L(f)$. Furthermore, by Lemma 2.2 used with $P = 0$,

$$s_Q \le c\|g_{\nu k}\|_{p',2Q},$$

which gives, with different constants c depending only on n, F, μ, and α, and $I_{\nu k}(F) := \{Q \in I_{\nu k} : Q \cap F \ne \emptyset\}$, with $I_{\nu k}$ defined a few lines before (3'),

$$\sum_{Q \in \mathcal{N}_\nu(F)} |s_Q|^{p'} \le c \sum_{k=1}^{\infty} \sum_{Q \in I_{\nu k}(F)} \int_{2Q} |g_{\nu k}|^{p'}\, d\mu \le c \sum_{k=1}^{K} \int |g_{\nu k}|^{p'}\, d\mu = c\|g_\nu\|^{p'}_{\overrightarrow{L_{p'}}}$$

and, by the construction of L_1 and Lemma 4.2,

$$\Big\{ \sum_{\nu=0}^{\infty} \Big(\sum_{Q \in \mathcal{N}_\nu(F)} |s_Q|^{p'} \Big)^{q'/p'} \Big\}^{1/q'} \le c\|g\|_{\ell^{q'}(\overrightarrow{L_{p'}})} \le c\|L\|.$$

It follows that (14)–(17) hold.

5. The dual of $B^{p',q'}_{-\alpha}(F), \alpha > 0$.

In the following theorem we prove that the dual of $B^{p',q'}_{-\alpha}(F)$ is $B^{p,q}_\alpha(F)$ for $\alpha > 0, 1 < p, q < \infty$, proving the reflexivity of $B^{p,q}_\alpha(F)$ for these indices. The first part of the theorem follows from the discussion at the beginning of Section 4.

THEOREM 5.1. *Assume that* $\alpha > 0$ *and* $1 < p, q < \infty$.

(i) *If* $f \in B^{p,q}_\alpha(F)$ *and* L *is defined by*

(18) $$L(g) := \sum_{\nu=0}^{\infty} \sum_{Q \in \mathcal{N}_\nu(F)} s_Q \int a_Q f\, d\mu$$

$$\textit{for } g = \sum_{\nu=0}^{\infty} \sum_{Q \in \mathcal{N}_\nu(F)} s_Q a_Q \in B^{p',q'}_{-\alpha}(F),$$

then

$$L \in (B^{p',q'}_{-\alpha}(F))' \quad \text{and} \quad \|L\| \le c\|f\|_{B^{p,q}_\alpha(F)},$$

where c is a constant depending only on n, F, μ, *and* α.

(ii) *If* $L \in (B^{p',q'}_{-\alpha}(F))'$, *then there is a unique* $f \in B^{p,q}_\alpha(F)$ *such that* (18) *holds and*

$$\|f\|_{B^{p,q}_\alpha(F)} \le c\|L\|, \tag{19}$$

where the constant c depends only on n, F, μ, *and* α.

Proof. As remarked above, (i) is already proved. To prove (ii) we start from a bounded linear functional L on $B^{p',q'}_{-\alpha}(F)$ and proceed in several steps.

Step 1. To find f we first consider atoms associated with cubes in $\mathcal{N}_0(F)$, i.e. cubes with edge length 1. For $Q \in \mathcal{N}_0(F)$, take any function $h_Q \in L^{p'}(\mu)$, $h_Q \ne 0$, such that $\operatorname{supp} h_Q \subset 2Q$. Then $h_Q/\|h_Q\|_{p'}$ is a $(-\alpha, p')$-atom associated with Q, which gives

$$|L(h_Q/\|h_Q\|_{p'})| \le \|L\|, \quad \text{i.e.} \quad |L(h_Q)| \le \|L\|\|h_Q\|_{p'}.$$

By Riesz's representation theorem there exists a function $f_Q \in L^p(\mu)$ with $\operatorname{supp} f_Q \subset 2Q$ such that $L(h_Q) = \int_{2Q} h_Q f_Q\,d\mu$ for $h_Q \in L^{p'}(\mu, 2Q)$. On overlapping cubes $2Q$ and $2Q'$ with $Q, Q' \in \mathcal{N}_0(F)$ we get $f_Q = f_{Q'}$ μ-a.e., which determines a unique f which is locally in $L^p(\mu)$ and such that $f = f_Q$ on $2Q$, and

$$L(h) = \int hf\,d\mu \quad \text{for } h \in L^{p'}(\mu), \text{with compact support.}$$

Step 2. Now that we have found f, we want to prove (19). This is done by defining, step by step, a special $g = \sum\sum s_Q a_Q \in B^{p',q'}_{-\alpha}(F)$. First we want to estimate $f - P_Q$ where $P_Q = P_Q(f)$ is defined by (2). We start with the case $\nu > 0$. We fix $Q \in \mathcal{N}_\nu(F)$ and introduce

$$\delta := \inf_{P \in \mathcal{P}_{[\alpha]}} \|f - P\|_{p,2Q},$$

and assume that $\delta > 0$. By Hahn–Banach's theorem there exists a $\Psi \in (L^p(\mu, 2Q))'$ such that $\Psi(f) = 1$, $\Psi(P) = 0$ for $P \in \mathcal{P}_{[\alpha]}$ and $\|\Psi\| = 1/\delta$. Hence, there exists a $\psi \in L^{p'}(\mu, 2Q)$ with $\operatorname{supp}\psi \subset 2Q$ satisfying $\|\psi\|_{p'} = 1/\delta$ and

$$\Psi(\varphi) = \int \psi\varphi\,d\mu \quad \text{for } \varphi \in L^p(\mu, 2Q).$$

Now we put

$$a_Q := 2^{\nu\alpha}\psi/\|\psi\|_{p'} = 2^{\nu\alpha}\delta\psi.$$

Since $\Psi(P) = 0$ for $P \in \mathcal{P}_{[\alpha]}$ we see that a_Q is a $(-\alpha, p')$-atom associated with Q. Furthermore,

$$1 = \Psi(f) = \int \psi f\,d\mu = \frac{2^{-\nu\alpha}}{\delta} \int a_Q f\,d\mu,$$

i.e.

$$\delta 2^{\nu\alpha} = \int a_Q f\,d\mu.$$

By Lemma 2.2 this gives

$$\Big(\int_{2Q} |f - P_Q|^p\,d\mu\Big)^{1/p} 2^{\nu\alpha} \le c\delta 2^{\nu\alpha} = c\int_{2Q} a_Q f\,d\mu,$$

and we have proved the desired estimate for $\nu > 0, \delta > 0$:

$$\Big(\int_{2Q} |f - P_Q|^p\,d\mu\Big)^{1/p} 2^{\nu\alpha} \le c\int_{2Q} a_Q f\,d\mu. \tag{20}$$

If $\delta = 0$ we let $a_Q = 0$ and note that (20) still holds, using Lemma 2.2 to deduce that $f = P_Q$ on $2Q$.

For $\nu = 0$ we choose a_Q such that $\operatorname{supp} a_Q \subset 2Q$, $a_Q f \ge 0$ μ-a.e. on $2Q$, and $|a_Q|^{p'} = c|f|^p$, where c is chosen such that $\|a_Q\|_{p'} = 1$. Then a_Q is a $(-\alpha, p')$-atom associated with Q and

$$\int a_Q f\,d\mu = \|f\|_{p,2Q},$$

which gives (20) for $\nu = 0$, since $P_Q = 0$ for $\nu = 0$.

Step 3. Let L, f, and a_Q for $Q \in \mathcal{N}_\nu(F)$ be as above and consider a fixed ν. Let c_ν be positive constants—which we shall choose in Step 4—and choose non-negative numbers s_Q such that

$$|s_Q|^{p'} := c_\nu\Big|\int a_Q f\,d\mu\Big|^p \quad \text{for } Q \in \mathcal{N}_\nu(F).$$

For $R > 0$ put

$$I(R) := \{Q \in \mathcal{N}_\nu(F) : Q \subset B(0, R)\}.$$

Then, by Step 1, Hölder's inequality with equality, and (20),

$$\begin{aligned}\Big|L\Big(\sum_{Q\in I(R)} s_Q a_Q\Big)\Big| &= \sum_{Q\in I(R)} s_Q \int a_Q f\,d\mu \\ &= \Big(\sum_{Q\in I(R)} |s_Q|^{p'}\Big)^{1/p'}\Big(\sum_{Q\in I(R)}\Big(\int_{2Q} a_Q f\,d\mu\Big)^p\Big)^{1/p} \\ &\ge \Big(\sum_{Q\in I(R)} |s_Q|^{p'}\Big)^{1/p'}\frac{1}{c}\Big(\sum_{Q\in I(R)}\int_{2Q} |f - P_Q|^p\,d\mu\Big)^{1/p} 2^{\nu\alpha}.\end{aligned}$$

But

$$\Big|L\Big(\sum_{Q\in I(R)} s_Q a_Q\Big)\Big| \le \|L\|\Big(\sum_{Q\in I(R)} |s_Q|^{p'}\Big)^{1/p'}.$$

By combining the last two estimates and letting R tend to infinity we obtain

$$(21)\qquad A_\nu := \Big(\sum_{Q\in\mathcal{N}_\nu(F)} \int_{2Q} |f-P_Q|^p\,d\mu\Big)^{1/p} 2^{\nu\alpha} \le c\|L\|$$

if $s_Q \ne 0$ for some $Q\in\mathcal{N}_\nu(F)$. If $s_Q=0$ for all $Q\in\mathcal{N}_\nu(F)$, then $f=P_Q$ on $2Q$ by (20) and (21) is trivially true.

Step 4. Let L, f, a_Q and s_Q for $Q\in\mathcal{N}_\nu(F)$ be as in Step 3; note that we are still free to choose the constants c_ν in the definition of s_Q in Step 3. We put

$$S'_\nu := \Big(\sum_{Q\in\mathcal{N}_\nu(F)} |s_Q|^{p'}\Big)^{1/p'}$$

and observe that by the definition of s_Q, the fact that a_Q are $(-\alpha,p')$-atoms, and (21) we obtain

$$(S'_\nu)^{p'} = c_\nu \sum_{Q\in\mathcal{N}_\nu(F)} \Big|\int a_Q f\,d\mu\Big|^p = c_\nu \sum_{Q\in\mathcal{N}_\nu(F)} \Big|\int a_Q(f-P_Q)\,d\mu\Big|^p$$

$$\le c_\nu \sum_{Q\in\mathcal{N}_\nu(F)} \|a_Q\|_{p'}^p \int_{2Q} |f-P_Q|^p\,d\mu \le c_\nu(c\|L\|)^p < \infty.$$

This gives, using the calculations in Step 3 and A_ν in (21),

$$L\Big(\sum_{Q\in\mathcal{N}_\nu(F)} s_Q a_Q\Big) = \sum_{Q\in\mathcal{N}_\nu(F)} s_Q \int a_Q f\,d\mu \ge \frac{1}{c} S'_\nu A_\nu.$$

Consequently, for any positive integer M,

$$\infty > \|L\|\Big(\sum_{\nu=0}^M |S'_\nu|^{q'}\Big)^{1/q'} \ge \Big|L\Big(\sum_{\nu=0}^M \sum_{Q\in\mathcal{N}_\nu(F)} s_Q a_Q\Big)\Big|$$

$$= \sum_{\nu=0}^M \sum_{Q\in\mathcal{N}_\nu(F)} s_Q \int a_Q f\,d\mu \ge \frac{1}{c}\sum_{\nu=0}^M S'_\nu A_\nu.$$

In order to estimate the above sum we now choose c_ν so that, for some constant c, $A_\nu^q = c(S'_\nu)^{q'}$. This gives equality in Hölder's inequality and we get

$$\frac{1}{c}\sum_{\nu=0}^M S'_\nu A_\nu = \frac{1}{c}\Big(\sum_{\nu=0}^M |S'_\nu|^{q'}\Big)^{1/q'} \Big(\sum_{\nu=0}^M A_\nu^q\Big)^{1/q}.$$

By combining this with the previous chain of inequalities we obtain

$$\Big(\sum_{\nu=0}^M A_\nu^q\Big)^{1/q} \le c\|L\|,$$

and by letting M tend to infinity we see that (19) holds if $S'_\nu \ne 0$ for some ν. If $S'_\nu = 0$ for all ν, then $f=0$ so (19) holds in this case also.

By Step 1, linearity, continuity and Lemma 3.2, we finally conclude that (18) holds.

Appendix

Proof Lemma 4.2. If L_1 is given by (13), then applying Hölder's inequality twice one obtains

$$|L_1(v)| \le \|v\|_{\ell^q(\overrightarrow{L_p})} \|g\|_{\ell^{q'}(\overrightarrow{L_{p'}})},$$

so L_1 defines an element in $(\ell^q(\overrightarrow{L_p}))'$ with $\|L_1\| \le \|g\|_{\ell^{q'}(\overrightarrow{L_{p'}})}$.

Let now a non-trivial, bounded linear functional L_1 on $\ell^q(\overrightarrow{L_p})$ be given. By restricting L_1 to the subspace $S_{\nu k}$ of $\ell^q(\overrightarrow{L_p})$ consisting of elements of the form $v=(0,0,\ldots,0,v_\nu,0,\ldots)$, $v_\nu=(0,0,\ldots,0,v_{\nu k},0,\ldots,0)$, L_1 can be considered as a functional on $L^p(\mu)$, and thus there exists a unique $g_{\nu k}\in L^{p'}(\mu)$ such that $L_1(v)=\int v_{\nu k} g_{\nu k}\,d\mu$ for $v\in S_{\nu k}$. Let $g=(g_\nu)_{\nu=0}^\infty$, $g_\nu=(g_{\nu 1},\ldots,g_{\nu K})$. We shall prove that $g\in\ell^{q'}(\overrightarrow{L_{p'}})$ with norm at most $\|L_1\|$. Then, since (13) holds for all v of the form $(v_0,v_1,\ldots,v_N,0,0,\ldots)$, the desired representation (13) follows by continuity.

If $s_{\nu k}\in L^p(\mu)$, and $s^N=(s_0,s_1,\ldots,s_N,0,0,\ldots)$, $s_\nu=(s_{\nu 1},\ldots,s_{\nu K})$, then

$$|L_1(s^N)| = \Big|\sum_{\nu=0}^N\sum_{k=1}^K \int s_{\nu k} g_{\nu k}\,d\mu\Big| \overset{1)}{\le} \sum_{\nu=0}^N\sum_{k=1}^K \|s_{\nu k}\|_p \|g_{\nu k}\|_{p'}$$

$$\overset{2)}{\le} \sum_{\nu=0}^N A_\nu B_\nu \overset{3)}{\le} \Big(\sum_{\nu=0}^N A_\nu^q\Big)^{1/q}\Big(\sum_{\nu=0}^N B_\nu^{q'}\Big)^{1/q'}$$

where $A_\nu = (\sum_{k=1}^K \|s_{\nu k}\|_p^p)^{1/p}$ and $B_\nu = (\sum_{k=1}^K \|g_{\nu k}\|_{p'}^{p'})^{1/p'}$. For $p>1$ we shall choose the functions $s_{\nu k}$ in such a way that we have equality in 1), 2), and 3). Since $(\sum_{\nu=0}^N A_\nu^q)^{1/q} = \|s^N\|_{\ell^q(\overrightarrow{L_p})}$, this gives

$$\|s^N\|_{\ell^q(\overrightarrow{L_p})}\Big(\sum_{\nu=0}^N B_\nu^{q'}\Big)^{1/q'} = |L_1(s^N)| \le \|L_1\|\|s^N\|_{\ell^q(\overrightarrow{L_p})}$$

from which we deduce that $\|g\|_{\ell^{q'}(\overrightarrow{L_{p'}})} = (\sum_{\nu=0}^\infty B_\nu^{q'})^{1/q'} \le \|L_1\|$ (at least for N large enough, s^N will be non-zero in $\ell^q(\overrightarrow{L_p})$).

Assume first that $p > 1$ and $q > 1$. Choose $s_{\nu k} \in L^p(\mu)$ in such a way that $s_{\nu k} g_{\nu k} \geq 0$ and $|s_{\nu k}|^p = c_{\nu k}|g_{\nu k}|^{p'}$ where $c_{\nu k}$ is a positive constant. This gives equality in 1), and choosing $c_{\nu k}$ not depending on k, $c_{\nu k} = c_\nu$, we have $\|s_{\nu k}\|_p^p = c_\nu \|g_{\nu k}\|_{p'}^{p'}$, which gives equality also in 2). By choosing c_ν in such a way that $A_\nu^q = B_\nu^{q'}$ we get equality in 3). This concludes the proof if $p > 1$ and $q > 1$.

If $q = 1$, to get equality in 3) (with the usual interpretation) we choose $c_\nu = 0$ if $\nu \neq \nu_0$ and $c_{\nu_0} > 0$ arbitrary, where ν_0 is such that $B_{\nu_0} = \max_{0\leq\nu\leq N} B_\nu$. If $p = 1$, then define, given $\varepsilon > 0, E_{\nu k} = \{x : |g_{\nu k}(x)| \geq (1-\varepsilon)\|g_{\nu k}\|_\infty\}$ and take $s_{\nu k}$ such that $s_{\nu k} g_{\nu k} \geq 0$ and $|s_{\nu k}| = c_{\nu k}\chi_{E_{\nu k}}|g_{\nu k}|$, where $\chi_{E_{\nu k}}$ is the characteristic function of $E_{\nu k}$. Then $\int s_{\nu k} g_{\nu k}\, d\mu \geq (1-\varepsilon)\|g_{\nu k}\|_\infty \|s_{\nu k}\|_1$, and taking $c_{\nu k} = 0$ except for $k = k_0$, where k_0 satisfies $\|g_{\nu k_0}\|_\infty = \max_{1\leq k\leq K}\|g_{\nu k}\|_\infty$, and $c_{\nu k_0} = c_\nu > 0$ we get equality in 2). This leads as above to

$$(1-\varepsilon)\|g\|_{\ell^{q'}(\overrightarrow{L_{p'}})} \leq \|L_1\|,$$

so we get the desired estimate.

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