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# A quasi-affine transform of an unbounded operator

by

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**Abstract.** Some results on quasi-affinity for bounded operators are extended to unbounded ones and normal extensions of an unbounded operator are discussed in connection with quasi-affinity.

**1. Introduction.** Sz.-Nagy and Foiaş introduced quasi-affinity (quasi-similarity) for bounded operators on Hilbert spaces ([13]). In [7], we introduced the same notion for unbounded operators in Hilbert spaces and studied which properties of unbounded operators are preserved under quasi-similarity. In particular, we observed that, for a given (unbounded) subnormal operator $T$ in a Hilbert space $\mathcal{H}$, if there is a self-adjoint operator in $\mathcal{H}$ which is a quasi-affine transform of $T$ then $T$ is also self-adjoint and they are unitarily equivalent. On the other hand, it is known [9] that, if $T$ is a bounded subnormal operator on a Hilbert space $\mathcal{H}$ and there is a bounded normal operator in $\mathcal{H}$ which is a quasi-affine transform of $T$, then $T$ is normal and they are unitarily equivalent.

We will generalize this result to unbounded subnormal operators and show that, if $A$ is a (possibly unbounded) subnormal operator in a Hilbert space $\mathcal{H}$ and there is a normal operator in $\mathcal{H}$ which is a quasi-affine transform of $A$, then $A$ has a unique normal extension $B$ in the same Hilbert space $\mathcal{H}$ and more precisely, for any normal extension $N$ of $A$ in a possibly larger Hilbert space, its restriction to $\mathcal{D}(N)\cap\mathcal{H}$ agrees with the normal operator $B$.

**2. Quasi-affinity.** In this paper, all operators (transformations) are assumed to be linear. Let $X$ be a bounded (everywhere defined) transformation from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$. If $X$ is injective and has dense range, then $X$ is said to be *quasi-invertible*. Let $A$ and $B$ be operators

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in $\mathcal{H}$ and $\mathcal{K}$ respectively and let $X$ be a bounded transformation from $\mathcal{H}$ to $\mathcal{K}$. Then the relation $X \cdot A \subseteq B \cdot X$ means that $X\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $X \cdot A\xi = B \cdot X\xi$ for all $\xi \in \mathcal{D}(A)$. Here $\mathcal{D}(A)$ denotes the domain of $A$.

LEMMA 2.1. *Let $A$ and $B$ be densely defined closed operators in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively and let $X$ be a bounded transformation from $\mathcal{H}$ to $\mathcal{K}$. Suppose there is a constant $\lambda$ such that both $\lambda - A$ and $\lambda - B$ are injective and have dense range. If*

$$X \cdot A \subseteq B \cdot X,$$

*then*

$$X \cdot (\lambda - A)^{-1} \subseteq (\lambda - B)^{-1} \cdot X.$$

*In particular, if $\lambda$ belongs to the resolvent sets of both $A$ and $B$, then*

$$X \cdot (\lambda - A)^{-1} = (\lambda - B)^{-1} \cdot X.$$

Proof. The lemma follows from a simple computation.

DEFINITION 2.2. Let $A$ and $B$ be densely defined operators in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. If there is a quasi-invertible transformation $X$ from $\mathcal{H}$ to $\mathcal{K}$ such that

$$X \cdot A \subseteq B \cdot X,$$

then we say that $A$ is a *quasi-affine transform* of $B$ with intertwining operator $X$, or $A$ is *quasi-affine* to $B$ with intertwining operator $X$.

A densely defined operator $A$ is called *formally hyponormal* if it satisfies

$$\mathcal{D}(A) \subseteq \mathcal{D}(A^*) \quad \text{and} \quad \|A\xi\| \geq \|A^*\xi\| \quad \text{for all } \xi \in \mathcal{D}(A).$$

A formally hyponormal operator $A$ is closable and its closure $\overline{A}$ is also formally hyponormal ([3], [7]). Clearly, if $\mathcal{D}(A) = \mathcal{H}$ then $A$ is bounded hyponormal (see [5] for the theory of bounded hyponormal operators).

For a densely defined operator $A$, we denote by $\varrho(A)$ the resolvent set of $A$.

LEMMA 2.3. *Let $A$ be a densely defined closed operator in a Hilbert space $\mathcal{H}$ with $\varrho(A) \neq \emptyset$, and let $B$ be an extension of $A$ in $\mathcal{H}$. If $B$ is formally hyponormal, then $A$ coincides with $\overline{B}$.*

Proof. We first note that $A$ is a quasi-affine transform of $B$ with trivial intertwining operator. Hence, $\varrho(A) \subseteq \varrho(\overline{B})$, by [7, Theorem 3.3]. Since $\lambda - A$ and $\lambda - \overline{B}$ have bounded inverses, the proposition follows from [7, Theorem 2.5].

## 3. Subnormal operators.

Let $A$ be a densely defined closed operator $A$ in a Hilbert space $\mathcal{H}$. If $A$ satisfies $A^* \cdot A = A \cdot A^*$; namely, $\mathcal{D}(A^*) = \mathcal{D}(A)$ and $\|A^*\xi\| = \|A\xi\|$ for all $\xi \in \mathcal{D}(A)$, then $A$ is said to be *normal*.



DEFINITION 3.1. A densely defined operator $A$ in $\mathcal{H}$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace and a normal operator $N$ in $\mathcal{K}$ such that

$$\mathcal{D}(N) \cap \mathcal{H} \supseteq \mathcal{D}(A) \quad \text{and} \quad A\xi = N\xi \quad \text{for all } \xi \in \mathcal{D}(A).$$

A subnormal operator is formally hyponormal ([7], [11]). We refer to [1], [6], [11], [12] and the references cited there for earlier works on unbounded subnormal operators.

PROPOSITION 3.2. *Let $A$ be a subnormal operator in a Hilbert space $\mathcal{H}$ and let $B$ be a normal operator in $\mathcal{H}$. Suppose $B$ is quasi-affine to $A$ with positive intertwining operator. Then $B$ is a unique normal extension of $A$ in $\mathcal{H}$. Moreover, for any normal extension $N$ of $A$ in a possibly larger Hilbert space, the restriction of $N$ to $\mathcal{H}$,*

$$N|_{\mathcal{D}(N)\cap\mathcal{H}},$$

*is equal to $B$.*

Proof. Let $X$ be a positive quasi-invertible operator such that $X \cdot B \subseteq A \cdot X$. Suppose $N$ is a normal extension of $A$ to a Hilbert space $\mathcal{K}$ with $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$. Define the extensions $X_0$ and $B_0$ of $X$ and $B$ respectively to $\mathcal{K}$ as follows:

$$X_0 = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \quad \text{w.r.t. } \mathcal{H} \oplus \mathcal{H}^\perp$$

and

$$\mathcal{D}(B_0) = \mathcal{D}(B) \oplus \mathcal{H}^\perp \quad \text{and} \quad B_0(\xi \oplus \eta) = B\xi \oplus \eta.$$

Then, by our assumption, $B_0$ is normal and

$$X_0 \cdot B_0 \subseteq N \cdot X_0.$$

We recall a decomposition of a normal operator into real and imaginary parts by the spectral theory for normal operators ([14], Theorem 7.32).

Put

$$N_1 = \frac{1}{2}\overline{N + N^*}, \quad N_2 = \frac{1}{2i}\overline{N - N^*},$$
$$B_1 = \frac{1}{2}\overline{B_0 + B_0^*}, \quad B_2 = \frac{1}{2i}\overline{B_0 - B_0^*},$$

where $i = \sqrt{-1}$. Then $N_1$ and $N_2$ (resp. $B_1$ and $B_2$) are strongly commuting self-adjoint operators with $N = N_1 + iN_2$ and $N^* = N_1 - iN_2$ (resp. $B_0 = B_1 + iB_2$ and $B_0^* = B_1 - iB_2$). By the Fuglede–Putnam theorem ([8]), we have $X_0 \cdot B_0^* \subseteq N^* \cdot X_0$, so that

$$\begin{aligned} 2N_1 \cdot X_0 &\supseteq (N_1 + iN_2)X_0 + (N_1 - iN_2)X_0 \\ &\supseteq X_0(B_1 + iB_2) + X_0(B_1 - iB_2) \supseteq 2X_0 \cdot B_1|_{\mathcal{D}(B_1)\cap\mathcal{D}(B_2)}. \end{aligned}$$

Since $\mathcal{D}(B_0) = \mathcal{D}(B_1) \cap \mathcal{D}(B_2)$ is a core for $B_1$, it follows that

$$N_1 \cdot X_0 \supseteq X_0 \cdot B_1,$$

and analogously,

$$N_2 \cdot X_0 \supseteq X_0 \cdot B_2.$$

Since $N_j$ and $B_j$ $(j = 1, 2)$ are self-adjoint, it follows from Lemma 2.1 that

$$(i - N_j)^{-1} \cdot X_0 = X_0 \cdot (i - B_j)^{-1},$$

$j = 1, 2$. Clearly, $(i - N_j)^{-1}$ and $(i - B_j)^{-1}$ $(j = 1, 2)$ are all bounded normal operators. Since $X_0$ is positive, it is not difficult to see, by the same argument as in the proof of [2, Lemma 4.1], that $\mathcal{H} \oplus \{0\}$ reduces $(i - B_j)^{-1}$ and $(i - N_j)^{-1}$, and

$$(i - N_j)^{-1}|_{\mathcal{H}\oplus\{0\}} = (i - B_j)^{-1}|_{\mathcal{H}\oplus\{0\}} \quad (j = 1, 2).$$

Take $\delta \oplus 0$ in $\mathcal{D}(B_j)$ $(j = 1, 2)$. Then there are $\delta_1$ in $\mathcal{H}$ and $\delta_2$ in $\mathcal{H}^\perp$ such that $(i - B_j)(\delta \oplus 0) = \delta_1 \oplus \delta_2$. Since

$$\delta \oplus 0 = (i - B_j)^{-1}(\delta_1 \oplus \delta_2) = (i - B_j)^{-1}(\delta_1 \oplus 0) + (i - B_j)^{-1}(0 \oplus \delta_2)$$

and

$$(i - B_j)^{-1}(\delta_1 \oplus 0) \in \mathcal{H} \oplus \{0\} \quad \text{and} \quad (i - B_j)^{-1}(0 \oplus \delta_2) \in 0 \oplus \mathcal{H}^\perp,$$

we have $(i - B_j)^{-1}(0 \oplus \delta_2) = 0$, and so $\delta_2 = 0$. Thus

$$(i - B_j)(\delta \oplus 0) = \delta_1 \oplus 0 \in \mathcal{H} \oplus \{0\}.$$

By using the equality $(i - N_j)^{-1}|_{\mathcal{H}\oplus\{0\}} = (i - B_j)^{-1}|_{\mathcal{H}\oplus\{0\}}$ $(j = 1, 2)$, we have $(i - N_j)^{-1}(i - B_j)(\delta \oplus 0) = \delta \oplus 0$, which means that

$$\mathcal{D}(B_j) \cap (\mathcal{H} \oplus \{0\}) \subseteq \mathcal{D}(N_j) \cap (\mathcal{H} \oplus \{0\})$$

and

$$N_j(\delta \oplus 0) = B_j(\delta \oplus 0)$$

for all $\delta \in \mathcal{H}$ with $\delta \oplus 0$ in $\mathcal{D}(B_j)$ $(j = 1, 2)$.

Repeating the above argument with $N_j$ replacing $B_j$, we have $\mathcal{D}(B_j) \cap (\mathcal{H} \oplus \{0\}) = \mathcal{D}(N_j) \cap (\mathcal{H} \oplus \{0\})$. Thus,

$$N_j|_{\mathcal{D}(N_j)\cap(\mathcal{H}\oplus\{0\})} = B_j|_{\mathcal{D}(B_j)\cap(\mathcal{H}\oplus\{0\})} \quad (j = 1, 2).$$

Therefore

$$\mathcal{D}(B_0) \cap (\mathcal{H} \oplus \{0\}) = \mathcal{D}(N_1) \cap \mathcal{D}(N_2) \cap (\mathcal{H} \oplus \{0\}) = \mathcal{D}(N) \cap (\mathcal{H} \oplus \{0\})$$

and

$$N(\delta \oplus 0) = B_0(\delta \oplus 0) = B\delta \oplus 0$$

for all $\delta \in \mathcal{D}(B)$. Identifying $\mathcal{H} \oplus \{0\}$ with $\mathcal{H}$, we have obtained

$$N|_{\mathcal{D}(N)\cap\mathcal{H}} = B.$$

This completes the proof of the proposition.



THEOREM 3.3. *Let $A$ be a subnormal operator in a Hilbert space $\mathcal{H}$. Suppose there exists a normal operator $B$ in $\mathcal{H}$ which is quasi-affine to $A$. Then $A$ has a unique normal extension $C$ in the same Hilbert space $\mathcal{H}$, which is unitarily equivalent to $B$. To be precise, for any normal extension $N$ of $A$ in a possibly larger Hilbert space, the restriction of $N$ to $\mathcal{D}(N) \cap \mathcal{H}$ coincides with $C$.*

Proof. Let $X$ be a quasi-invertible operator such that $X \cdot B \subseteq A \cdot X$. Let $X = U \cdot P$ be the polar decomposition of $X$. Then $U$ is unitary and $P$ is a positive, injective operator on $\mathcal{H}$ satisfying

$$(U^* \cdot A \cdot U) \cdot P \supseteq P \cdot B.$$

Suppose $N$ is a normal extension of $A$ to a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace; $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$. Put

$$U_0 = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} \quad \text{w.r.t. } \mathcal{H} \oplus \mathcal{H}^\perp,$$

where $I$ is the identity operator on $\mathcal{H}^\perp$. Then it follows that $U_0^* \cdot N \cdot U_0$ is a normal extension of a subnormal operator $U^* \cdot A \cdot U$ to $\mathcal{K}$. By Proposition 3.2 and identifying $\mathcal{H} \oplus \{0\}$ with $\mathcal{H}$,

$$U_0^* \cdot N \cdot U_0|_{\mathcal{D}(U_0^* \cdot N \cdot U_0)\cap\mathcal{H}} = B.$$

It follows that

$$N|_{\mathcal{D}(N)\cap\mathcal{H}} = U \cdot B \cdot U^*.$$

Thus $N|_{\mathcal{D}(N)\cap\mathcal{H}}$ is unitarily equivalent to the normal operator $B$. This completes the proof of the theorem.

COROLLARY 3.4. *Under the same assumption as in Theorem 3.3, if $\varrho(A) \neq \emptyset$ then the closure of $A$ is normal and is unitarily equivalent to $B$.*

Proof. By Theorem 3.3, there is a normal extension $C$ in $\mathcal{H}$ of $A$ and so, $\overline{A} \subseteq C$. Since $C$ is normal, $\overline{A} = C$, by Lemma 2.3.

Remark 3.5. Kyung Hee Jin showed in [4, Theorem 3 and Corollary 4], by using the result of Stampfli and Wadhwa [10] for bounded hyponormal operators, that subnormality in the corollary above can be replaced by the weaker condition of "formal hyponormality" (he also discussed some conditions of quasi-affinity under which a given subnormal operator is normal); that is, let $A$ be a formally hyponormal operator with $\varrho(A) \neq \emptyset$ and suppose there exists a normal operator $B$ in $\mathcal{H}$ which is a quasi-affine transform of $A$. Then the closure of $A$ is normal and is unitarily equivalent to $B$.

In this result, in case the assumption that $A$ has non-empty resolvent set is dropped, what can we say about $A$? That is, it seems to be of interest to know whether such an operator $A$ has the property similar to the one stated in Theorem 3.3.

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# The dual of Besov spaces on fractals

by

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**Abstract.** For certain classes of fractal subsets $F$ of $\mathbb{R}^n$, the Besov spaces $B_\alpha^{p,q}(F)$ have been studied for $\alpha > 0$ and $1 \le p,q \le \infty$. In this paper the Besov spaces $B_\alpha^{p,q}(F)$ are introduced for $\alpha < 0$, and it is shown that the dual of $B_\alpha^{p,q}(F)$ is $B_{-\alpha}^{p',q'}(F)$, $\alpha \ne 0$, $1 < p,q < \infty$, where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.

**1. Introduction and notation.** The Besov spaces $B_\alpha^{p,q}(F)$ consisting of functions defined on a fractal subset $F$ of $\mathbb{R}^n$ have been studied for $\alpha > 0$ and $1 \le p,q \le \infty$ by the present authors, e.g. in [5]. In this paper we introduce Besov spaces $B_\alpha^{p,q}(F)$ for negative indices $\alpha$, and show that the dual of $B_\alpha^{p,q}(F)$, $\alpha > 0$, $1 \le p,q < \infty$ or $\alpha < 0$, $1 < p,q < \infty$, is $B_{-\alpha}^{p',q'}(F)$, where $p'$ and $q'$ are the dual indices of $p$ and $q$. This is well known if $F = \mathbb{R}^n$ (see e.g. [6, p. 178]). Duality theory involving the Lipschitz spaces $B_\alpha^{\infty,\infty}(F)$, $\alpha > 0$, where $F$ is a fractal set (and in fact an arbitrary closed set if $\alpha < 1$) was given in [3].

Our definition of $B_\alpha^{p,q}(F), \alpha < 0$, is in terms of atomic decompositions and is inspired by the atomic decomposition of distributions in $B_\alpha^{p,q}(\mathbb{R}^n)$ given in [1]. If $F = \mathbb{R}^n$ and $\alpha < 0$, our decomposition reduces to one given in [1], except that we use atoms normed in $L^p$ rather than smooth atoms normed in $L^\infty$. Atomic decompositions of functions in $B_\alpha^{p,q}(F), \alpha > 0$, $F$ a fractal set, were given in [4].

For the definition of $B_\alpha^{p,q}(F)$ we refer to Section 2 ($\alpha > 0$) and Section 3 ($\alpha < 0$). The duality results are given in Sections 4 and 5. In particular, Theorems 4.3 and 5.1 give precise statements describing the duality. Throughout the paper, the assumption on $F$ is that $F$ is a $d$-set preserving Markov's inequality. We now define these concepts, referring to [5, Chapter II] for the general theory.

Let $F$ be a closed subset of $\mathbb{R}^n$ and $0 < d \le n$, and denote by $B(x,r)$ the closed ball with center $x$ and radius $r$. A positive Borel measure $\mu$ with

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