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Compactness and countable compactness in weak topologies

by

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Abstract. A bounded closed convex set K in a Banach space X is said to have quasi-normal structure if each bounded closed convex subset H of K for which $\operatorname{diam}(H) > 0$ contains a point u for which $\|u - x\| < \operatorname{diam}(H)$ for each $x \in H$. It is shown that if the convex sets on the unit sphere in X satisfy this condition (which is much weaker than the assumption that convex sets on the unit sphere are separable), then relative to various weak topologies, the unit ball in X is compact whenever it is countably compact.

1. Introduction. This paper concerns topologies weaker than the norm topology on a Banach space, and a concept, introduced independently by Soardi [17] and Wong [19], called quasi-normal structure. A bounded convex subset K of a Banach space X is said to have *quasi-normal structure* (called close-to-normal structure in [19]) if each bounded closed convex subset H of K for which $\operatorname{diam}(H) > 0$ contains a point u satisfying $\|u - v\| < \operatorname{diam}(H)$ for every $v \in H$. We show here that for various weak topologies in spaces whose convex subsets of the unit sphere have quasi-normal structure, countable compactness implies compactness.

Quasi-normal structure is a very mild assumption which to date has received little attention. It is known, however, that a bounded convex subset K of a Banach space X has quasi-normal structure if either (a) X is separable ([17], [19]), (b) X is strictly convex ([19]), or (c) K is weakly sequentially compact and X has Kadec–Klee norm ([19]). In fact, it is shown in [4] and [20] (cf. also [6]) that if X is separable, then X has an equivalent norm which has a much stronger property called normal structure. Recall that a Banach space X is said to have *normal structure* if every bounded convex subset H of X which contains more than one point contains a point x_0 such that

$$\sup\{\|x_0 - u\| : u \in H\} < \sup\{\|x - y\| : x, y \in H\}.$$

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It appears to be an open problem whether every reflexive space has an equivalent norm with normal structure. However, Troyanski [18] has shown that every weakly compactly generated Banach space admits a locally uniformly rotund (hence Kadec–Klee) norm. Thus, in particular, every reflexive Banach space has an equivalent norm relative to which all bounded closed convex sets have quasi-normal structure. (For more information on these topics we refer the reader to Deville–Godefroy–Zizler [5]. Also see [9] and [14].)

We emphasize that our basic assumption here only requires that convex subsets of the *unit sphere* (hence of any sphere) of the space have quasi-normal structure. (The unit sphere of X is the set $S = \{x \in X : \|x\| = 1\}$.)

For purposes of this paper we shall call a family $\mathcal{S}$ of *convex* subsets of a Banach space X which contains $\emptyset$ and is closed under intersections a *convexity structure*. (In a metric space setting the assumption of convexity is dropped.) If $\mathcal{S}$ also contains the closed balls of X then $\mathcal{S}$ is a subbase for a topology τ on X for which the norm closed balls are τ-closed. In this case the members of $\mathcal{S}$ are precisely the τ-closed convex subsets of X. An example of such a topology is of course the weak topology itself, where $\mathcal{S}$ is the family of all closed convex sets. On the other hand, the coarsest such topology is the ball topology b_X introduced by Corson and Lindenstrauss in 1966 ([3]) and recently studied extensively by Godefroy and Kalton [8].

In the separable spaces any convexity structure $\mathcal{S}$ consisting of closed sets is compact whenever it is countably compact, because such spaces are Lindelöf (cf. also [13]). Our main result shows that this fact is true in a much stronger sense when the convexity structures are appropriately restricted. In fact, it is not clear that our underlying assumptions are in any way related to separability.

Throughout we suppose $\mathcal{S}$ is a given convexity structure on X, and for convenience we restrict $\mathcal{S}$ (hence the topology τ generated by $\mathcal{S}$) to the unit ball B of X. We shall assume that $\mathcal{S}$ contains the closed balls of B, and we shall also need to assume that $\mathcal{S}$ *contains its closed r-neighborhoods*. By this we mean that for each $D \in \mathcal{S}$ and $r > 0$, the set $\bigcup_{x\in D} B(x;r) \cap B \in \mathcal{S}$. (Here and throughout we use $B(x;r)$ to denote the closed ball centered at x with radius r.) While the assumption that $\mathcal{S}$ contains the closed balls of X is quite mild, the significance of the r-neighborhood assumption is less obvious (although a convexity structure which consists of all closed convex sets and is countably compact obviously satisfies this condition). On the other hand, the interplay between the weak topology and the underlying geometry of the space is further highlighted by the abstract treatment of Section 3.

Finally, for $x \in B$ and $D \in \mathcal{S}$, define $\operatorname{dist}(x,D) = \inf\{\|x-y\| : y \in D\}$, and let

$$P(x,D) = \{z \in D : \|x-z\| = \operatorname{dist}(x,D)\}.$$

The sets $\{P(x,D) : x \in B,\ D \in \mathcal{S}\}$ are called the *proximinal sets* of $\mathcal{S}$. If $D \in \mathcal{S}$ is τ-countably compact and $D \neq \emptyset$, note that

$$P(x,D) = \bigcap_{n=1}^{\infty} \{z \in D : \|x-z\| \le \operatorname{dist}(x,D) + 1/n\} \neq \emptyset.$$

In particular, the proximinal sets in $\mathcal{S}$ are always convex subsets of *spheres*. Consequently, the proximinal sets in $\mathcal{S}$ will have quasi-normal structure if the convex subsets of the unit sphere have this property.

2. Main result.

Our main result is the following.

THEOREM 1. *Let X be a Banach space, let $\mathcal{S}$ be a convexity structure on X which contains the closed balls of X as well as its closed r-neighborhoods, and let τ be the topology generated by $\mathcal{S}$. Then if the proximinal sets of $\mathcal{S}$ have quasi-normal structure, an element K of $\mathcal{S}$ is τ-compact iff it is τ-countably compact.*

A very special case of the theorem occurs if X has strictly convex norm or, more generally, if the convex subsets of the unit sphere are separable (although in this instance the theorem has a simpler proof (see [2])). On the other hand, if τ is the weak topology on X then the above is a special case of the Eberlein–Shmul'yan theorem. However, our result offers a new approach and could be new for other topologies, in particular for $\tau = b_X$.

Recall that a Banach space X is said to have the *finite-infinite intersection property* ($IP_{f,\infty}$) if for every collection $\{B_\alpha : \alpha \in \mathcal{I}\}$ of closed balls in X such that $\bigcap B_\alpha \neq \emptyset$ there is a finite subset $\mathcal{F} \in \mathcal{I}$ such that $\bigcap\{B_\alpha : \alpha \in \mathcal{F}\} \neq \emptyset$. If the index set $\mathcal{I}$ is assumed always to be countable, we say that X has the *finite-countably infinite intersection property* ($cIP_{f,\infty}$). It is observed in [8, Theorem 2.8] that the unit ball B of X is b_X-compact iff X has the $IP_{f,\infty}$. Compactness of B in the ball topology is interesting because if b_X is Hausdorff, this is equivalent to saying that X is isometric to a dual space ([7]).

The following is an immediate corollary to Theorem 1.

COROLLARY 2. *Let X be a Banach space for which the convex subsets of the unit sphere have quasi-normal structure. Suppose that r-neighborhoods of ball intersections in X are themselves ball intersections. Then the unit ball B of X is b_X-compact iff X has $cIP_{f,\infty}$.*

Proof. If $\mathcal{S}$ is the convexity structure generated by intersections of the closed balls in X, then the assumption on the unit sphere of X implies that all the proximinal sets in $\mathcal{S}$ have quasi-normal structure.

Proof of Theorem 1. Assume K is τ-countably compact, and let

$$\Sigma = \{D : D = K \cap S,\ S \in \mathcal{S}\}.$$

Note that Σ is itself a convexity structure. It suffices to show that $\bigcap_\alpha D_\alpha \neq \emptyset$ for an arbitrary descending chain $\{D_\alpha : \alpha \in A\}$ of nonempty sets in Σ. The conclusion will then follow from Alexander's subbase theorem (Kelley [10], p. 139).

Assume the theorem is false, let Γ be the *smallest* ordinal (obviously uncountable) for which there exists a chain $\{D_\alpha : \alpha \in \Gamma\}$ of sets in Σ such that $\bigcap_{\alpha\in\Gamma} D_\alpha = \emptyset$, and consider the family Λ of all Σ-subchains of $\{D_\alpha\}$; specifically, $\{B_\alpha\} \in \Lambda$ if $\{B_\alpha\}$ is a descending chain of nonempty sets in Σ with $B_\alpha \subseteq D_\alpha$, $\alpha \in \Gamma$. We order Λ as follows. For $\{B_\alpha\}, \{C_\alpha\} \in \Lambda$, we say $\{C_\alpha\} \preceq \{B_\alpha\}$ if $C_\alpha \subseteq B_\alpha$, $\alpha \in \Gamma$.

Now let $\{B_\alpha\} \in \Lambda$ and $\beta \in \Gamma$, define

$$r_\beta(\{B_\alpha\}) = \inf_{x\in B_\beta}\{\lim_\alpha \operatorname{dist}(x, B_\alpha)\},$$

and (using the fact that $\{r_\beta(\{B_\alpha\})\}$ is increasing with β) let

$$r(\{B_\alpha\}) = \lim_\beta r_\beta(\{B_\alpha\}).$$

Also, let

$$\overline{r}(\{D_\alpha\}) = \sup\{r(\{B_\alpha\}) : \{B_\alpha\} \preceq \{D_\alpha\}\}.$$

Now let $\{D_\alpha^0\} = \{D_\alpha\}$, and having defined $\{D_\alpha^n\} \in \Lambda$, choose $\{D_\alpha^{n+1}\} \preceq \{D_\alpha^n\}$ so that

$$r(\{D_\alpha^{n+1}\}) \geq \overline{r}(\{D_\alpha^n\}) - \frac{1}{n+1}.$$

By countable compactness,

$$\emptyset \neq D_\alpha^\infty := \bigcap_{n=1}^\infty D_\alpha^n.$$

We assert that the chain $\{D_\alpha^\infty\}$ has the property that $r(\{B_\alpha\}) = r(\{D_\alpha^\infty\})$ for every $\{B_\alpha\} \preceq \{D_\alpha^\infty\}$. Indeed, let $\{B_\alpha\} \preceq \{C_\alpha\}$ and fix $\beta \in \Gamma$. Then, since $B_\alpha \subseteq C_\alpha$, if $x \in B$, then

$$\operatorname{dist}(x, B_\alpha) \geq \operatorname{dist}(x, C_\alpha),$$

so

$$\lim_\alpha \operatorname{dist}(x, B_\alpha) \geq \lim_\alpha \operatorname{dist}(x, C_\alpha),$$

from which

$$\inf_{x\in B_\beta}\{\lim_\alpha \operatorname{dist}(x, B_\alpha)\} \geq \inf_{x\in B_\beta}\{\lim_\alpha \operatorname{dist}(x, C_\alpha)\} \geq \inf_{x\in C_\beta}\{\lim_\alpha \operatorname{dist}(x, C_\alpha)\}.$$

Therefore $r(\{B_\alpha\}) \geq r(\{C_\alpha\})$ whenever $\{B_\alpha\} \preceq \{C_\alpha\}$. It follows that if $\{B_\alpha\} \preceq \{D_\alpha^\infty\}$, then

$$r(\{D_\alpha^{n+1}\}) \leq r(\{D_\alpha^\infty\}) \leq r(\{B_\alpha\}) \leq \overline{r}(\{D_\alpha\}) \leq r(\{D_\alpha^{n+1}\}) + \frac{1}{n+1}$$

from which the conclusion follows.

(1) *Therefore (replacing $\{D_\alpha\}$ with $\{D_\alpha^\infty\}$), without loss of generality we may assume that whenever $\{B_\alpha\} \preceq \{D_\alpha\}$,*

$$r = r(\{D_\alpha\}) \equiv r(\{B_\alpha\}).$$

Now observe that since $r(\{D_\alpha\}) = \lim_\beta r_\beta(\{D_\alpha\})$, for each positive integer n there exists $\beta_n \in \Gamma$ such that if $\beta \geq \beta_n$, then $r_\beta(\{D_\alpha\}) \geq r(\{D_\alpha\}) - 1/n$. Consequently (since Γ is uncountable), there exists a smallest $\beta \in \Gamma$ such that $r_\gamma(\{D_\alpha\}) \equiv r(\{D_\alpha\})$ for $\gamma \geq \beta$. By replacing $\{D_\alpha\}$ with $\{D_\alpha \cap D_\beta\}_{\alpha\in\Gamma}$ we may take $\beta = 1$ and consequently we may assume that for each $\gamma \in \Gamma$,

$$r_\gamma(\{D_\alpha\}) = r(\{D_\alpha\}) = r.$$

Now suppose $\{B_\alpha\} \preceq \{D_\alpha\}$. Then

$$r \leq \inf_{x\in B_1}\{\lim_\alpha \operatorname{dist}(x, B_\alpha)\} = r_1(\{B_\alpha\}) \leq \inf_{x\in B_\beta}\{\operatorname{dist}(x, B_\alpha)\}$$
$$\leq \lim_\beta(\inf_{x\in B_\beta}\{\lim_\alpha \operatorname{dist}(x, B_\alpha)\}) = r.$$

(2) *By* (1) *and the above we may assume that whenever $\{B_\alpha\} \preceq \{D_\alpha\}$,*

$$r = \inf_{x\in B_1}\{\lim_\alpha \operatorname{dist}(x, B_\alpha)\} = r(\{B_\alpha\}).$$

Now let $\{B_\alpha\} \preceq \{D_\alpha\}$ and, for each $x \in B_1$, let $d_x = \lim_\alpha \operatorname{dist}(x, B_\alpha)$. Then $r_1(\{B_\alpha\}) = \inf\{d_x : x \in B_1\} = r$. Let

$$C_n = \{x \in B_1 : d_x \leq r + 1/n\}.$$

Then for $x \in B_1$, $d_x \leq r + 1/n$ iff $\operatorname{dist}(x, B_\alpha) \leq r + 1/n$ for all $\alpha \in \Gamma$ iff for all $\alpha \in \Gamma$ there exists $u \in B_\alpha$ such that $\|x - u\| \leq r + 1/n$ iff $x \in B(u; r + 1/n) \subseteq \bigcup_{u\in B_\alpha} B(u; r + 1/n)$. Therefore

$$C_n = \Big(\bigcap_{\alpha\in\Gamma}\Big(\bigcup_{u\in B_\alpha} B(u; r + 1/n)\Big)\Big) \cap B_1.$$

Consequently (since Σ contains its closed r-neighborhoods), $C_n \in \Sigma$. Since $C_n \neq \emptyset$ for each n, $C(\{B_\alpha\}) := \bigcap_{n=1}^\infty C_n$ is a nonempty *convex* subset of B_1. Note in particular that

$$C(\{B_\alpha\}) = \{x \in B_1 : d_x = r\}.$$

The fact just established and observation (2) yield

(3) *If $\{C_\alpha\} \preceq \{B_\alpha\} \preceq \{D_\alpha\}$, then $C(\{C_\alpha\}) \subseteq C(\{B_\alpha\}) \subseteq C(\{D_\alpha\})$.*

We are now ready to complete the proof. First note that if $r = 0$ and $x \in C(\{B_\alpha\})$, then for each $\alpha \in \Gamma$, $d_x = 0 = \lim_\alpha \operatorname{dist}(x, D_\alpha) \geq \operatorname{dist}(x, D_\alpha)$; hence $x \in \bigcap_\alpha D_\alpha$, contradicting our initial assumption. So we assume $r > 0$ and show that this also leads to a contradiction.

Let $x_0 \in C(\{D_\alpha\})$ be fixed and define $H_\alpha^1 = B(x_0; r) \cap D_\alpha$. Since $d_{x_0} = r$, $\operatorname{dist}(x_0, D_\alpha) \leq r$ for each α, so $H_\alpha^1 \neq \emptyset$. Also, since $\lim_\alpha \operatorname{dist}(x_0, D_\alpha) = r$

and Γ is uncountable, there exists $\alpha_1 \in \Gamma$, $\alpha_1 \geq 1$, such that for $\alpha \geq \alpha_1$, $\mathrm{dist}(x_0, D_\alpha) = r$. Consequently, if $x \in H_\alpha^1$ for $\alpha \geq \alpha_1$ then $\|x - x_0\| = r$. Next consider the chain $\{H_\alpha^1\}_{\alpha \geq \alpha_1}$. Notice that one may view this as a chain in Λ by taking $H_\alpha^1 \equiv H_{\alpha_1}^1$ for $\alpha < \alpha_1$. Now choose $x_1 \in C(\{H_\alpha^1\}) \cap H_{\alpha_1}^1$. Then $\|x_1 - x_0\| = r$, and by repeating the argument just given, replacing x_0 with x_1 and $\{D_\alpha\}$ with $\{H_\alpha^1\}$, one may obtain $\{H_\alpha^2\}$, $\alpha_2 \geq 2$, and $x_2 \in C(\{H_\alpha^2\}) \cap H_{\alpha_2}^2$ such that $\|x_2 - x_1\| = \|x_2 - x_0\| = r$. The idea is to continue this process by transfinite induction. Let $\gamma \in \Gamma$ and suppose that for each $\beta < \gamma$ elements x_β, $\alpha_\beta \geq \beta$, and $\{H_\alpha^\gamma\} \in \Lambda$ have been chosen so that:

1) $x_\beta \in H_{\alpha_\beta}^\beta \cap C(\{H_\alpha^\beta\})$;
2) $\mu \leq \beta < \gamma \Rightarrow H_\alpha^\beta \subseteq H_\alpha^\mu$;
3) $\mu < \beta$ and $x \in H_\alpha^\beta \Rightarrow \|x - x_\mu\| = r$.

If $\gamma = \xi + 1$ let $H_\alpha^\gamma = B(x_\xi; r) \cap H_\alpha^\xi$, and choose $\alpha_\gamma \geq \gamma$ so that if $\alpha \geq \alpha_\gamma$ and $x \in H_\alpha^\xi$, then $\|x - x_\xi\| = r$. Now set $H_\alpha \equiv H_{\alpha_\gamma}$ for $\alpha < \gamma$ and choose $x_\gamma \in H_{\alpha_\gamma}^\gamma \cap C(\{H_\alpha^\gamma\})$. If γ is a limit ordinal let α' be the smallest ordinal larger than each of the ordinals α_β. (Note that $\{\alpha_\beta\}$ cannot be cofinal in Γ because $\bigcap_{\alpha \in \Gamma} D_\alpha = \emptyset$, but since $\gamma < \Gamma$, $\bigcap_{\beta < \gamma} D_{\alpha_\beta} \neq \emptyset$.) For $\alpha \geq \alpha'$ define $H_\alpha^\gamma = \bigcap_{\beta < \gamma} H_\alpha^\beta$. Since $\gamma < \Gamma$, $H_\alpha^\gamma \neq \emptyset$. Set $\alpha_\gamma = \alpha'$, let $H_\alpha \equiv H_{\alpha_\gamma}$ for $\alpha < \gamma$, and choose $x_\gamma \in C(\{H_\alpha^\gamma\}) \cap H_{\alpha_\gamma}^\gamma$. Since $\alpha_\gamma \geq \gamma$, this completes the induction.

The fact that $\{x_\gamma : \gamma \in \Gamma\}$ lies in $C(\{D_\alpha\})$ follows from observation (3). Moreover, $\|x_\alpha - x_\gamma\| = r$ if $\alpha \neq \gamma$. Now let $H = \overline{\mathrm{conv}}\{x_\alpha : \alpha \in \Gamma\}$. Then if $x \in H$, clearly $\|x - x_\alpha\| \leq r$. On the other hand, since $x_\alpha \in D_\alpha$,

$$\lim_\alpha \|x - x_\alpha\| \geq \inf_{x \in D_1} \{\lim_\alpha \mathrm{dist}(x, D_\alpha)\} = r.$$

Thus $\lim_\alpha \|x - x_\alpha\| = r$, so for some $\beta \in \Gamma$, $\|x - x_\beta\| = r = \mathrm{diam}(H)$. Since $\{x_\gamma\}$ lies, for example, on the proximinal set $\{H_{\alpha_1}^1\}$, this contradicts the quasi-normal structure hypothesis of the theorem and establishes the proof.

3. Abstract convexity structures.

It is possible to formulate an abstract version of the theorem just proved. We begin by describing Penot's framework of [16].

Let (M, d) be a bounded metric space. A family of subsets Σ of M is said to be a *convexity structure* if Σ *contains the closed balls of* M and is closed under intersections. Metric fixed point theory has been studied extensively in such a framework (e.g., [1], [11], [12], [15]).

As in the Banach space case, such a convexity structure Σ is a subbase for a topology τ on M relative to which the closed balls of M are τ-closed. If τ is countably compact, then, as before, proximinal sets of (nonempty) sets in Σ are nonempty (and in Σ). We say that a set D in Σ is *quasi-normal* if $H \subseteq D$ and $H \in \Sigma$ implies $\mathrm{diam}(H) = 0$ or there exists $u \in H$ such

that $d(u, v) < \mathrm{diam}(H)$ for every $v \in H$. By making modifications in the argument just given (most of which are obvious) we are able to prove an abstract version of Theorem 1. Of course, as before, we must assume here that Σ contains its closed r-neighborhoods.

THEOREM 3. *Let (M, d) be a bounded metric space, let Σ be a countably compact convexity structure on M which contains its closed r-neighborhoods, and let τ be the topology on M generated by Σ as a subbase. Then if the proximinal sets in Σ are either separable or quasi-normal, M is τ-compact.*

Proof. Aside from the reference to the fact that the sets C_β^n, C_β, and H are convex, the linear structure of X plays no role in the proof of Theorem 1. To prove Theorem 3 we ignore the fact that these sets are convex and proceed as follows. As in the proof of Theorem 1 one can prove observations (1), (2), and (3), and in particular prove that the sets $C(\{B_\alpha\}) = \{x \in B_1 : d_x = r\}$ are all nonempty and in Σ. (Recall that $C(\{B_\alpha\}) = (\bigcap_{\alpha \in \Gamma}\{\bigcup_{u \in B_\alpha} B(u; r)\}) \cap B_1$.) Therefore it is possible to follow precisely the proof of Theorem 1 (replacing the norm $\|\cdot\|$ with the distance d) to where the set H is introduced. Note that if the proximinal sets in Σ are separable, then the mere existence of an uncountable set $\{x_\gamma : \gamma \in \Gamma\}$ for which $d(x_\alpha, x_\gamma) \equiv r > 0$ if $\alpha \neq \gamma$ yields a contradiction. If the proximinal sets in Σ are quasi-normal we define

$$H = \bigcap\{D : D \in \Sigma,\ D \supseteq \{x_\gamma\}\}.$$

Then, since $\{x_\gamma\} \subseteq C(\{D_\alpha\})$ and $C(\{D_\alpha\}) \in \Sigma$, $H \subseteq C(\{D_\alpha\})$. Also, $H \subseteq \bigcap_{\alpha \in \Gamma} B(x_\alpha; r)$, so if $x \in H$, $d(x, x_\alpha) \leq r$. At this point one only need observe that $\mathrm{diam}(H) = r$ in order to reach the conclusion as in the proof of Theorem 1. However, $\mathrm{diam}(H) = r$ follows easily from the fact that $\mathrm{diam}\{x_\gamma : \gamma \in \Gamma\} = r$ and the fact that Σ contains the closed balls of M (cf., [11], Part III).

The following corollary of Theorem 3 also appears to be new. Recall that a convexity structure Σ is said to be *uniformly normal* if there exists $c \in (0, 1)$ such that for every $D \in \Sigma$ for which $\mathrm{diam}(D) > 0$, there exists $u \in D$ such that $\sup\{d(u, v) : v \in D\} \leq c\,\mathrm{diam}(D)$.

COROLLARY 4. *Let (M, d) be a bounded complete metric space and suppose Σ is a convexity structure on M which is uniformly normal and which contains its closed r-neighborhoods. Then Σ (and the topology generated by Σ) is compact.*

Proof. Since Σ is uniformly normal, and thus every set in Σ is obviously quasi-normal, this corollary is immediate from Theorem 3 and the known fact that a uniformly normal convexity structure is countably compact (Khamsi [12]).

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Mild integrated C-existence families

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Abstract. We study mild n times integrated C-existence families without the assumption of exponential boundedness. We present several equivalent conditions for these families. Hille–Yosida type necessary and sufficient conditions are given for the exponentially bounded case.

1. Introduction. Motivated by the study of the abstract Cauchy problem, two generalizations of strongly continuous semigroups, integrated semigroups and C-semigroups, have recently been introduced and received extensive attention (see [1–3, 5–7, 11–14, 16]). However, there are limitations to both integrated semigroups and C-semigroups. In order to cover more cases, [8] defined a pair of families of operators, one of which yields uniqueness, while the other yields existence of solutions of the abstract Cauchy problem, for all initial data in the image of C.

In this paper, we concentrate on mild C-existence families without assuming exponential boundedness. Section 2 offers a supplement for n times integrated C-semigroups. Section 3 contains the general definition and equivalent conditions for mild n times integrated C-existence families. Section 4 is devoted to the study of a Hille–Yosida type theorem, in which some equivalent conditions are found for exponentially bounded mild once integrated C-existence families. Finally, in Section 5, we provide some examples.

All operators are linear on a complex Banach space X. For an operator A, $D(A)$ and $\mathrm{Im}(A)$ will stand for the domain and image of A, respectively. We shall write $[D(A)]$ for the normed space $D(A)$ equipped with the graph norm: for $x \in D(A)$, $|||x||| = \|x\| + \|Ax\|$. The space $[D(A)]$ is complete if and only if A is closed in X. Finally, $B(X)$ is the algebra of all bounded linear operators T with $D(T) = X$, and $C \in B(X)$ will be fixed throughout this paper.

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