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Construction of standard exact sequences of power series spaces

by

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Abstract. The following result is proved: Let $\Lambda_R^p(\alpha)$ denote a power series space of infinite or of finite type, and equip $\Lambda_R^p(\alpha)$ with its canonical fundamental system of norms, $R \in \{0, \infty\}$, $1 \le p < \infty$. Then a tamely exact sequence

$$(*) \qquad 0 \to \Lambda_R^p(\alpha) \to \Lambda_R^p(\alpha) \to \Lambda_R^p(\alpha)^{\mathbb{N}} \to 0$$

exists iff α is strongly stable, i.e. $\lim_n \alpha_{2n}/\alpha_n = 1$, and a linear-tamely exact sequence $(*)$ exists iff α is uniformly stable, i.e. there is A such that $\limsup_n \alpha_{Kn}/\alpha_n \le A < \infty$ for all K. This result extends a theorem of Vogt and Wagner which states that a topologically exact sequence $(*)$ exists iff α is stable, i.e. $\sup_n \alpha_{2n}/\alpha_n < \infty$.

An important tool in structure theory of power series spaces is the existence of exact sequences of the form

$$(*) \qquad 0 \to \Lambda_R^p(\alpha) \to \Lambda_R^p(\alpha) \to \Lambda_R^p(\alpha)^{\mathbb{N}} \to 0.$$

Here $\Lambda_R^p(\alpha)$ denotes a power series space of infinite type if $R = \infty$ and of finite type if $R = 0$, respectively, $1 \le p \le \infty$. A topologically exact sequence $(*)$ exists if and only if α is stable, i.e. $\sup_n \alpha_{2n}/\alpha_n < \infty$; this result has been proved for the nuclear case in [11] and in [6] for the general case. The existence of such sequences has been used to characterize the subspaces, quotient spaces and complemented subspaces of stable power series spaces of infinite type (cf. [11]) and of finite type (cf. [7], [8], [6]).

The purpose of this note is the investigation of the existence of tamely and linear-tamely exact sequences of the form $(*)$ (for the concept of tameness see below, or [1], [9], [4], [5]). We shall prove the following main result: Let $\Lambda_R^p(\alpha)$ be equipped with its canonical fundamental system of norms, $R \in \{0, \infty\}$, $1 \le p < \infty$. Then a tamely exact sequence $(*)$ exists if and only if α is strongly stable, i.e. $\lim_n \alpha_{2n}/\alpha_n = 1$, and a linear-tamely exact sequence $(*)$ exists iff α is uniformly stable, i.e. there is A such that $\limsup_n \alpha_{Kn}/\alpha_n \le A < \infty$ for all K. We notice that we do not need any nu-

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clearity assumptions. Moreover, we construct sequences which enjoy an additional lifting property (which makes e.g. so-called three-spaces-techniques available).

In a forthcoming paper, this result will be combined with a tame splitting theorem proved in [4] to set up a tame resp. linear-tame structure theory for power series spaces based on a common proof both for the cases $R = \infty$ and $R = 0$; this is in advantage compared with the topological situation where different methods have been applied for $R = \infty$ and $R = 0$, respectively, since a general topological splitting result for power series spaces of finite type fails (cf. [11], [7], [8], [6]).

The first section contains preliminaries and the notation. In the second section, we prove necessary conditions for the existence of tamely and linear-tamely exact sequences of the form (∗). In Section 3 the basic lemma of Vogt and Wagner [11], 2.2, is generalized to the case $1 \le p < \infty$ without nuclearity assumptions; further we give precise continuity estimates and prove the above mentioned lifting property of the sequence; this section is of more technical nature.

Section 4 contains the most important part of our construction and the main results. By means of the easy Lemma 2.3 we here only have to consider strongly stable sequences α. A delicate combinatorial construction, in particular a carefully created bijection $\mathbb{N}^2 \to \mathbb{N}$, combined with Lemma 3.2 gives the result.

1. A Fréchet space E equipped with a fixed fundamental system of continuous seminorms $\| \ \|_0 \le \| \ \|_1 \le \| \ \|_2 \le \ldots$ is called a *graded Fréchet space*, the sequence of seminorms is called a *grading*. Graded subspaces and graded quotient spaces are endowed with the induced seminorms. If E, F, G are graded Fréchet spaces and $i : E \to F$, $q : F \to G$ are linear maps, then the sequence

$$0 \to E \xrightarrow{i} F \xrightarrow{q} G \to 0$$

is called *linear-tamely exact* if i is injective, q is surjective, $\operatorname{im} i = \ker q$ and there exist $a \ge 1, b \ge 0$ and constants $c_n > 0$ such that

(1) $$\|ie\|_n \le c_n \|e\|_{an+b}, \quad \|e\|_n \le c_n \|ie\|_{an+b},$$

(2) $$\|qf\|_n \le c_n \|f\|_{an+b}, \quad \inf\{\|\psi\|_n : q\psi = g\} \le c_n \|g\|_{an+b}$$

for all n and all $e \in E$, $f \in F$, $g \in G$. The sequence is then called *(a)-tamely exact*, and *tamely exact* if $a = 1$. A linear bijection $i : E \to F$ is called a *linear-tame* or *(a)-tame* isomorphism if (1) holds, it is called a *tame isomorphism* if (1) holds for $a = 1$.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For any sequence $0 \le \alpha_0 \le \alpha_1 \le \alpha_2 \le \ldots \nearrow \infty$,

$R \in [0, \infty]$, and $1 \le p \le \infty$ we define

$$\Lambda_R^p(\alpha) = \{x = (x_0, x_1, \ldots) \subset \mathbb{K} : |x|_t < \infty \text{ for all } t < R\}$$

where $|x|_t = (\sum_j |x_j|^p e^{pt\alpha_j})^{1/p}$ if $1 \le p < \infty$ and $|x|_t = \sup_j |x_j| e^{t\alpha_j}$ if $p = \infty$. The space $\Lambda_R^p(\alpha)$ is called a *power series space* of *infinite type* if $R = \infty$ and of *finite type* if $R < \infty$, respectively. Any sequence $r_0 < r_1 < r_2 < \ldots \nearrow R$ defines a grading on $\Lambda_R^p(\alpha)$ by $\| \ \|_k = | \ |_{r_k}$ and makes it a graded Fréchet space (as which we shall always consider $\Lambda_R^p(\alpha)$ in this paper). Note that

$$\|x\|_k = \Big(\sum_j |x_j|^p e^{pr_k\alpha_j}\Big)^{1/p} \quad \text{if } 1 \le p < \infty \quad \text{and}$$

$$\|x\|_k = \sup_j |x_j| e^{r_k \alpha_j} \quad \text{if } p = \infty.$$

Most important are the gradings defined by $r_k = k$ on $\Lambda_\infty^p(\alpha)$ and by $r_k = -1/k$ on $\Lambda_0^p(\alpha)$, respectively.

The space $\Lambda_R^p(\alpha)$ (or α) is called *stable* if $\sup_n \alpha_{2n}/\alpha_n < \infty$, *strongly stable* if $\lim_n \alpha_{2n}/\alpha_n = 1$, and *uniformly stable* if there exists A such that $\limsup_n \alpha_{Kn}/\alpha_n \le A < \infty$ for all K.

We shall use Kolmogorov numbers (cf. [3]). For any linear space E and absolutely convex sets $A \subset B \subset E$ we define

$$\delta_n(A, B) = \inf\{\delta > 0 : A \subset \delta B + F,\ F \subset E \text{ a subspace with } \dim F \le n\}.$$

For $U_t = \{x \in \Lambda_R^p(\alpha) : |x|_t \le 1\}$ we then have (see e.g. [3], 9.3.1)

$$\delta_n(U_{r_2}, U_{r_1}) = e^{(r_1 - r_2)\alpha_n}, \quad r_1 < r_2 < R.$$

Throughout this paper, we shall not assume that $\Lambda_R^p(\alpha)$ is nuclear.

2. In this section we prove necessary conditions for the existence of tamely or linear-tamely exact sequences of the form

(∗) $$0 \to \Lambda_R^p(\alpha) \to \Lambda_R^p(\alpha) \xrightarrow{q} \Lambda_R^p(\alpha)^{\mathbb{N}} \to 0.$$

The space $\Lambda_R^p(\alpha)^{\mathbb{N}}$ is equipped with the grading

$$\|x\|_k = \Big(\sum_{i=1}^k \|x^i\|_k^p\Big)^{1/p} \quad \text{if } 1 \le p < \infty \quad \text{and}$$

$$\|x\|_k = \sup_{i=1}^k \|x^i\|_k \quad \text{if } p = \infty$$

for $x = (x^i)_{i=1}^\infty \in \Lambda_R^p(\alpha)^{\mathbb{N}}$, $x^i \in \Lambda_R^p(\alpha)$. Further, for fixed $K \in \mathbb{N}$ the space

$\Lambda_R^p(\alpha)^K$ is endowed with the grading

$$\|x\|_k = \Big(\sum_{i=1}^K \|x^i\|_k^p\Big)^{1/p} \quad \text{if } 1 \le p < \infty \quad \text{and}$$

$$\|x\|_k = \sup_{i=1}^K \|x^i\|_k \quad \text{if } p = \infty.$$

2.1. Lemma. *Let $1 \le p \le \infty, R \in \{0,\infty\}$, $r_k = k$ if $R = \infty$ resp. $r_k = -1/k$ if $R = 0$.*

(i) *If $(*)$ is tamely exact, then α is strongly stable.*

(ii) *If $(*)$ is linear-tamely exact, then α is uniformly stable.*

Proof. We write $\Lambda = \Lambda_R^p(\alpha)$. Let $K \ge 2$ be fixed. Let $U_k \subset \Lambda$, $V_k \subset \Lambda^{\mathbb{N}}$ and $W_k \subset \Lambda^K$ denote the corresponding neighborhoods of zero $\{x : \|x\|_k \le 1\}$. Let $\pi_K : \Lambda^{\mathbb{N}} \to \Lambda^K$ be defined by $(x_k)_{k=1}^\infty \mapsto (x_k)_{k=1}^K$. We have

$$\pi_K(V_{k+K}) \subset W_k \subset \pi_K(V_k).$$

We now suppose that for suitable $a, c \ge 1$, $b, d \ge 0$ and constants $c_k > 0$ we have

$$q(U_{ak+b}) \subset c_k V_k \quad \text{and} \quad V_{ck+d} \subset c_k q(U_k)$$

where $a = c = 1$ in the case (i). For the Kolmogorov numbers we conclude that

$$\begin{aligned}\delta_n(W_{ck+d}, W_m) &\le c_{k,m}\delta_n((\pi_K \circ q)(U_k), (\pi_K \circ q(U_{am+aK+b})))\\ &\le c_{k,m}\delta_n(U_k, U_{am+aK+b})\end{aligned}$$

for $k \ge am + aK + b$. The space $\Lambda_R^p(\alpha)^K$ is canonically isomorphic to $\Lambda_R^p(\beta)$ with $\beta = (\alpha_0, \ldots, \alpha_0, \alpha_1, \ldots, \alpha_1, \alpha_2, \ldots)$ where each α_j occurs exactly K times; in particular, we have $\beta_{Kn} = \alpha_n$. It follows that

$$e^{(r_m - r_{ck+d})\beta_n} \le c_{k,m} e^{(r_{am+aK+b} - r_k)\alpha_n}$$

and therefore

$$\frac{\alpha_n}{\beta_n} \le \frac{r_{ck+d} - r_m}{r_k - r_{am+aK+b}} + \frac{c'_{k,m}}{\beta_n(r_k - r_{am+aK+b})}, \qquad k > am + aK + b.$$

If $r_k = k$ then we put $m = 0$ and obtain $\limsup_n \alpha_n/\beta_n \le c$, and thus $\limsup_n \alpha_{Kn}/\alpha_n \le c$.

For $r_k = -1/k$ we calculate

$$\frac{r_{ck+d} - r_m}{r_k - r_{am+aK+b}} = a\frac{1 + \frac{d}{ck} - \frac{m}{ck}}{1 - \frac{am}{k} - \frac{aK+b}{k}}\,\frac{1 + \frac{aK+b}{am}}{1 + \frac{d}{ck}}$$

and obtain by choosing $k \gg m$ and both very large that $\limsup_n \alpha_n/\beta_n \le a$, hence $\limsup_n \alpha_{Kn}/\alpha_n \le a$, which proves the assertion.

With the same proof we can obtain the following more detailed results.

2.2. Lemma. *Let $r_0 < r_1 < r_2 < \ldots \nearrow R \in \{0,\infty\}$ and $1 \le p \le \infty$.*

(a) *If $\Lambda_R^p(\alpha) \times \Lambda_R^p(\alpha)$ is tamely isomorphic to a graded quotient space of $\Lambda_R^p(\alpha)$ then*

(i) *$\limsup_n \alpha_{2n}/\alpha_n \le \liminf_k r_{k+d}/r_k$ if $R = \infty$.*

(ii) *$\limsup_n \alpha_{2n}/\alpha_n \le \liminf_m r_m/r_{m+b}$ if $R = 0$.*

In particular, α is strongly stable if $\lim_k r_{k+d}/r_k = 1$ and $R = \infty$ resp. $\lim_m r_m/r_{m+b} = 1$ and $R = 0$.

(b) *If $\Lambda_R^p(\alpha)^{\mathbb{N}}$ is linear-tamely isomorphic to a graded quotient space of $\Lambda_R^p(\alpha)$ then*

(i) *$\limsup_n \alpha_{Kn}/\alpha_n \le \liminf_k r_{ck+d}/r_k$ for all K if $R = \infty$.*

(ii) *$\limsup_n \alpha_{Kn}/\alpha_n \le \liminf_m r_m/r_{(a+1)m}$ for all K if $R = 0$.*

In particular, α is uniformly stable if $\liminf_k r_{ck+d}/r_k < \infty$ and $R = \infty$ resp. $\liminf_m r_m/r_{(a+1)m} < \infty$ and $R = 0$.

We end this section with comparing the two conditions: strong and uniform stability.

2.3. Lemma (cf. [2], 4.4, 4.5). (i) *There exists a sequence $\alpha \nearrow \infty$ which is uniformly stable but not strongly stable.*

(ii) *If $\alpha \nearrow \infty$ is uniformly stable and $\limsup_n \alpha_{Kn}/\alpha_n \le c < D$ for all K then there exists a strongly stable sequence $\widetilde{\alpha} \nearrow \infty$ such that $(1/D)\alpha_n \le \widetilde{\alpha}_n \le D\alpha_n$ for large n.*

Proof. (i) We can choose any increasing sequence α with $\alpha_{2^n} = \beta_n$ and $\beta = (A, A^2, A^2, \ldots, A^j, \ldots, A^j, \ldots)$ where $A > 1$ and each A^j occurs exactly j times.

(ii) We put $\beta_n = \alpha_{2^n}$ and define $\overline{\beta}_n$ by $\overline{\beta}_n = D^k$ iff $D^k \le \beta_n < D^{k+1}$. We set $m_k = \min\{n : \beta_n \ge D^k\}$ and $L(k) = m_{k+1} - m_k$. By assumption we have $\lim_k L(k) = \infty$. We set $Q(n) = \max\{k : m_k \le n\}$ and define $\widetilde{\beta}_n = D^{Q(n)+(n-m_{Q(n)})/L(Q(n))}$. Then we get $(1/D)\beta_n \le \overline{\beta}_n \le \widetilde{\beta}_n \le D\overline{\beta}_n \le D\beta_n$ for large n and $\widetilde{\beta}_{n+1}/\widetilde{\beta}_n = D^{1/L(Q(n))}$ for large n (consider the cases $Q(n+1) - Q(n) \in \{0,1\}$), hence $\lim_n(\widetilde{\beta}_{n+1}/\widetilde{\beta}_n) = 1$. Finally, we put $\widetilde{\alpha}_{2^n} = \widetilde{\beta}_n$, and for $1 \le i < 2^n$ we set $\widetilde{\alpha}_{2^n+i} = (1-\tau)\widetilde{\alpha}_{2^n} + \tau\widetilde{\alpha}_{2^{n+1}}$ if $\alpha_{2^n+i} = (1-\tau)\alpha_{2^n} + \tau\alpha_{2^{n+1}}$ and obtain the assertion.

We shall make use of the following easy remark.

2.4. Remark. Let $r_0 < r_1 < r_2 < \ldots \nearrow R \in [0,\infty]$ and $1 \le p \le \infty$.

(i) If $\lim_n \alpha_n/\beta_n = 1$ then $\Lambda_R^p(\alpha)$ and $\Lambda_R^p(\beta)$ are tamely isomorphic.

(ii) Put $A = \limsup_n\{\alpha_n/\beta_n, \beta_n/\alpha_n\}$. If $A \le \inf_{k\ge k_0} r_{ck+d}/r_k$ and $R = \infty$ resp. $A \le \inf_{k\ge k_0} r_k/r_{ck+d}$ and $R = 0$ then $\Lambda_R^p(\alpha)$ and $\Lambda_R^p(\beta)$ are (c)-tamely, i.e. linear-tamely isomorphic.

3. We now set up the construction of standard exact sequences of the form $(*)$.

Let J be an index set and $A = (a_{j,m})_{j\in J, m\in\mathbb{N}}$ be a Köthe matrix, i.e. a matrix satisfying $0 \le a_{j,m} \le a_{j,m+1}$ for all j, m and $\sup_m a_{j,m} > 0$ for all j. For $1 \le p \le \infty$ we define

$$\lambda^p(A) = \{x = (x_j)_{j\in J} \subset \mathbb{K} : \|x\|_m < \infty \text{ for all } m\}$$

where $\|x\|_m = (\sum_j |x_j|^p a_{j,m}^p)^{1/p}$ if $1 \le p < \infty$ and $\|x\|_m = \sup_j |x_j| a_{j,m}$ if $p = \infty$.

3.1. Lemma (cf. [11], 2.1). *Let $A = (a_{i,m})_{i\in I, m\in\mathbb{N}}$ and $B = (b_{j,m})_{j\in J, m\in\mathbb{N}}$ be Köthe matrices, let $M_j \subset I$, $j \in J$, be disjoint subsets of I and $\inf_{i\in M_j} a_{i,m} = b_{j,m}$ for all j and m. Let $1 \le p, q \le \infty$ with $1/p + 1/q = 1$. If $p > 1$ we assume that for every m there is $s(m)$ such that*

$$c_m = \sup_j \Big(\sum_{i\in M_j} \Big(\frac{a_{i,m}}{a_{i,s(m)}}\Big)^q\Big)^{1/q} < \infty,$$

if $p = 1$ we put $s(m) = m$ and $c_m = 1$. Then

$$Q\xi = \Big(\sum_{i\in M_j} \xi_i\Big)_{j\in J}, \quad \xi = (\xi_i)_{i\in I} \in \lambda^p(A),$$

defines a continuous linear and surjective map $Q : \lambda^p(A) \to \lambda^p(B)$ satisfying

$$\|Q\xi\|_m \le c_m \|\xi\|_{s(m)} \quad \text{and} \quad \inf\{\|\xi\|_m : Q\xi = \eta\} \le \|\eta\|_m.$$

Proof. We prove the case $1 \le p < \infty$, the same arguments give the case $p = \infty$. First we have

$$\|Q\xi\|_m^p = \sum_j \Big|\sum_{i\in M_j} \xi_i\Big|^p b_{j,m}^p \le \sum_j \Big(\sum_{i\in M_j} |\xi_i| a_{i,m}\Big)^p \le c_m^p \|\xi\|_{s(m)}^p.$$

Let $\eta = (\eta_j)_{j\in J} \in \lambda^p(B)$, let $m \in \mathbb{N}$ and $\varepsilon > 0$. For each j with $\eta_j \ne 0$ we choose $\varepsilon_j > 0$ and $i_j \in M_j$ with $a_{i_j,m} \le b_{j,m} + \varepsilon_j$. We put $\xi_i = \eta_j$ if $i = i_j$ and $\xi_i = 0$ otherwise. Then we have $Q\xi = \eta$ and

$$\|\xi\|_m = \Big(\sum_i |\xi_i|^p a_{i,m}^p\Big)^{1/p} \le \Big(\sum_j |\eta_j|^p (b_{j,m} + \varepsilon_j)^p\Big)^{1/p} \le \|\eta\|_m + \varepsilon$$

if $\varepsilon_j > 0$ are chosen suitably small. This gives the result.

3.2. Lemma (cf. [11], 2.2). *Let $1 \le p < \infty$, $1 < q \le \infty$ and $1/p + 1/q = 1$. Let $A = (a_{i,j,k;m})_{i,j,k\in\mathbb{N}, m\in\mathbb{N}}$ be a Köthe matrix, put $A_k = (a_{1,j,k;m})_{j\in\mathbb{N}, m\in\mathbb{N}}$ for every $k \in \mathbb{N}$ and $A_K = ((a_{i,j,k;m}^p + a_{i+1,j,k;m}^p)^{1/p})_{i,j,k\in\mathbb{N}, m\in\mathbb{N}}$. Suppose that*

(1) *$a_{i,j,k;k} = 1$ for all i, j, k.*
(2) *$a_{i,j,k;m} \ge a_{i+1,j,k;m}$ for all i, j and $m \le k$,*
$a_{i,j,k;m} \le a_{i+1,j,k;m}$ for all i, j and $m \ge k$.
(3) *$\lim_i a_{i,j,k;m} = 0$ for all j and $m < k$.*
(4) *For every m there is $s_r(m)$ such that*

$$c_{m,r} = \sup_{j,k} \Big(\sum_i \Big(\frac{a_{i,j,k;m}}{a_{i,j,k;s_r(m)}}\Big)^r\Big)^{1/r} < \infty, \quad r = \min\{p, q\}.$$

Then there exists an exact sequence

$$0 \to \lambda^p(A_K) \xrightarrow{i} \lambda^p(A) \xrightarrow{Q} \prod_{k=1}^{\infty} \lambda^p(A_k) \to 0.$$

Moreover, putting $s_\infty(m) = m$, $c_{m,\infty} = 1$, and defining $s_p(m)$, $s_q(m)$, $c_{m,p}$, $c_{m,q}$ according to (4) for $q < \infty$, we obtain the continuity estimates

$$\|Qx\|_m \le c_{m,q} \|x\|_{s_q(m)}, \quad \inf\{\|z\|_m : Qz = y\} \le \|y\|_m,$$
$$\|i\xi\|_m \le 2^{(p-1)/p} \|\xi\|_m, \quad \|\xi\|_m \le 2^{1/p} c_{m,p} c_{s_p(m),q} \|i\xi\|_{s_q(s_p(m))}.$$

Proof. (a) For $x = (x_{i,j,k}) \in \lambda^p(A)$ we put $Qx = (\sum_i x_{i,j,k})_{j,k\in\mathbb{N}}$. With

$$b_{j,k;m} = \inf_i a_{i,j,k;m} = \begin{cases} 0, & m < k, \\ a_{1,j,k;m}, & m \ge k, \end{cases}$$

we obtain by means of the previous lemma a continuous linear and surjective map $Q : \lambda^p(A) \to \prod_{k=1}^{\infty} \lambda^p(A_k)$ satisfying the desired estimates, where $\prod_{k=1}^{\infty} \lambda^p(A_k)$ is equipped with the seminorms

$$\|y\|_m = \Big(\sum_{k=1}^m \sum_j |y_{j,k}|^p a_{1,j,k;m}^p\Big)^{1/p}.$$

Note that the assertion on Q also holds for $p = \infty$ and

$$\|y\|_m = \sup_{k=1}^{m} \sup_j |y_{j,k}| a_{1,j,k;m}.$$

(b) By definition of Q we have

$$\ker Q = \Big\{(x_{i,j,k}) \in \lambda^p(A) : \sum_i x_{i,j,k} = 0 \text{ for all } j, k\Big\}.$$

Let $e_{i,j,k}$ denote the canonical unit vectors in $\lambda^p(A)$. We shall prove that the vectors $g_{i,j,k} = e_{i,j,k} - e_{i+1,j,k} \in \ker Q$ form a basis of $\ker Q$. For $x = (x_{i,j,k}) \in \ker Q$ and $\eta_{i,j,k} = \sum_{\nu=1}^{i} x_{\nu,j,k}$ we have

$$\sum_{i=1}^n \eta_{i,j,k} g_{i,j,k} = \sum_{i=1}^n x_{i,j,k} e_{i,j,k} - \Big(\sum_{\nu=1}^n x_{\nu,j,k}\Big) e_{n+1,j,k}$$

and we obtain from (2) the inequalities

$$\Big\| \sum_{i=1}^{n} \eta_{i,j,k} g_{i,j,k} \Big\|_m^p = \sum_{i=1}^{n} |x_{i,j,k}|^p a^p_{i,j,k;m} + \Big| \sum_{\nu=1}^{n} x_{\nu,j,k} \Big|^p a^p_{n+1,j,k;m}$$
$$= \sum_{i=1}^{n} |x_{i,j,k}|^p a^p_{i,j,k;m} + \Big| \sum_{\nu=n+1}^{\infty} x_{\nu,j,k} \Big|^p a^p_{n+1,j,k;m}$$
$$\le (1 + c^p_{m,q}) \sum_{i=1}^{\infty} |x_{i,j,k}|^p a^p_{i,j,k;s_q(m)}.$$

We put $T_n x = \sum_{i,j,k\le n} \eta_{i,j,k} g_{i,j,k}$, $x \in \ker Q$. The above inequality shows that

$$\|T_n x\|_m \le (1 + c^p_{m,q})^{1/p} \|x\|_{s_q(m)}$$

and hence the equicontinuity of $\{T_n\}_n$. We show that $\lim_n T_n x = x$ for all x in a total subset of $\ker Q$, and thus for all $x \in \ker Q$ by equicontinuity. Since $1 \le p < \infty$, the vectors $x \in \ker Q$ of the form $x = \sum_i \xi_i e_{i,j,k}$ form a total subset, and for such an x and $m \ge k$, $n \ge j, k$ we have

$$\|x - T_n x\|_m^p = \sum_{i=n+2}^{\infty} |\xi_i|^p a^p_{i,j,k;m} + \Big| \xi_{n+1} + \sum_{\nu=1}^{n} \xi_\nu \Big|^p a^p_{n+1,j,k;m}$$
$$\le 2^{p-1} \sum_{i=n+1}^{\infty} |\xi_i|^p a^p_{i,j,k;m} + 2^{p-1} \Big| \sum_{\nu=n+1}^{\infty} \xi_\nu \Big|^p a^p_{n+1,j,k;m}$$
$$\le 2^{p-1}(1 + c^p_{m,q}) \sum_{\nu=n+1}^{\infty} |\xi_\nu|^p a^p_{\nu,j,k;s_q(m)} \to 0 \quad (n \to \infty).$$

(c) We define $i\xi = \sum_{i,j,k} \xi_{i,j,k} g_{i,j,k}$ for $\xi = (\xi_{i,j,k}) \in \lambda^p(A_K)$ and obtain a continuous linear map $i : \lambda^p(A_K) \to \lambda^p(A)$ satisfying

$$\|i\xi\|_m \le 2^{(p-1)/p} \Big(\sum_{i,j,k} |\xi_{i,j,k}|^p (a^p_{i,j,k;m} + a^p_{i+1,j,k;m}) \Big)^{1/p} \le 2^{(p-1)/p} \|\xi\|_m.$$

It is clear that i is injective and $\operatorname{im} i \subset \ker Q$. Let $x = (x_{i,j,k}) \in \ker Q$ and put $\eta_{i,j,k} = \sum_{\nu=1}^{i} x_{\nu,j,k}$. We show that $\eta = (\eta_{i,j,k}) \in \lambda^p(A_K)$, which implies $i\eta = x \in \operatorname{im} i$ by means of (b). For $* = i, i+1$ we have

$$|\eta_{i,j,k}| a_{*,j,k;m} = \begin{cases} |\sum_{\nu=i+1}^{\infty} x_{\nu,j,k}| a_{*,j,k;m} \le c_{m,q} (\sum_{\nu=i+1}^{\infty} |x_{\nu,j,k}|^p a^p_{\nu,j,k;s_q(m)})^{1/p}, & m \ge k, \\ |\sum_{\nu=1}^{i} x_{\nu,j,k}| a_{*,j,k;m} \le c_{m,q} (\sum_{\nu=1}^{i} |x_{\nu,j,k}|^p a^p_{\nu,j,k;s_q(m)})^{1/p}, & m \le k, \end{cases}$$

and therefore $\sum_{j,k} \sup_i |\eta_{i,j,k}|^p a^p_{*,j,k;m} \le c^p_{m,q} \|x\|^p_{s_q(m)}$. We conclude from our nuclearity assumption (4) that

$$\|\eta\|_m^p = \sum_{i,j,k} |\eta_{i,j,k}|^p (a^p_{i,j,k;m} + a^p_{i+1,j,k;m}) \le 2 c^p_{m,p} c^p_{s_p(m),q} \|x\|^p_{s_q(s_p(m))}.$$

We end this section with proving an additional lifting property of the sequence in the previous lemma. Let

$$U_m = \Big\{ \eta \in \prod_{k=1}^{\infty} \lambda^p(A_k) : \|\eta\|_m \le 1 \Big\} \quad \text{and}$$
$$V_m = \{ \xi \in \lambda^p(A) : \|\xi\|_m \le 1 \}.$$

3.3. Lemma. *For $1 \le p \le \infty$ we have*

$$2Q(V_m \cap rV_{m+1}) \supset U_m \cap rU_{m+1} \quad \textit{for all } m \textit{ and } r > 0.$$

Proof. Let $m \in \mathbb{N}$ and $r \ge 1$, let $\eta \in \prod_{k=1}^{\infty} \lambda^p(A_k)$ satisfy $\|\eta\|_m \le 1$ and $\|\eta\|_{m+1} \le r$, let $\varepsilon > 0$. We put $i(j,k) = 1$ if $k \le m$ and choose $i(j,k)$ very large if $k > m$. We define $\xi_{i,j,k} = \eta_{j,k}$ if $i = i(j,k)$ and $\xi_{i,j,k} = 0$ if $i \ne i(j,k)$. Then $Q\xi = \eta$. For $1 \le p < \infty$ we obtain

$$\|\xi\|_m = \Big(\sum_{j,k} |\eta_{j,k}|^p a^p_{i(j,k),j,k;m} \Big)^{1/p} \le 1 + \varepsilon,$$
$$\|\xi\|_{m+1} = \Big(\sum_{j,k} |\eta_{j,k}|^p a^p_{i(j,k),j,k;m+1} \Big)^{1/p} \le r + \varepsilon$$

provided that $i(j,k)$ is large enough for $k > m$. Here we have used (1) and (3) of 3.2. The same proof gives the case $p = \infty$.

4. In this section we apply Lemma 3.2 to construct the desired sequences of the form (*). By reason of 2.3(ii) and 2.4 we only have to consider strongly stable sequences α.

Let $1 \le \alpha_1 \le \alpha_2 \le \ldots \nearrow \infty$ and assume that $\lim_n \alpha_{2n}/\alpha_n = 1$. We put $n_1 = 0$, $A_1 = \alpha_1$, $A_2 = \max\{\alpha_3, 2\}$ and $n_2 = \max\{n : \alpha_n \le A_2\}$. If A_i and n_i are already defined, then we put

$$A_{i+1} = \max\{\alpha_{3n_i}, i+1\} \quad \text{and} \quad n_{i+1} = \max\{n : \alpha_n \le A_{i+1}\}.$$

We observe that $n_{i+1} \ge 3n_i$, hence $n_i < n_{i+1}$, and $A_{i+1}/A_i \le \max\{\alpha_{3n_i}/\alpha_{n_i}, (i+1)/i\}$, $i \ge 2$. We define

$$\beta_n = A_i \quad \text{if } n_i < n \le n_{i+1}.$$

For $n_i < n \le n_{i+1}$ we obtain

$$\beta_n = A_i \le \alpha_n \le \alpha_{n_{i+1}} \le A_{i+1} = \frac{A_{i+1}}{A_i} \beta_n,$$

hence $\lim_n \alpha_n/\beta_n = 1$ and $\lim_n \beta_{2n}/\beta_n = 1$. We set $m_i = n_{i+1} - n_i$ and see that $m_i \geq 2n_i$ and $m_i \leq m_{i+1}$, $i \geq 1$. We define $N : \mathbb{N}^2 \to \mathbb{N}$ by

$$N(i,k) = n_{i_k+i-1} + k \quad \text{where } i_k = \min\{j \in \mathbb{N} : m_j \geq k\}.$$

Then N is a bijection: The injectivity follows from $n_{i_k+i-1} + k \leq n_{i_k+i}$ which is true for $i = 1$ by definition of i_k and follows for $i > 1$ since m_i is increasing; if, conversely, $n_j < m \leq n_{j+1}$, then we can put $k = m - n_j$ and $i = j + 1 - i_k$ since $i_k \leq j$ because $m_j \geq k$, which proves surjectivity.

We define a second bijection $M : \mathbb{N}^2 \to \mathbb{N}$ by $M(j,k) = 2^{k-1} + (j-1)2^k$ and obtain a bijection $n : \mathbb{N}^3 \to \mathbb{N}$, namely $n(i,j,k) = N(i, M(j,k))$.

Now let $r_0 < r_1 < r_2 < \ldots$ and define

$$a_{i,j,k;m} = e^{(r_m - r_k)\beta_{n(i,j,k)}}.$$

Then conditions (1)–(3) of Lemma 3.2 are clear, and (4) follows (e.g. for $s_r(m) = m+1$) since

$$\beta_{n(i,j,k)} \geq \beta_{n_i+1} \geq A_i \geq i.$$

From Lemma 3.2 we hence obtain a tamely exact sequence. We now prove that the Köthe sequence spaces $\lambda^p(\cdot)$ in that sequence are tamely isomorphic to $\Lambda_R^p(\alpha)$ where $R = \sup_k r_k \geq 0$ and $1 \leq p \leq \infty$.

We equip $\Lambda_R^p(\alpha)$ and $\Lambda_R^p(\beta)$ with the gradings $\|\ \|_k = |\ |_{r_k}$. Since $\lim_n \alpha_n/\beta_n = 1$ we see that $\Lambda_R^p(\alpha)$ and $\Lambda_R^p(\beta)$ are tamely isomorphic. A diagonal transformation with $(e^{r_k\beta_{n(i,j,k)}})_{(i,j,k)}$ induces an isometric isomorphism $\lambda^p(A) \cong \Lambda_R^p(\beta)$. For every k, the diagonal transformation with $(e^{r_k\beta_{n(1,j,k)}})_j$ gives an isometric isomorphism $\lambda^p(A_k) \cong \Lambda_R^p((\beta_{n(1,j,k)})_j)$.

We note that $n_j \leq (3/2)(n_j - n_{j-1}) = (3/2)m_{j-1}$ implies that $(2/3)n_{i_m} \leq m_{i_m-1} \leq m$ by the definition of i_m. We have $n(1,j,k) = n_{i_M} + M$ where $M = M(j,k)$; hence $n_{i_m} \leq 2m$ implies that $2^{k-1}(2j-1) \leq n(1,j,k) \leq 3 \cdot 2^{k-1}(2j-1)$. Since β is strongly stable, we conclude that $\lambda^p(A_k)$ and $\Lambda_R^p(\beta)$ are tamely isomorphic, and so are $\prod_{k=1}^\infty \lambda^p(A_k)$ and $\Lambda_R^p(\beta)^{\mathbb{N}}$.

It remains to determine the kernel of the sequence. By definition we have

$$\beta_{n(i,j,k)} = A_{i_M+i-1}, \quad \beta_{n(i+1,j,k)} = A_{i_M+i}, \qquad M = M(j,k).$$

We distinguish the cases $0 < R \leq \infty$ and $R = 0$; if $R > 0$ we assume that $r_0 \geq 0$. We get for $R > 0$,

$$\begin{aligned} e^{(r_m - r_k)\beta_{n(i,j,k)}} &\leq a_{i,j,k;m} + a_{i+1,j,k;m} \\ &= e^{(r_m-r_k)\beta_{n(i,j,k)}} + e^{(r_m-r_k)\frac{A_{i_M+i}}{A_{i_M+i-1}}\beta_{n(i,j,k)}} \\ &\leq 2e^{(\frac{A_{i_M+i}}{A_{i_M+i-1}}r_m - r_k)\beta_{n(i,j,k)}} \leq c_m e^{(r_{m+1}-r_k)\beta_{n(i,j,k)}} \end{aligned}$$

since $A_{i_M+i}/A_{i_M+i-1} \leq r_{m+1}/r_m$ for all except finitely many (M,i). For $R = 0$ we get

$$\begin{aligned} e^{(r_m - r_k)\beta_{n(i+1,j,k)}} &\leq a_{i,j,k;m} + a_{i+1,j,k;m} \leq 2e^{r_m\beta_{n(i,j,k)} - r_k\beta_{n(i+1,j,k)}} \\ &\leq 2e^{(\frac{A_{i_M+i-1}}{A_{i_M+i}}r_m - r_k)\beta_{n(i+1,j,k)}} \leq c_m e^{(r_{m+1}-r_k)\beta_{n(i+1,j,k)}} \end{aligned}$$

since $(A_{i_M+i-1}/A_{i_M+i})r_m \leq r_{m+1}$ for all except finitely many (M,i). In both cases, a suitable diagonal transformation shows that $\lambda^p(A_K)$ and $\Lambda_R^p(\beta)$ are tamely isomorphic.

We have proved:

4.1. Theorem. *Let α be strongly stable, let $r_0 < r_1 < r_2 < \ldots \nearrow R \in [0,\infty]$, let $1 \leq p < \infty$. Then there exists a tamely exact sequence*

$$(*) \qquad 0 \to \Lambda_R^p(\alpha) \xrightarrow{i} \Lambda_R^p(\alpha) \xrightarrow{q} \Lambda_R^p(\alpha)^{\mathbb{N}} \to 0.$$

Moreover, we can obtain the following continuity estimates. We write $q = (q_k)_{k=1}^\infty$. For every m and $\varepsilon > 0$ there is a constant $D = D_{m,\varepsilon} > 0$ such that

$$|i\xi|_{r_m-\varepsilon} \leq D|\xi|_{r_m+\varepsilon}, \qquad |\xi|_{r_m-\varepsilon} \leq D|i\xi|_{r_m+\varepsilon},$$

$$\sum_{k=1}^m |q_k x|_{r_m-\varepsilon} \leq D|x|_{r_m+\varepsilon}, \qquad \inf\{|z|_{r_m-\varepsilon} : qz = y\} \leq D\sum_{k=1}^m |y^k|_{r_m+\varepsilon}$$

for $\xi, x \in \Lambda_R^p(\alpha)$ and $y = (y^k)_{k=1}^\infty \in \Lambda_R^p(\alpha)^{\mathbb{N}}$. Furthermore, for

$$V_t = \{\xi \in \Lambda_R^p(\alpha) : |\xi|_t \leq 1\} \quad \text{and}$$

$$U_{m,t} = \Big\{\eta = (\eta^k)_k \in \Lambda_R^p(\alpha)^{\mathbb{N}} : \sum_{k=1}^m |\eta^k|_t \leq 1\Big\}$$

we have

$$Dq(V_{r_m-\varepsilon} \cap rV_{r_{m+1}-\varepsilon}) \supset U_{m,r_m+\varepsilon} \cap rU_{m+1,r_{m+1}+\varepsilon} \quad \text{for all } r > 0.$$

For the linear-tame case, by means of 2.3 and 2.4, from 4.1 we obtain

4.2. Theorem. *Let α be uniformly stable, let $r_0 < r_1 < r_2 < \ldots \nearrow R \in \{0,\infty\}$, let $1 \leq p < \infty$. Assume that*

$$\limsup_n \frac{\alpha_{Kn}}{\alpha_n} < \begin{cases} \liminf_k r_{ck+d}/r_k & \text{for all } K \text{ if } R = \infty, \\ \liminf_k r_k/r_{ck+d} & \text{for all } K \text{ if } R = 0. \end{cases}$$

Then there exists a (c^2)-tamely, i.e. linear-tamely exact sequence

$$(*) \qquad 0 \to \Lambda_R^p(\alpha) \to \Lambda_R^p(\alpha) \to \Lambda_R^p(\alpha)^{\mathbb{N}} \to 0.$$

4.3. Corollary. *Let $R \in \{0,\infty\}$. Assume that there is b such that $\lim_k r_{k+b}/r_k = 1$. Then the following conditions on α are equivalent:*

(i) $\Lambda^p_R(\alpha) \times \Lambda^p_R(\alpha)$ *is tamely isomorphic to a graded quotient space of* $\Lambda^p_R(\alpha)$ *for (some or any)* $1 \le p \le \infty$.

(ii) *There exists a tamely exact sequence* $(*)$ *for (some or any)* $1 \le p < \infty$.

(iii) α *is strongly stable.*

4.4. Corollary. *Let* $R \in \{0, \infty\}$. *Assume that there are* $c \ge 1$ *and* $d \ge 0$ *such that*

$$1 < \liminf_k \frac{r_{ck+d}}{r_k} < \infty \quad \text{if } R = \infty$$

$$\text{resp.} \quad 1 < \liminf_k \frac{r_k}{r_{ck+d}} < \infty \quad \text{if } R = 0.$$

Then the following conditions on α *are equivalent:*

(i) $\Lambda^p_R(\alpha)^{\mathbb{N}}$ *is linear-tamely isomorphic to a graded quotient space of* $\Lambda^p_R(\alpha)$ *for (some or any)* $1 \le p \le \infty$.

(ii) *There exists a linear-tamely exact sequence* $(*)$ *for (some or any)* $1 \le p < \infty$.

(iii) α *is uniformly stable.*

Of course, the assumptions on r_k in 4.3, 4.4 are satisfied for the standard gradings $r_k = k$ if $R = \infty$ resp. $r_k = -1/k$ if $R = 0$.

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