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Mild integrated C-existence families

by

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Abstract. We study mild n times integrated C-existence families without the assumption of exponential boundedness. We present several equivalent conditions for these families. Hille–Yosida type necessary and sufficient conditions are given for the exponentially bounded case.

1. Introduction. Motivated by the study of the abstract Cauchy problem, two generalizations of strongly continuous semigroups, integrated semigroups and C-semigroups, have recently been introduced and received extensive attention (see [1–3, 5–7, 11–14, 16]). However, there are limitations to both integrated semigroups and C-semigroups. In order to cover more cases, [8] defined a pair of families of operators, one of which yields uniqueness, while the other yields existence of solutions of the abstract Cauchy problem, for all initial data in the image of C.

In this paper, we concentrate on mild C-existence families without assuming exponential boundedness. Section 2 offers a supplement for n times integrated C-semigroups. Section 3 contains the general definition and equivalent conditions for mild n times integrated C-existence families. Section 4 is devoted to the study of a Hille–Yosida type theorem, in which some equivalent conditions are found for exponentially bounded mild once integrated C-existence families. Finally, in Section 5, we provide some examples.

All operators are linear on a complex Banach space X. For an operator A, $D(A)$ and $\mathrm{Im}(A)$ will stand for the domain and image of A, respectively. We shall write $[D(A)]$ for the normed space $D(A)$ equipped with the graph norm: for $x \in D(A)$, $|||x||| = \|x\| + \|Ax\|$. The space $[D(A)]$ is complete if and only if A is closed in X. Finally, $B(X)$ is the algebra of all bounded linear operators T with $D(T) = X$, and $C \in B(X)$ will be fixed throughout this paper.

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2. A supplement for integrated C-semigroups. By definition, a strongly continuous family $\{S(t)\}_{t\ge 0}$ of bounded operators is a *C-semigroup* if

(i) $S(0)=C$,
(ii) $S(t)S(s)=CS(t+s)$ for all $s,t\ge 0$.

The following definition is an obvious generalization of the definition of a once integrated C-semigroup in [11].

DEFINITION 2.1. Suppose $n\in\mathbb{N}$. A strongly continuous family $\{S(t)\}_{t\ge 0}$ of bounded operators, with $S(0)=0$, is an *n times integrated C-semigroup* if $S(t)$ commutes with C for $t\ge 0$ and

$$S(t)S(s)x=\frac{1}{(n-1)!}\Big[\int_t^{t+s}(t+s-r)^{n-1}S(r)Cx\,dr - \int_0^s (t+s-r)^{n-1}S(r)Cx\,dr\Big] \quad \text{for } x\in X. \tag{2.1}$$

$\{S(t)\}_{t\ge 0}$ is said to be *nondegenerate* if $S(t)x=0$ for all $t\ge 0$ implies $x=0$. A C-semigroup is called a 0 *times integrated C-semigroup*.

LEMMA 2.2. *Suppose $n\in\mathbb{N}\cup\{0\}$. If the n times integrated C-semigroup $\{S(t)\}_{t\ge 0}$ is nondegenerate, then C is injective.*

Proof. Assume the contrary. If $n\in\mathbb{N}$, from (2.1) there exists $x\in X$ such that $x\ne 0$ and $S(t)S(s)x=0$ for all $t,s\ge 0$. This implies $S(s)x\in\bigcap_{t\ge 0}\{y: S(t)y=0\}=\{0\}$ for all $s\ge 0$ by the nondegeneracy of $\{S(t)\}_{t\ge 0}$. Hence $x=0$ by the same reason. If $n=0$, the lemma follows in the same way.

The family $\{S(t)\}_{t\ge 0}$ satisfying $S(t)\equiv 0$ for all $t\ge 0$ shows that the converse of Lemma 2.2 is not true. Here we assume $n\ge 1$.

If the n times integrated C-semigroup $\{S(t)\}_{t\ge 0}$ is nondegenerate, then

$$S(t)x-\frac{t^n}{n!}Cx=\int_0^t S(r)y\,dr$$

has at most one solution y for each $x\in X$. This enables us to define the generator as follows (cf. [17]).

DEFINITION 2.3. Suppose $n\in\mathbb{N}\cup\{0\}$. The *generator* A of the nondegenerate n times integrated C-semigroup $\{S(t)\}_{t\ge 0}$ is the following operator:

$$x\in D(A) \quad\text{and}\quad y=Ax \Leftrightarrow S(t)x=\frac{t^n}{n!}Cx+\int_0^t S(r)y\,dr \text{ for all } t\ge 0.$$

The implication (i)⇒(ii) of the following theorem for exponentially bounded n times integrated semigroups (that is, for $C=I$) appeared in [1, Prop. 3.3, p. 338], while the implication (ii)⇒(i) seems to be new.

THEOREM 2.4. *Supose $n\in\mathbb{N}\cup\{0\}$, A is linear and $\{S(t)\}_{t\ge 0}$ is a strongly continuous family of bounded operators. Then the following are equivalent.*

(i) *$\{S(t)\}_{t\ge 0}$ is a nondegenerate n times integrated C-semigroup generated by A.*

(ii) *C is injective and commutes with $S(t)$ for $t\ge 0$, A is closed and satisfies: $CA\subseteq AC$, $S(t)A\subseteq AS(t)$ for each $t\ge 0$ and $\int_0^t S(r)x\,dr\in D(A)$ for each $x\in X$, moreover,*

$$S(t)x-\frac{t^n}{n!}Cx=A\int_0^t S(r)x\,dr. \tag{2.2}$$

A is maximal with respect to (2.2) *and the inclusion $S(t)A\subseteq AS(t)$.*

Proof. (i)⇒(ii). We confine the proof to the case $n\in\mathbb{N}$, that of $n=0$ can be found in [8]. From Lemma 2.2, C is injective, from Definition 2.3, A is closed, and from (2.1), $S(t)$ and $S(s)$ commute. Then for every $x\in D(A)$, the equality

$$S(t)x-\frac{t^n}{n!}Cx=\int_0^t S(r)Ax\,dr$$

implies that

$$S(t)S(s)x-\frac{t^n}{n!}CS(s)x=\int_0^t S(r)S(s)Ax\,dr,$$

hence $S(s)x\in D(A)$ and $AS(s)x=S(s)Ax$ by the definition of A. Moreover,

$$S(t)Cx-\frac{t^n}{n!}C^2x=C\Big[S(t)x-\frac{t^n}{n!}Cx\Big]=\int_0^t S(r)CAx\,dr,$$

so that $Cx\in D(A)$ and $ACx=CAx$. Thus we have proved that $CA\subseteq AC$ and $S(t)A\subseteq AS(t)$ for $t\ge 0$.

To show (2.2), let $s,t\ge 0$ and $x\in X$. Then

$$\begin{aligned} S(t)\int_0^s S(r)x\,dr &= \int_0^s S(t)S(r)x\,dr \\ &= \frac{1}{(n-1)!}\int_0^s\Big[\int_t^{t+r}(t+r-u)^{n-1}S(u)Cx\,du \\ &\quad - \int_0^r (t+r-u)^{n-1}S(u)Cx\,du\Big]\,dr \end{aligned}$$

$$= \frac{1}{n!}\int_t^{t+s}(t+s-u)^n S(u)Cx\,du$$
$$-\frac{1}{n!}\int_0^s (t+s-u)^n S(u)Cx\,du + \frac{t^n}{n!}\int_0^s S(u)Cx\,du.$$

Hence

$$S(t)\int_0^s S(u)x\,du - \frac{t^n}{n!}C\int_0^s S(u)x\,du$$
$$= \frac{1}{n!}\Big[\int_t^{t+s}(t+s-u)^n S(u)Cx\,du - \int_0^t (t+s-u)^n S(u)Cx\,du\Big].$$

Interchange the roles of s and t to obtain

$$S(s)\int_0^t S(u)x\,du - \frac{s^n}{n!}C\int_0^t S(u)x\,du$$
$$= \frac{1}{n!}\Big[\int_s^{t+s}(t+s-u)^n S(u)Cx\,du - \int_0^t (t+s-u)^n S(u)Cx\,du\Big].$$

Therefore,

$$S(t)\int_0^s S(u)x\,du - \frac{t^n}{n!}C\int_0^s S(u)x\,du$$
$$= S(s)\int_0^t S(u)x\,du - \frac{s^n}{n!}C\int_0^t S(u)x\,du$$
$$= \int_0^t S(u)\Big[S(s)x - \frac{s^n}{n!}Cx\Big]\,du.$$

By the definition of A,

$$\int_0^s S(u)x\,du \in D(A) \quad\text{and}\quad A\int_0^s S(u)x\,du = S(s)x - \frac{s^n}{n!}Cx.$$

To show the maximality of A, let A_1 be another closed operator satisfying (2.2) and $S(t)A_1 \subseteq A_1S(t)$. Then for $x \in D(A_1)$,

$$S(t)x - \frac{t^n}{n!}Cx = \int_0^t S(r)A_1x\,dr.$$

By the definition of A, $x \in D(A)$ and $A_1x = Ax$, so that $A_1 \subseteq A$.

(ii)⇒(i). Since C is injective, it follows easily from (2.2) that $\{S(t)\}_{t\ge0}$ is nondegenerate. For $x \in D(A)$, $S(t)Ax = AS(t)x$ by hypotheses. This

together with (2.2) implies that

$$S(t)x - \frac{t^n}{n!}Cx = \int_0^t S(r)Ax\,dr,$$

hence

$$\frac{d}{dt}S(t)x = \frac{t^{n-1}}{(n-1)!}Cx + S(t)Ax. \tag{2.3}$$

Now we assume $x \in X$ and $t \ge r \ge 0$. Then (2.3) implies

$$\frac{d}{dr}\Big[S(t-r)\int_0^r S(u)x\,du\Big]$$
$$= -\frac{(t-r)^{n-1}}{(n-1)!}C\int_0^r S(u)x\,du - S(t-r)A\int_0^r S(u)x\,du + S(t-r)S(r)x$$
$$= -\frac{(t-r)^{n-1}}{(n-1)!}\int_0^r S(u)Cx\,du + \frac{r^n}{n!}S(t-r)Cx.$$

Integrating with respect to r from 0 to s for $s \le t$ yields

$$S(t-s)\int_0^s S(r)x\,dr = -\int_0^s \frac{(t-r)^{n-1}}{(n-1)!}\Big(\int_0^r S(u)Cx\,du\Big)\,dr$$
$$+ \int_{t-s}^t \frac{(t-r)^n}{n!}S(r)Cx\,dr.$$

Applying integration by parts to the last integral and multiplying both sides of the resulting equality by A, we find

$$S(t-s)S(s)x = \frac{s^n}{n!}S(t-s)Cx + S(t-s)A\int_0^s S(r)x\,dr \tag{2.4}$$
$$= \frac{s^n}{n!}S(t-s)Cx - \int_0^s \frac{(t-r)^{n-1}}{(n-1)!}\Big[S(r)Cx - \frac{r^n}{n!}C^2x\Big]\,dr$$
$$- \frac{s^n}{n!}\Big[S(t-s)Cx - \frac{(t-s)^n}{n!}C^2x\Big]$$
$$+ \int_{t-s}^t \frac{(t-r)^{n-1}}{(n-1)!}\Big[S(r)Cx - \frac{r^n}{n!}C^2x\Big]\,dr$$
$$= \int_{t-s}^t \frac{(t-r)^{n-1}}{(n-1)!}S(r)Cx\,dr - \int_0^s \frac{(t-r)^{n-1}}{(n-1)!}S(r)Cx\,dr.$$

Here we make use of the fact that the coefficient of C^2x equals

$$\int_0^s \left[\frac{(t-r)^{n-1}}{(n-1)!}\frac{r^n}{n!} - \frac{(t-r)^n}{n!}\frac{r^{n-1}}{(n-1)!}\right] dr + \frac{s^n}{n!}\frac{(t-s)^n}{n!}$$
$$= -\int_0^s \frac{d}{dr}\left[\frac{(t-r)^n}{n!}\frac{r^n}{n!}\right] dr + \frac{s^n}{n!}\frac{(t-s)^n}{n!} = 0.$$

Replace t by $t+s$ in (2.4) to obtain (2.1). From the maximality of A, it is easy to verify that A is the generator of $\{S(t)\}_{t\ge0}$.

Remark. It is worth noticing that the proof of (i)⇒(ii) in Theorem 2.4 is different from Arendt's proof of his Proposition 3.3 in [1, p. 338], based on the equality $(\lambda - A)^{-1} = \int_0^\infty \lambda^n e^{-\lambda t} S(t)\,dt$ ($\lambda > 0$ is large enough) and the resolvent equation $(\lambda-A)^{-1} - (\mu-A)^{-1} = (\mu-\lambda)(\lambda-A)^{-1}(\mu-A)^{-1}$.

3. Mild n times integrated C-existence families. Theorem 2.4 suggests to introduce the concept of mild n times integrated C-existence families via (2.2).

DEFINITION 3.1. Suppose that $n \in \mathbb{N}\cup\{0\}$ and A is closed. A strongly continuous family $\{S(t)\}_{t\ge0}$ of bounded operators is a *mild n times integrated C-existence family* for A if the following two conditions are satisfied.

(i) $\int_0^t S(r)x\,dr \in D(A)$ and $t \to A\int_0^t S(r)x\,dr$ is a continuous map from $[0,\infty)$ into X for all $x\in X$.

(ii) $S(t)x - (t^n/n!)Cx = A\int_0^t S(r)x\,dr$ for all $x \in X$.

If $n = 0$, we simply call $\{S(t)\}_{t\ge0}$ a *mild C-existence family* [8].

For Theorems 3.3 and 4.1, we shall rely on the following form of Arendt's integrated version of Widder's theorem.

LEMMA 3.2 (Arendt [1, Corollary 1.2]). *Suppose that G is a Banach space, $f:(0,\infty)\to G$ and $w\ge 0$. Then the following are equivalent.*

(i) *f is infinitely differentiable with $\|(r-w)^{n+1}f^{(n)}(r)\| \le Mn!$ for some $M\ge0$, all $r>w$ and $n=0,1,2,\ldots$*

(ii) *There exists $F:[0,\infty)\to G$ satisfying $F(0)=0$,*

(3.1) $$\limsup_{h\downarrow0}\frac{1}{h}\|F(t+h)-F(t)\| \le Me^{wt}$$

for all $t\ge0$ such that

$$f(r) = \int_0^\infty re^{-rt}F(t)\,dt \quad \text{for all } r>w.$$

THEOREM 3.3. *Suppose that A is closed, $n\in\mathbb{N}\cup\{0\}$ and there exist $w\ge0$ and $s\in\mathbb{C}$ such that $s-A$ and $r-A$ are injective for all $r>w$, and $\mathrm{Im}(C)\subseteq \mathrm{Im}((s-A)^n)$. Then the following are equivalent.*

(i) *There exists a mild n times integrated C-existence family $\{S(t)\}_{t\ge0}$ for A.*

(ii) *There exists a mild $(s-A)^{-n}C$-existence family $\{W(t)\}_{t\ge0}$ for A.*

If $W(t)$ in (ii) *is exponentially bounded with $\|W(t)\| = O(e^{wt})$ then* (ii) *and hence* (i) *are both equivalent to*

(iii) *for all $r>w$, $\mathrm{Im}((r-A)(s-A)^n) \supseteq \mathrm{Im}(C)$ and for $x\in X$,*

(3.2) $$(r-A)^{-1}(s-A)^{-n}Cx = \int_0^\infty e^{-rt}W(t)x\,dt.$$

Proof. Following the notation of [8, Theorem 4.2], we shall use

(1) $h_s(t) = e^{st}$, $P_k(t) = t^k$ ($k\in\mathbb{N}$, $s,t\in\mathbb{R}$),

(2) $*$ is convolution

$$(F*g)(t) = \int_0^t F(t-y)g(y)\,dy,$$

where $F:[0,\infty)\to B(X)$ is strongly continuous, and $g:[0,\infty)\to X$ is continuous,

(3) $$H_n = \sum_{k=1}^n (-1)^{n-k}\frac{P_{n-k}}{(n-k)!}h_s(s-A)^{-k}C.$$

(i)⇒(ii). For $x\in X$ and $t\ge0$, let

$$W(t)x = \left[\left((-1)^n\sum_{k=1}^n\binom{n}{k}s^k\frac{h_s}{(k-1)!}P_{k-1}\right)*S\right](t)x$$
$$+(-1)^nS(t)x + H_n(t)x.$$

Since

$$A\int_0^t\left(\int_0^r \frac{s^ke^{su}u^{k-1}}{(k-1)!}S(r-u)x\,du\right)dr$$
$$= A\int_0^t\left(\int_u^t S(r-u)x\,dr\right)\frac{s^ke^{su}u^{k-1}}{(k-1)!}\,du$$
$$= \int_0^t\left[S(t-u)x - \frac{(t-u)^n}{n!}Cx\right]\frac{s^ke^{su}u^{k-1}}{(k-1)!}\,du$$
$$= \int_0^t \frac{s^ke^{su}u^{k-1}}{(k-1)!}S(t-u)x\,du - \int_0^t\frac{s^ke^{su}u^{k-1}(t-u)^n}{(k-1)!n!}Cx\,du,$$

and

$$A\int_0^t S(u)x\,du = S(t)x - \frac{t^n}{n!}Cx,$$

and, in view of the equality $A(s-A)^{-1} = s(s-A)^{-1} - I$,

$$\begin{aligned} A\int_0^t H_n(u)x\,du &= \sum_{k=1}^n \int_0^t \frac{(-1)^{n-k}u^{n-k}}{(n-k)!}e^{su}\,du\,[A(s-A)^{-k}]Cx \\ &= \sum_{k=1}^n \frac{(-1)^{n-k}t^{n-k}e^{st}}{(n-k)!}(s-A)^{-k}Cx - (s-A)^{-n}Cx \\ &\quad + (-1)^n \int_0^t \frac{u^{n-1}e^{su}}{(n-1)!}\,du\,Cx, \end{aligned}$$

one has

$$A\int_0^t W(r)x\,dr = W(t)x - (s-A)^{-n}Cx + f(t)Cx,$$

where

$$\begin{aligned} f(t) &= (-1)^{n-1}\sum_{k=1}^n \binom{n}{k} s^k \int_0^t \frac{e^{su}u^{k-1}(t-u)^n}{(k-1)!n!}\,du \\ &\quad + (-1)^{n-1}\frac{t^n}{n!} + (-1)^n \int_0^t \frac{u^{n-1}e^{su}}{(n-1)!}\,du. \end{aligned}$$

Since

$$\int_0^t e^{su}u^{k-1}(t-u)^n\,du = (h_s P_{k-1}) * P_n(t),$$

one obtains for $r > \max\{\operatorname{Re} s, 0\}$,

$$\int_0^\infty e^{-rt}f(t)\,dt = \frac{(-1)^{n-1}}{r(r-s)^n} + \frac{(-1)^n}{r(r-s)^n} = 0$$

by several steps of calculation. Hence $f(t) \equiv 0$. It follows that $\{W(t)\}_{t\ge0}$ is a mild $(s-A)^{-n}C$-existence family for A.

(ii)⇒(i). Assume that $\{W(t)\}_{t\ge0}$ is a mild $(s-A)^{-n}C$-existence family for A. Set

$$S(t)x = (s-A)^n J^n W(t)x,$$

where $Jf(t) = \int_0^t f(r)dr$ for $f : [0,\infty) \to X$ a continuous function. Since

$$\begin{aligned} A\int_0^t S(r)x\,dr &= A(s-A)^n J^{n+1}W(t)x = (s-A)^n J^n A\int_0^t S(r)x\,dr \\ &= (s-A)^n J^n[W(t)x - (s-A)^{-n}Cx] = S(t)x - \frac{t^n}{n!}Cx, \end{aligned}$$

$\{S(t)\}_{t\ge0}$ is a mild n times integrated C-existence family for A.

Now assume that $\|W(t)\|$ is $O(e^{wt})$.

(ii)⇒(iii). Since for $r > w$ and $x \in X$,

$$\begin{aligned} (s-A)^{-n}Cx &= \int_0^\infty re^{-rt}(s-A)^{-n}Cx\,dt \\ &= \int_0^\infty re^{-rt}W(t)x\,dt - \int_0^\infty re^{-rt}A\int_0^t W(t)x\,dr\,dt \\ &= \int_0^\infty re^{-rt}W(t)x\,dt - A\int_0^\infty re^{-rt}\int_0^t W(t)x\,dr\,dt \\ &= (r-A)\int_0^\infty e^{-rt}W(t)x\,dt, \end{aligned}$$

it follows that $\operatorname{Im}((r-A)(s-A))^n) \supseteq \operatorname{Im}(C)$ and (3.2) holds.

(iii)⇒(ii). Without loss of generality, we may assume that $w = 0$, for if $B = A - w$ and $\{W(t)\}_{t\ge0}$ is a mild $(s-A)^{-n}C$-existence family for B, then $\{e^{wt}W(t)\}_{t\ge0}$ is the required existence family for A. We shall simply write $C_1 = (s-A)^{-n}C$ and $\widetilde{W}(t)x = \int_0^t W(r)x\,dr$. Since $w = 0$, $\|\widetilde{W}(t)\|$ is $O(e^t)$ and $\widetilde{W}(t)$ is Lipschitz continuous. From (3.2),

$$(r-A)^{-1}C_1x = r\int_0^\infty e^{-rt}\widetilde{W}(t)x\,dt. \tag{3.3}$$

This implies the existence of $M \ge 0$ such that

$$\left\|\left(\frac{d}{dr}\right)^m \frac{1}{r}(r-A)^{-1}C_1x\right\| \le \frac{Mm!\|x\|}{(r-1)^{m+1}} \tag{3.4}$$

and

$$\left\|\left(\frac{d}{dr}\right)^m (r-A)^{-1}C_1x\right\| \le \frac{Mm!\|x\|}{(r-1)^{m+1}} \tag{3.5}$$

for $m \ge 0$ and $r > 1$. To obtain (3.5), we make use of Lemma 3.2. From (3.4), (3.5) and the identity

$$A\left[\frac{1}{r}(r-A)^{-1}C_1x\right] = (r-A)^{-1}C_1x - \frac{1}{r}C_1x,$$

one has

$$\left\|\left(\frac{d}{dr}\right)^m\left[\frac{1}{r}(r-A)^{-1}C_1x\right]\right\|_{[D(A)]}\le\frac{M_1m!\|x\|}{(r-1)^{m+1}}$$

for all $r>1$, with some $M_1\ge 0$.

By Lemma 3.2 again, there exists a Lipschitz continuous function $W_x:[0,\infty)\to[D(A)]$ such that $W_x(0)=0$ and for $r>0$,

$$\begin{aligned}(3.6)\quad &\frac{1}{r+1}(r+1-A)^{-1}Cx\\ &=r\int_0^\infty e^{-rt}W_x(t)\,dt=\left[1-\frac{1}{r+1}\right](r+1)\int_0^\infty e^{-(r+1)t}e^tW_x(t)\,dt\\ &=(r+1)\int_0^\infty e^{-(r+1)t}e^tW_x(t)\,dt\\ &\quad-(r+1)\int_0^\infty e^{-(r+1)t}\Big(\int_0^t e^sW_x(s)\,ds\Big)\,dt\\ &=(r+1)\int_0^\infty e^{-(r+1)t}\Big[e^tW_x(t)-\int_0^t e^sW_x(s)\,ds\Big]\,dt.\end{aligned}$$

Integrate in (3.3) by parts and compare the result with (3.6) to obtain

$$\int_0^t\widetilde{W}(s)x\,ds=e^tW_x(t)-\int_0^t e^sW_x(s)\,ds\in D(A).$$

Therefore

$$\begin{aligned}\int_0^\infty r^2e^{-rt}tC_1x\,dt=C_1x&=(r-A)\int_0^\infty re^{-rt}\widetilde{W}(t)x\,dt\\ &=\int_0^\infty r^2e^{-rt}\Big[\widetilde{W}(t)x-A\int_0^t\widetilde{W}(s)x\,ds\Big]\,dt.\end{aligned}$$

By the uniqueness of Laplace transform,

$$(3.7)\qquad tC_1x=\widetilde{W}(t)x-A\int_0^t\widetilde{W}(s)x\,ds.$$

A being closed, differentiating (3.7) with respect to t yields

$$\int_0^t W(s)x\,ds=\widetilde{W}(t)x\in D(A)\quad\text{and}\quad W(t)x-C_1x=A\int_0^t W(s)x\,ds.$$

Thus $\{W(t)\}_{t\ge0}$ is a mild $(s-A)^{-n}C$-existence family for A.

Remark. It is well known that $W(t)$ is exponentially bounded if and only if $S(t)$ is exponentially bounded (cf. [8, Theorem 4.2]). Thus if one of $W(t)$, $S(t)$ is exponentially bounded, then from the proof of (iii)⇒(ii) of Theorem 3.3 both

$$\int_0^t W(s)x\,ds\in D(A)\quad\text{and}\quad\int_0^t S(s)x\,ds\in D(A)$$

are true for all $x\in X$. Hence condition (3) of [8, Definition 4.1] can be removed. Similarly, the assumption

$$\int_0^t w_1(s)x\,ds\in D(A)\quad\text{for all }x\in X$$

in [8, Theorem 2.8(c) and Definition 3.2(a)] can also be removed.

Moreover, when $\{S(t)\}_{t\ge0}$ is exponentially bounded with $\|S(t)\|=O(e^{wt})$, from the equality $A\int_0^t S(s)x\,ds=S(t)x-(t^n/n!)Cx$, it is easy to verify that

$$(3.8)\qquad (r-A)^{-1}Cx=r^n\int_0^\infty e^{-rt}S(t)x\,dt\quad(r>w).$$

We need this equality later.

4. Hille–Yosida type theorem.

This section is devoted to a Hille–Yosida type theorem for mild once integrated C-existence families. What we shall obtain are equivalent conditions. A sufficient condition was considered in [8, Theorem 5.1].

THEOREM 4.1. *Suppose A is closed and there exists $w\ge0$ such that $r-A$ is injective for all $r>w$. Then the following are equivalent.*

(i) *$\mathrm{Im}((r-A)^m)\supseteq\mathrm{Im}(C)$ and for every $w'>w$, there exists $M\ge0$ such that*

$$\|(r-A)^{-m}C\|\le M(r-w')^{-m}$$

for $m\ge1$ and $r>w'$.

(ii) *For all $s>w$, there exists a mild $(s-A)^{-1}C$-existence family $\{W(t)\}_{t\ge0}$ for A such that*

$$(4.1)\qquad \limsup_{h\downarrow0}\frac{1}{h}\|W(t+h)-W(t)\|=O(e^{w't})$$

for every $w'>w$.

(iii) *There exists a mild once integrated C-existence family $\{S(t)\}_{t\ge0}$ for A such that*

$$(4.2)\qquad \limsup_{h\downarrow0}\frac{1}{h}\|S(t+h)-S(t)\|=O(e^{w't})$$

for every $w'>w$.

Proof. (i)⇒(ii) follows from Lemma 3.2 and [8, Theorem 5.1].

(ii)⇒(iii). The existence of $W(t)$ implying that of $S(t)$ has been proved in Theorem 3.3(ii)⇒(i). It remains to show that (4.1) implies (4.2). Since

$$S(t)x = (s-A)JW(t)x = (s-A)\int_0^t W(r)x\,dr = s\int_0^t W(r)x\,dr - W(t)x + (s-A)^{-1}Cx,$$

we have

$$S(t+h)x - S(t)x = s\int_t^{t+h} W(r)x\,dr - [W(t+h)x - W(t)x].$$

Hence

$$\limsup_{h\to 0}\frac{1}{h}\|S(t+h)-S(t)\| \le |s|\|W(t)\| + \limsup_{h\to 0}\frac{1}{h}\|W(t+h)-W(t)\| = O(e^{w't}). \tag{4.3}$$

(iii)⇒(i) is a consequence of Lemma 3.2 and (3.9) for $n=1$.

Remark. If $w>0$, we may put $w'=w$ in Theorem 4.1(i), (ii) and obtain more precise equivalent conditions. In fact, if (4.1) holds for $w'=w$, then $\|W(t)\| = O(e^{wt})$. This implies (4.3) and hence (4.2) with w' replaced by w.

5. Examples. In this section, we give two examples to show the applications of the theorems in the previous sections. The following obvious lemma is needed for Example 5.2.

LEMMA 5.1. *Assume that A is closed, $D(A)$ is dense in X, and $C\in B(X)$ is injective. Then the following are equivalent.*

(i) *A generates a once integrated C-semigroup $\{S(t)\}_{t\ge 0}$ that is Lipschitz continuous on bounded intervals.*

(ii) *A generates a C-semigroup.*

Proof. (i)⇒(ii). Let

$$Z = \{x : S(t)x\in C^1([0,\infty),X)\}.$$

Then Z is closed. Let $\{x_n\}\subset Z$ be such that $\{x_n\}\to x$. Since, for t in every bounded interval $I\subset[0,\infty)$, $\|S'(t)(x_n-x_m)\|\le M\|x_n-x_m\|$ for some $M>0$, $\{S'(t)x_n\}$ uniformly converges on I to a continuous X-valued

function $y(t)$. Taking limits in

$$S(t)x_n = \int_0^t S'(s)x_n\,ds$$

yields

$$S(t)x = \int_0^t y(s)\,ds.$$

Hence $S(t)x$ is continuously differentiable and $x\in Z$. Since $D(A)$ ($\subseteq Z$) is dense in X and Z is closed, we have $Z=X$. Differentiating both sides of the equality

$$S(t)x - tCx = A\int_0^t S(s)x\,ds \quad (\text{for all } x\in X)$$

yields

$$S'(t)x - Cx = A\int_0^t S'(s)x\,ds. \tag{5.1}$$

From Theorem 2.4, $\{S'(t)\}_{t\ge 0}$ is a C-semigroup generated by A.

(ii)⇒(i). Obvious.

The following example is a version of [8, Example 7.1].

EXAMPLE 5.2. Let

$$X = \{\text{continuous } f:\mathbb{R}\to\mathbb{C} \text{ satisfying } \lim_{x\to\infty} f(x)e^{x^2} = 0\}.$$

Then X, endowed with the norm

$$\|f\| = \sup_{x\in\mathbb{R}}|F(x)e^{x^2}|,$$

is a Banach space. Let

$$A = \frac{d}{dx} \quad (A \text{ has maximal domain in } X),$$
$$(C_1f)(x) = e^{-x^2}f(x)$$

and let

$$[S(t)f](x) = \int_0^t e^{-(x+s)^2}f(x+s)\,ds.$$

Then $\{S(t)\}_{t\ge 0}$ is a mild once integrated C_1-existence family for A. In fact, for every $f \in X$,

$$A \int_0^t S(r)f\,dr = \frac{d}{dx}\int_0^t \Big[\int_0^r e^{-(x+s)^2} f(x+s)\,ds\Big]\,dr$$
$$= \int_0^t e^{-(x+s)^2} f(x+s)\,ds - tC_1 f = S(t)f - tC_1 f.$$

Now, $\{S(t)\}_{t\ge0}$ is not exponentially bounded. Let $f(x) = e^{-2x^2}$. Then $f \in X$, $\|f\| = 1$ and for sufficiently large $t > 0$,

$$\|S(t)f\| = \sup_{x\in\mathbb{R}} |e^{x^2}(S(t)f)(x)| \ge e^{t^2} \int_0^t e^{-(s-t)^2} f(s-t)\,ds$$
$$= e^{t^2}\int_0^t e^{-3s^2}\,ds \ge Ke^{t^2}$$

for some $K > 0$. On the other hand, for any C, A does not generate a once integrated C-semigroup with Lipschitz continuity on bounded intervals. If it did, then from the density of $D(A)$ in X, A would generate a C-semigroup by Lemma 5.1. This is impossible by [8, Example 7.1].

Furthermore, we cannot even say that, for any C and any $s \in \mathbb{C}$, there is a mild $(s-A)^{-1}C$-existence family for A. In fact, since $f(x) = ke^{sx}$ is the solution of the equation $sf - df/dx = 0$, f is not in X for $k \ne 0$. Hence $s-A$ is injective. For $g \in X$, $f(x) = e^{sx}[\int_0^x e^{-su}Cg(u)\,du + k]$ is the solution of the equation

$$[(s-A)f](x) = Cg(x).$$

If $f \in X$, then k should equal both

$$-\int_0^\infty e^{-su}Cg(u)\,du \quad\text{and}\quad -\int_0^{-\infty} e^{-su}Cg(u)\,du.$$

This might be impossible for some g. Thus the inclusion $\mathrm{Im}(s-A) \supseteq \mathrm{Im}(C)$ may not be true in general.

The following example is a version of [8, Example 7.5].

EXAMPLE 5.3. Suppose that X_1 and X_2 are Banach spaces, L is a closed operator from a subspace of X_2 into X_1, and G_i is the generator of an exponentially bounded once integrated nondegenerate semigroup on X_i for $i = 1,2$. We assume that $\|S_i(t)\| = O(e^{w_i t})$ for some $w_i \ge 0$ and that $D(G_2) \subseteq D(L)$. Then there exists an exponentially bounded mild once integrated C-existence family for A where

$$A = \begin{bmatrix} G_1 & (s-G_1)^{-1}L \\ 0 & G_2 \end{bmatrix}, \quad C = \begin{bmatrix} I & L(s-G_2)^{-1} \\ 0 & (s-G_2)^{-1} \end{bmatrix}$$

for $s > \max\{w_1, w_2\}$.

Proof. From Theorem 3.3, G_i generates an exponentially bounded $(s-G_i)^{-1}$-semigroup $W_i(\cdot)$ for $i = 1,2$. Set

$$W(t) = \begin{bmatrix} W_1(t) & (W_1 * L(s-G_2)^{-1}W_2)(t) \\ 0 & W_2(t)(s-G_2)^{-1} \end{bmatrix}.$$

Using the equalities

$$(r-G_i)^{-1}(s-G_i)^{-1}x = \int_0^\infty e^{-rt}W_i(t)x\,dt \quad (x \in X_i,\ i = 1,2),$$

$$\int_0^\infty e^{-rt}\int_0^t [(W_1 * L(s-G_2)^{-1}W_2)(u)x\,du]\,dt$$
$$= \int_0^\infty e^{-rt}W_1(t)\,dt\,[L(s-G_2)^{-1}]\int_0^\infty e^{-rt}W_2(t)x\,dt$$
$$= (r-G_1)^{-1}(s-G_1)^{-1}L(s-G_2)^{-2}(r-G_2)^{-1}x$$

for $r > \max\{w_1, w_2\}$ and $x \in X_2$, and

$$(r-A)^{-1} = \begin{bmatrix} (r-G_1)^{-1} & (r-G_1)^{-1}(s-G_1)^{-1}L(r-G_2)^{-1} \\ 0 & (r-G_2)^{-1} \end{bmatrix},$$

one obtains, after several steps of computation,

$$(r-A)^{-1}(s-A)^{-1}C = \int_0^\infty e^{-rt}W(t)x\,dt \quad (x \in X_1,\ x \in X_2).$$

Then $S(t) = (s-A)JW(t)$ is the required mild once integrated C-existence family for A by Theorem 3.3 again.

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Banach spaces which admit a norm with the uniform Kadec–Klee property

by

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Abstract. Several results are established about Banach spaces $\mathfrak{X}$ which can be renormed to have the uniform Kadec–Klee property. It is proved that all such spaces have the complete continuity property. We show that the renorming property can be lifted from $\mathfrak{X}$ to the Lebesgue–Bochner space $L_2(\mathfrak{X})$ if and only if $\mathfrak{X}$ is super-reflexive. A basis characterization of the renorming property for dual Banach spaces is given.

1. Introduction. A sequence $\{x_n\}$ in a Banach space $\mathfrak{X}$ is *separated* (respectively, *ε-separated*) if $\inf\{\|x_n - x_m\| : n \neq m\} > 0$ (respectively, $\geq \varepsilon$). Recall that $\mathfrak{X}$ has the *Kadec–Klee property* if every separated weakly convergent sequence $\{x_n\}$ in the closed unit ball of $\mathfrak{X}$ converges to an element of norm strictly less than one. We say that $\mathfrak{X}$ has the *uniform Kadec–Klee* (*UKK*) *property* (or that $\mathfrak{X}$ has a *UKK norm*) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that every ε-separated weakly convergent sequence $\{x_n\}$ in the closed unit ball of $\mathfrak{X}$ converges to an element of norm less than $1 - \delta$. This notion was introduced by Huff in [15]. Clearly, if $\mathfrak{X}$ has the Schur property (that is, if weak and norm sequential convergence are the same) or if $\mathfrak{X}$ is uniformly convex then $\mathfrak{X}$ has the UKK property. While uniformly convex spaces are necessarily reflexive, it turns out that many classical non-reflexive spaces, e.g. the Hardy spaces H_1 of analytic functions on the ball or on the polydisk [1], the Lorentz spaces $L_{p,1}(\mu)$ [5, 9], and the trace class C_1 [13, 25], all have UKK norms.

The question of characterizing the Banach spaces which are isomorphic to uniformly convex spaces has been studied intensively. This paper takes

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