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A rigid space admitting compact operators

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Abstract. A rigid space is a topological vector space whose endomorphisms are all simply scalar multiples of the identity map. The first complete rigid space was published in 1981 in [2]. Clearly a rigid space is a trivial-dual space, and admits no compact endomorphisms. In this paper a modification of the original construction results in a rigid space which is, however, the domain space of a compact operator, answering a question that was first raised soon after the existence of complete rigid spaces was demonstrated.

Rigid topological vector spaces made their first appearance in the literature in 1977 with an example by Waelbroeck, in the paper [7]. This first example was, however, an incomplete space, and the first complete rigid space appeared in Kalton's and Roberts' paper [2], based on a construction of Roberts from 1977 which followed Waelbroeck's space. The published version differed slightly in form from the unpublished example, and extended the result to obtain a complete space that was not only rigid, but also quotient-rigid and a subspace of $L_0[0,1]$ (quotient-rigid meaning that every quotient of the space inherits the rigid character). Roberts' original method of construction appeared in print in 1984 in the book [1], and it is this version that is used as the starting point for the rigid space in this paper. The following construction is based on part of the author's PhD dissertation, [5].

The question of the existence of compact operators with trivial-dual domain spaces was answered not long before the appearance of rigid spaces, in the paper [3] of 1975. Of course, the lack of continuous linear functionals on a space implies the absence of the simplest compact operators, those of finite rank. Following the work in 1973 of Pallaschke [4] and Turpin [6], in which a variety of spaces (including $L_p[0,1]$ for $0 < p < 1$) were shown to have no compact endomorphisms, Pełczyński asked whether any trivial-dual space possessed compact endomorphisms. Since these early papers, some progress has been made in constructing trivial-dual spaces admitting

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compact operators, and a few large classes of spaces have been shown to be bereft of compact operators, but a general characterization of trivial-dual spaces admitting compact operators is still not known. A characterization of separable p-Banach spaces admitting compact operators appears in [5], and this characterization motivated the construction of the rigid space to follow (which is a p-Banach space, for some fixed $0 < p < 1$).

The paper is organized into three sections, the first consisting of some definitions and simple lemmas which shall be needed, the second containing the construction of the space, and the third the verification that the space is rigid and does admit compact operators.

1. Definitions and lemmas. The rigid space to be constructed is a p-Banach space; that is, it is a complete locally bounded and locally p-convex space. Although the topologies of p-Banach spaces are often described by p-norms (which are sub-additive and p-homogeneous), the construction to follow relies on a generalization of a p-norm called a super p-norm.

DEFINITION. Assume $0 < p < 1$ is fixed and X is a vector space. A map $\|\cdot\| : X \to [0,\infty)$ is a *super p-norm* if

(1) $\|x\| = 0 \Leftrightarrow x = 0$,
(2) $\|x+y\| \le \|x\| + \|y\|$ and
(3) for any $x \in X$ and any $\alpha \in \mathbb{R}$ with $|\alpha| \le 1$, $\|\alpha x\| \le |\alpha|^p \|x\|$.

If condition (1) fails to hold, $\|\cdot\|$ is a *super p-seminorm*, and if (3) is replaced with the weaker condition

(3) $\|\alpha x\| \le \|x\|$ whenever $|\alpha| \le 1$ and $\|\alpha x\| \to 0$ whenever $|\alpha| \to 0$

then $\|\cdot\|$ is an *F-norm*.

The following definition introduces a method of constructing new F-norms from old ones which will be employed often.

DEFINITION. Let $\{(X_\alpha, \|\cdot\|_\alpha)\}$ be a collection of F-normed vector spaces, and let $X = \operatorname{span}\{X_\alpha\}$. For $x \in X$, let

$$\|x\| = \inf\Big\{\sum_{i=1}^{n} \|x_{\alpha_i}\|_{\alpha_i} : x = \sum_{i=1}^{n} x_{\alpha_i},\ x_{\alpha_i} \in X_{\alpha_i},\ n \in \mathbb{N}\Big\}.$$

$\|\cdot\|$ will be referred to as inf-norm$\{(X_\alpha, \|\cdot\|_\alpha)\}$.

LEMMA 1.1. *The real-valued map $\|\cdot\| : X \to [0,\infty)$ defined in the above definition is an F-seminorm on the space X. Further, if each $\|\cdot\|_\alpha$ is a super p-norm (for some fixed p), then $\|\cdot\|$ will be a super p-seminorm.*

The proof of Lemma 1.1 is almost immediate, and really only requires confirmation that $\|\cdot\|$ is subadditive. A discussion of inf-norms may be found in [1]. Note, however, that without some further assumptions on the collection of F-normed vector spaces, $\|\cdot\|$ may not separate points in X, as it may be possible to decompose a given $x \in X$ as $x = x_{\alpha_1} + \ldots + x_{\alpha_n}$ in such a way that $\sum \|x_{\alpha_i}\|_{\alpha_i}$ is arbitrarily small. The next lemma gives us some control over the resultant inf-norm.

LEMMA 1.2. *Let $X_1 \subset X_2 \subset \ldots$ be a sequence of finite-dimensional vector spaces, each space endowed with an F-norm $\|\cdot\|_n$. Assume that $\|\cdot\|_n \ge \|\cdot\|_{n-1}$ on X_{n-1}. Let $\|\cdot\| = \text{inf-norm}\langle(X_n, \|\cdot\|_n)\rangle$. Then*

$$x \in X_{n_0} \Rightarrow \|x\| = \inf\Big\{\sum_{i=1}^{n_0} \|x_i\|_i : x = \sum_{i=1}^{n_0} x_i,\ x_i \in X_i\Big\}.$$

Further, this infimum is realized, so there exist $x_1 \in X_1, \ldots, x_{n_0} \in X_{n_0}$ for which

$$x = \sum_{i=1}^{n_0} x_i \quad \textit{and} \quad \|x\| = \sum_{i=1}^{n_0} \|x_i\|_i.$$

Proof. Let $x \in X_{n_0}$ and let $\varepsilon > 0$. By the definition of $\|\cdot\|$, there is an $m \in \mathbb{N}$ and $x_1 \in X_1, \ldots, x_m \in X_m$ so that

$$x = \sum_{i=1}^{m} x_i \quad \text{and} \quad \sum_{i=1}^{m} \|x_i\|_i < \|x\| + \varepsilon.$$

Suppose that $m > n_0$. Then by the embedded nature of the sequence of subspaces,

$$x_m = x - \sum_{i=1}^{m-1} x_i \in X_{m-1}$$

and so by hypothesis $\|x_m\|_m \ge \|x_m\|_{m-1}$. If we define $y_i = x_i$ for $1 \le i \le m-2$ and $y_{m-1} = x_{m-1} + x_m$, we have

$$\|x\| + \varepsilon > \sum_{i=1}^{m} \|x_i\|_i = \sum_{i=1}^{m-2} \|x_i\|_i + \|x_{m-1}\|_{m-1} + \|x_m\|_m$$
$$\ge \sum_{i=1}^{m-2} \|x_i\|_i + \|x_{m-1} + x_m\|_{m-1} = \sum_{i=1}^{m-1} \|y_i\|_i.$$

Continuing by induction, we obtain $y_1 \in X_1, \ldots, y_{n_0} \in X_{n_0}$ with

$$x = \sum_{i=1}^{n_0} y_i \quad \text{and} \quad \sum_{i=1}^{n_0} \|y_i\|_i < \|x\| + \varepsilon.$$

As ε is arbitrary, this establishes the first claim.

To see that the infimum is realized, we use the finite dimensionality of the spaces and a compactness argument, as follows. For this same $x \in X_{n_0}$, let

$$S_x = \Big\{(x_1, \ldots, x_{n_0}) : x = \sum_{i=1}^{n_0} x_i \text{ and } \sum_{i=1}^{n_0} \|x_i\|_i \le 2\|x\|\Big\}.$$

Then S_x is compact, and the map taking

$$(x_1, \ldots, x_{n_0}) \mapsto \sum_{i=1}^{n_0} \|x_i\|_i$$

is a continuous function from S_x into $\mathbb{R}$, and hence attains its minimum. ∎

Note that this implies $\|\cdot\|$ is an F-norm in this case, for given $x \in X_{n_0}$ there are elements $x_1, \ldots, x_{n_0}$ so that

$$x = \sum_{i=1}^{n_0} x_i \quad \text{and} \quad \|x\| = \sum_{i=1}^{n_0} \|x_i\|_i$$

and it is clear that $\|x\| = 0 \Leftrightarrow \|x_i\|_i = 0$ for each i, and thus $x_i = 0$ for each i and so $x = 0$. Hence $\|\cdot\|$ separates points on $\bigcup X_n$, and thus on the $\|\cdot\|$-completion of $\bigcup X_n$.

It is clear that if each $\|\cdot\|_n$ is a super p-norm, then $\|\cdot\|$ will be a super p-norm as well. The next two lemmas and the remark following them will give us a bit more control over the inf-norms to be used in the construction.

LEMMA 1.3. *Let $\|\cdot\|_1$ be a norm on a finite-dimensional vector space X, and let $\|\cdot\|_2$ be a p-norm on X, for some fixed $0 < p < 1$. Then there is a $\delta_1 > 0$ so that $\|\cdot\|_1 \le \|\cdot\|_2$ on $\{x \in X : \|x\|_2 \le \delta_1\}$, and a $\delta_2 > 0$ so that $\|\cdot\|_1 \ge \|\cdot\|_2$ on $\{x \in X : \|x\|_1 \ge \delta_2\}$.*

Proof. Let $m_1 = \max\{\|x\|_1 : \|x\|_2 = 1\}$. By the finite dimensionality of X, $0 < m_1 < \infty$. Let $\delta_1 = m_1^{p/(p-1)}$. Then given $y \in X$ with $\|y\|_2 = \delta_1$, there is a positive scalar λ and an $x \in X$ so that $y = \lambda x$ and $\|x\|_2 = 1$. Then $\lambda^p = \delta_1$, so $\lambda = m_1^{1/(p-1)}$. Hence

$$\|y\|_1 = \|\lambda x\|_1 = \lambda\|x\|_1 \le \lambda m_1 = \lambda^p = \|\lambda x\|_2 = \|y\|_2.$$

The fact that $\|\cdot\|_1 \le \|\cdot\|_2$ on $\{x \in X : \|x\|_2 \le \delta_1\}$ now follows from considering the type of homogeneity of the two F-norms.

The second claim is proved similarly, by first letting $m_2 = \max\{\|x\|_2 : \|x\|_1 = 1\}$ and then letting $\delta_2 = m_2^{-1/(p-1)}$. ∎

LEMMA 1.4. *With X, $\|\cdot\|_1$, $\|\cdot\|_2$, δ_1 and δ_2 defined as in Lemma 1.3, construct the new F-norm $\|\cdot\| = \text{inf-norm}\{(X, \|\cdot\|_1), (X, \|\cdot\|_2)\}$. Then $\|\cdot\|$ is homogeneous on the set $\{x \in X : \|x\| \le \varepsilon\}$ for any $0 < \varepsilon < \delta_1$. Also, $\|\cdot\|$ is asymptotically sub-p-homogeneous, i.e. given $C > 1$ there is a constant M so that $\|x\| \ge M$ and $|\lambda| \ge 1$ implies $\|\lambda x\| \le C|\lambda|^p\|x\|$.*

Proof. To prove the first claim, fix such an ε and choose $x \in X$ with $\|x\| \le \varepsilon$. Then

$$\inf\{\|x_1\|_1 + \|x_2\|_2 : x = x_1 + x_2\} \le \varepsilon < \delta_1,$$

and so we may as well assume the infimum is taken over those x_2's for which $\|x_2\|_2 < \delta_1$. Then by Lemma 1.3,

$$\|x\| = \inf\{\|x_1\|_1 + \|x_2\|_2\} \ge \inf\{\|x_1\|_1 + \|x_2\|_1\} \ge \inf\{\|x_1 + x_2\|_1\} = \|x\|_1.$$

By the definition of inf-norms, $\|\cdot\| \le \|\cdot\|_1$ on all of X, so we have $\|\cdot\| = \|\cdot\|_1$ on $\{x \in X : \|x\| \le \varepsilon\}$, i.e. $\|\cdot\|$ is homogeneous on this set.

Now let $C > 1$ be fixed. To prove the second claim, we will first show that for some constant M, $\|\cdot\| \ge C^{-1}\|\cdot\|_2$ on $\{x \in X : \|x\| \ge M\}$. Suppose this is not true for any M; that is, suppose there are arbitrarily large M's for which we can find an x with $\|x\| = M$ and $\|x\| < C^{-1}\|x\|_2$. Then let $a = \max\{\|x\|_2 : \|x\|_1 \le \delta_2\}$, and assume the existence of $M > (C^{-1}/(1-C^{-1}))a$ and x with $\|x\| = M$ and $\|x\| < C^{-1}\|x\|_2$. By a compactness argument as in Lemma 1.2, there is a decomposition of x as $x = x_1 + x_2$ with $\|x\| = \|x_1\|_1 + \|x_2\|_2$. Note that $\|x_1\|_1 \le \delta_2$, for otherwise by Lemma 1.3 we would have

$$\|x\| = \|x_1\|_1 + \|x_2\|_2 > \|x_1\|_2 + \|x_2\|_2 \ge \|x\|_2,$$

a contradiction to the fact that $\|\cdot\| \le \|\cdot\|_2$ on all of X. Then $\|x\|_2 - \|x_2\|_2 \le \|x - x_2\|_2 = \|x_1\|_2 \le a$, so $\|x\|_2 \le \|x_2\|_2 + a$. Also, $\|x_2\|_2 \le M$ since $\|x_1\|_1 + \|x_2\|_2 = M$. This gives us

$$M = \|x\| < C^{-1}\|x\|_2 \le C^{-1}(\|x_2\|_2 + a) \le C^{-1}(M + a) < M,$$

a contradiction, indicating such an M must exist.

Now, to complete the argument, choose any x with $\|x\| \ge M$ and any scalar $|\lambda| \ge 1$. Then

$$\|\lambda x\| \le \|\lambda x\|_2 = |\lambda|^p\|x\|_2 \le C|\lambda|^p\|x\|. \ ∎$$

Remark 1.1. Let us call an F-norm which is homogeneous on a neighborhood of 0 a *crimped* F-norm. Then if $\|\cdot\|_1, \ldots, \|\cdot\|_n$ are all crimped F-norms, defined respectively on finite-dimensional spaces $X_1, \ldots, X_n$, an argument similar to that of Lemma 1.4 shows that $\|\cdot\| = \text{inf-norm}\{(X_1, \|\cdot\|_1), \ldots, (X_n, \|\cdot\|_n)\}$ is homogeneous on some 0-neighborhood of the $\|\cdot\|$-completion of $\bigcup_{i=1}^{n} X_i$, i.e. $\|\cdot\|$ is also crimped. In addition, if each $\|\cdot\|_i$ is asymptotically sub-p-homogeneous, then $\|\cdot\|$ will be asymptotically sub-p-homogeneous as well.

As already mentioned, the rigid space to be constructed is a p-Banach space, though the norm to be built up will be a super p-norm in the style of Lemma 1.2. However, the unit ball of a super p-normed space is clearly absolutely p-convex, and just as clearly a bounded neighborhood of $\{0\}$, and so a complete super p-normed space is in fact a p-Banach space.

One last basic lemma is required before we begin the construction of the space. Since super p-norms do not necessarily have any sort of homogeneity with regard to scalars, continuity of linear operators on such spaces is not as easily discussed as in, say, p-normed spaces. The following inequality will, however, be sufficient for our purposes.

LEMMA 1.5. *Let $0 < p \le 1$ and let $(X, \|\cdot\|)$ be a super p-normed space. Let $T \in \mathcal{L}(X)$ and suppose that $\|x\| = 1 \Rightarrow \|Tx\| < 1/2$. Then for any non-zero x,*

$$\|x\| \le 1 \Rightarrow \|Tx\| < \|x\|^p.$$

Proof. We first need a simple fact about F-norms: if α is a scalar with $|\alpha| \le 1$, and if x is an element of X, then

$$\|\alpha x\| \ge \frac{1}{2}|\alpha|\|x\|. \tag{1.1}$$

This can be seen by choosing $n \in \mathbb{N}$ so that $|\alpha| \ge 1/n \ge |\alpha|/2$ and then noting that

$$\|\alpha x\| \ge \left\|\frac{1}{n}x\right\| \ge \frac{1}{n}\|x\| \ge \frac{1}{2}|\alpha|\,\|x\|.$$

Now let $x \in X$ with $\|x\| \le 1$. So for some $y \in X$, $x = \alpha y$ where $\|y\| = 1$ and $0 \le \alpha \le 1$. Then

$$\|x\| = \|\alpha y\| \ge \frac{1}{2}|\alpha|\,\|y\| = \frac{1}{2}\alpha$$

where the middle inequality is due to (1.1). Hence, $\alpha \le 2\|x\|$. This then gives us

$$\|Tx\| = \|\alpha Ty\| < (\alpha^p)\left(\frac{1}{2}\right) \le (2^p\|x\|^p)\left(\frac{1}{2}\right) \le \|x\|^p,$$

the desired inequality. ∎

2. Construction of the space. The rigid space to be made will be the completion of the union of a monotonically increasing sequence of finite-dimensional subspaces, $X_1 \subset X_2 \subset \ldots$ Each X_n will represent the algebraic sum of finite-dimensional subspaces $V_1, V_2, \ldots, V_n$, where each V_i is a subspace of V, the space of finitely non-zero real-valued sequences. To begin, let $\langle e_n \rangle$ be the usual basis of V, i.e. e_n is the sequence with a 1 in the nth slot and zeros elsewhere. Let

$$A = \{e_1 + e_n\}_{n=2}^{\infty} \cup \{e_1 - e_n\}_{n=2}^{\infty} \cup \{e_1\}$$

and let $\langle a_n \rangle$ be a sequence in A such that for each $a \in A$, $a = a_n$ for infinitely many n. In what follows, F_n will represent a finite collection of elements of V and V_n will be the span of F_n. Each V_n will be endowed with a super p-norm $|\cdot|_n$, where we are assuming that p is a fixed number in $(0,1)$. X_n will be the space $V_1 + \ldots + V_n$, and X_n will be given the super p-norm $\|\cdot\|_n = \text{inf-norm}\{(V_1, |\cdot|_1), \ldots, (V_n, |\cdot|_n)\}$. For each n and non-negative r, let $B_r^n = \{x \in X_n : \|x\|_n < r\}$.

The remainder of this section is concerned with showing the existence of the above objects, along with a sequence of positive numbers $\langle \varepsilon_n \rangle$ and numbers $p = p_1 < p_2 < \ldots < 1$ so that the following five conditions hold:

(1) $\|\cdot\|_n \ge \|\cdot\|_{n-1}/2$ on X_{n-1},

(2) $a_n \in V_n$ and $(V_1 + \ldots + V_{n-1}) \cap V_n \subset \mathbb{R}a_n$,

(3) $\operatorname{co} B_{\varepsilon_n^p}^{n-1} \subset B_{\varepsilon_{n-1}/4^{n-2}}^{n-1}$,

(4) If $F_n = \{x_1, \ldots, x_m\}$ then $|x_i|_n < \varepsilon_n/4^{n-1}$ and $a_n = (1/m)\sum_{i=1}^m x_i$,

(5) If $M_{n-1} = \max\{|F_1|, \ldots, |F_{n-1}|\}$ then $M_{n-1}^{1-p_n} < 2$.

Note that condition (3) implies that at the least $\varepsilon_n^p < \varepsilon_{n-1}$, and so $\langle \varepsilon_n \rangle$ decreases to 0.

The existence of the sequence $\langle (V_n, F_n, |\cdot|_n, \varepsilon_n, p_n) \rangle$ with the above properties is verified via induction on n. To begin, let $p_1 = p$ and choose $0 < \varepsilon < 1$. It is now relatively easy to find V_1, F_1 and $|\cdot|_1$ appropriately. For instance, let $V_1 = \operatorname{span}\{a_1\}$ and let $F_1 = \{a_1\}$. Now define $|\cdot|_1^{\mathrm{t}}$ on V_1 by $|\lambda a_1|_1^{\mathrm{t}} = \beta|\lambda|^{p_1}$, where $0 < \beta < \varepsilon_1$ is fixed. The "t" in $|\cdot|_1^{\mathrm{t}}$ stands for "temporary", as each $|\cdot|_n^{\mathrm{t}}$ needs to be replaced with $|\cdot|_n$, a crimped version of itself, in order to obtain property (1) above. Accordingly, let $|\cdot|$ be any norm on V_1, and let $|\cdot|_1 = \text{inf-norm}\{(V_1, |\cdot|), (V_1, |\cdot|_1^{\mathrm{t}})\}$. Then $|\cdot|_1$ is still a super p-norm, and since $|\cdot|_1 \le |\cdot|_1^{\mathrm{t}}$, (4) holds by the choice of β. The other properties hold vacuously.

Now assume that $(V_k, F_k, |\cdot|_k, \varepsilon_k, p_k)$ have been selected up to $k = n-1$. The space X_{n-1} is finite-dimensional, and hence is locally convex. By definition then, X_{n-1} has a base at $\{0\}$ of convex neighborhoods, and it is thus possible to choose an $\varepsilon_n > 0$ so that condition (3) holds. Define M_{n-1} as in condition (5) and choose $p_{n-1} < p_n < 1$ so that (5) holds. On the one-dimensional space $\mathbb{R}a_n$ define a super p-norm $|\cdot|_\beta$ by $|\lambda a_n|_\beta = \beta|\lambda|^{p_n}$, where $\beta > 0$ is yet to be chosen. Note that by the subadditivity of all the F-norms so far created,

$$\begin{aligned}\text{inf-norm}\{(V_1, |\cdot|_1), \ldots, (V_{n-1}, |\cdot|_{n-1}), (\mathbb{R}a_n, |\cdot|_\beta)\}\\ = \text{inf-norm}\{(X_{n-1}, \|\cdot\|_{n-1}), (\mathbb{R}a_n, |\cdot|_\beta)\}.\end{aligned} \tag{2.1}$$

We now have two cases to consider.

Case 1: $a_n \notin V_1 + \ldots + V_{n-1}$. First, suppose $x \in X_{n-1}$ and $x = v + \lambda a_n$, where $v \in X_{n-1}$. Then $\lambda a_n = x - v \in X_{n-1}$, and so $\lambda = 0$. It is clear then that on X_{n-1},

(2.2) $$\text{inf-norm}\{(X_{n-1}, \|\cdot\|_{n-1}), (\mathbb{R}a_n, |\cdot|_\beta)\} = \|\cdot\|_{n-1}$$

for any $\beta > 0$. Now V_n, F_n and $|\cdot|_n$ can be easily chosen, by letting $F_n = \{a_n\}$, $V_n = \operatorname{span} F_n$ and $|\cdot|_n^{\mathrm{t}} = |\cdot|_\beta$, where $0 < \beta < \varepsilon_n/4^{n-1}$. As before, replace $|\cdot|_n^{\mathrm{t}}$ with a crimped version, $|\cdot|_n$. Then by (2.1) and (2.2) (with β replaced by n and $\mathbb{R}a_n$ replaced with V_n in both cases), $\|\cdot\|_n = \|\cdot\|_{n-1}$ on all of X_{n-1}, and so condition (1) is trivially satisfied. Conditions (3) and (5) hold by the choice of ε_n and p_n above, and condition (2) is satisfied by the definition of F_n and V_n. Finally, fixing β in the range specified allows condition (4) to be satisfied.

Case 2: $a_n \in V_1 + \ldots + V_{n-1}$. By the induction hypothesis, each $|\cdot|_i$ is a crimped super p-norm, for $1 \le i \le n-1$. By Remark 1.1 then, $\|\cdot\|_{n-1}$ is a crimped super p-norm, and there is a $\delta > 0$ for which $\|\cdot\|_{n-1}$ is homogeneous on $\{x \in X_{n-1} : \|x\|_{n-1} \le \delta\}$. Also, since $p_1 < \ldots < p_{n-1}$, $\|\cdot\|_{n-1}$ is asymptotically sub-p_{n-1}-homogeneous, and there is an M for which $\|\lambda x\|_{n-1} \le 2|\lambda|^{p_{n-1}}\|x\|_{n-1}$ whenever $x \in \{x \in X_{n-1} : \|x\|_{n-1} \ge M\}$ and $|\lambda| \ge 1$. Note that $|\cdot|_\beta$ increases monotonically without bound on the non-zero elements of $\mathbb{R}a_n$ as β increases, and so by Dini's Theorem there is a β for which $|\cdot|_\beta \ge \|\cdot\|_{n-1}$ on $\{x \in \mathbb{R}a_n : \delta \le \|x\|_{n-1} \le M\}$. Since $\|\cdot\|_{n-1}$ is homogeneous on $\{x \in \mathbb{R}a_n : \|x\|_{n-1} \le \delta\}$ and $|\cdot|_\beta$ is p_n-homogeneous, we also have $|\cdot|_\beta \ge \|\cdot\|_{n-1}$ on $\{x \in \mathbb{R}a_n : \|x\|_{n-1} \le \delta\}$. Finally, if $x \in \mathbb{R}a_n$, $\|x\|_{n-1} = M$ and $|\lambda| \ge 1$, we have

$$\|\lambda x\|_{n-1} \le 2|\lambda|^{p_{n-1}}\|x\|_{n-1} < 2|\lambda|^{p_n}\|x\|_{n-1} \le 2|\lambda|^{p_n}|x|_\beta = 2|\lambda x|_\beta,$$

that is, $|x|_\beta \ge \|x\|_{n-1}/2$ whenever $\|x\|_{n-1} \ge M$. In sum, $|\cdot|_\beta \ge \|\cdot\|_{n-1}/2$ on all of $\mathbb{R}a_n$.

Recall that p_n has already been chosen. Let $(\ell_{p_n}, \|\cdot\|_{p_n})$ denote the sequence space ℓ_{p_n} endowed with the p_n-norm $\|\cdot\|_{p_n}$. On ℓ_{p_n} let

$$f_i = \left(\frac{\varepsilon_n/4^{n-1}}{2\beta}\right)^{1/p_n} e_i,$$

where, as usual, $\langle e_n\rangle$ represent the coordinate vectors. Choose $m \in \mathbb{N}$ so that

$$\left(\frac{\varepsilon_n/4^{n-1}}{2\beta}\right) m^{1-p_n} \ge 1.$$

Now let V_n be any m-dimensional subspace of V such that $a_n \in V_n$ and such that $(V_1 + \ldots + V_{n-1}) \cap V_n = \mathbb{R}a_n$. Define a bijective and linear map $T : \ell_{p_n}^m \to V_n$ in such a way that

$$T\left(\frac{1}{m}\sum_{i=1}^m f_i\right) = a_n.$$

(Any such map with this one restriction will suffice.) Now, for $1 \le i \le m$, let $x_i = Tf_i$ and let $F_n = \{x_1, \ldots, x_m\}$. Define $|\cdot|_n^{\mathrm{t}}$ on $V_n = \operatorname{span} F_n$ in the following manner: for each $x \in \ell_{p_n}^m$, let $|Tx|_n^{\mathrm{t}} = \beta\|x\|_{p_n}$. Then for any $x_i \in F_n$,

$$|x_i|_n^{\mathrm{t}} = |Tf_i|_n^{\mathrm{t}} = \beta\|f_i\|_{p_n} = \left(\frac{1}{2}\right)\left(\frac{\varepsilon_n}{4^{n-1}}\right) < \frac{\varepsilon_n}{4^{n-1}}.$$

Also, by the choice of the map T,

$$a_n = \frac{1}{m}\sum_{i=1}^m x_i$$

and so condition (4) is satisfied for $|\cdot|_n^{\mathrm{t}}$.

It now only remains to show that condition (1) holds. First, note that

$$|a_n|_n^{\mathrm{t}} = \beta\left\|\frac{1}{m}\sum_{i=1}^m f_i\right\|_{p_n} = \left(\frac{\varepsilon_n/4^{n-1}}{2}\right) m^{1-p_n} \ge \beta = |a_n|_\beta.$$

Since $|\cdot|_n^{\mathrm{t}}$ and $|\cdot|_\beta$ are both p_n-homogeneous, this implies that $|\cdot|_n^{\mathrm{t}} \ge |\cdot|_\beta$ on all of $\mathbb{R}a_n$. We have already shown that $|\cdot|_\beta \ge \|\cdot\|_{n-1}/2$ on $\mathbb{R}a_n$, and that $\|\cdot\|_{n-1}$ is homogeneous on a neighborhood of 0 in X_{n-1}. The homogeneity of $\|\cdot\|_{n-1}$ for small elements means we can find a norm $|\cdot|$ on V_n (necessarily dominating $\|\cdot\|_{n-1}/2$ on $\mathbb{R}a_n$) so that $|\cdot|_n = \text{inf-norm}\{(V_n, |\cdot|), (V_n, |\cdot|_n^{\mathrm{t}})\}$ satisfies $|\cdot|_n \ge \|\cdot\|_{n-1}/2$ on $\mathbb{R}a_n$. Note that condition (4) still holds for $|\cdot|_n$.

Suppose now that $x \in X_{n-1}$ and $x = v + v_n$, where $v \in X_{n-1}$ and $v_n \in V_n$. Then $v_n = x - v \in X_{n-1}$ and so it must be that $v_n = \lambda a_n$ for some scalar λ, since $(V_1 + \ldots + V_{n-1}) \cap V_n \subset \mathbb{R}a_n$. Then by (2.1) and the above,

$$\begin{aligned}\|\cdot\|_n &= \text{inf-norm}\{(V_1, |\cdot|_1), \ldots, (V_{n-1}, |\cdot|_{n-1}), (V_n, |\cdot|_n)\}\\ &= \text{inf-norm}\{(V_1, |\cdot|_1), \ldots, (V_{n-1}, |\cdot|_{n-1}), (\mathbb{R}a_n, |\cdot|_n)\}\\ &= \text{inf-norm}\{(X_{n-1}, \|\cdot\|_{n-1}), (\mathbb{R}a_n, |\cdot|_n)\}\\ &\ge \text{inf-norm}\left\{(X_{n-1}, \|\cdot\|_{n-1}), \left(\mathbb{R}a_n, \frac{1}{2}\|\cdot\|_{n-1}\right)\right\} \ge \frac{1}{2}\|\cdot\|_{n-1}.\end{aligned}$$

By induction then, the existence of the sequence $\langle(V_n, F_n, |\cdot|_n, \varepsilon_n, p_n)\rangle$ has been verified, and we can now define the space X. First define another super p-norm $|||\cdot|||_n = 4^{n-1}\|\cdot\|_n$ on each X_n. Next let $|||\cdot||| = \text{inf-norm}\langle(X_n, |||\cdot|||_n)\rangle$, and finally define X to be the $|||\cdot|||$-completion of $\bigcup X_n$. Then $(X, |||\cdot|||)$ is the desired space.

In the next section, the claims that $(X, |||\cdot|||)$ is rigid and admits compact operators will be justified, but we should first show that X as just defined actually exists; that is, the super p-norm $|||\cdot|||$ is not trivial. This follows from condition (1). If $x \in X_{n-1}$, then

$$|||x|||_n = 4^{n-1}\|x\|_n \ge (4^{n-1})\left(\frac{1}{2}\|x\|_{n-1}\right) = 2(4^{n-2}\|x\|_{n-1}) = 2|||x|||_{n-1},$$

and so the hypotheses of Lemma 1.2 are certainly satisfied.

3. Verification of the space. The goal of this section is to prove the following.

THEOREM. *The space $(X, |||\cdot|||)$ is a rigid space admitting non-trivial compact operators.*

We will show that $(X, |||\cdot|||)$ is rigid first. Accordingly, let $T \in \mathcal{L}(X)$ be an arbitrary continuous linear endomorphism. The first goal is to show that given any $a \in A$ there is a scalar λ (depending on a) so that $Ta = \lambda a$. We will then show that all the λ's are in fact the same constant, and since the linear span of A is dense in X this will imply $T = \lambda I$, proving the claim.

By considering a scalar multiple of T if necessary, we can assume that $|||x||| = 1 \Rightarrow |||Tx||| < 1/2$, and so Lemma 1.5 will apply. Fix $n \ge 2$ and let F_n be represented by $\{x_1, \ldots, x_m\}$. By condition (4) we know that $|x_i|_n < \varepsilon_n/4^{n-1}$ for each i, and so

$$|||x_i||| \le |||x_i|||_n = 4^{n-1}\|x_i\|_n \le 4^{n-1}|x_i|_n < \varepsilon_n.$$

Thus by Lemma 1.5, $x_i \in F_n \Rightarrow |||Tx_i||| < \varepsilon_n^p$. We can then choose $\delta > 0$ so that

$$\delta < \min\left\{\frac{\varepsilon_n^p}{m^{1-p}}, \varepsilon_n^p - |||Tx_1|||, \ldots, \varepsilon_n^p - |||Tx_m|||\right\}.$$

By the definition of the sequence $\langle a_k\rangle$ and the construction of the sequence of subspaces $\langle V_k\rangle$, V is dense in X. Therefore we can choose a collection $\{z_1, \ldots, z_m\} \subset V$ in such a way that $|||Tx_i - z_i||| < \delta$ for all $1 \le i \le m$. Define a map $S : V_n \to V$ by $Sx_i = z_i$. Then

$$|||Sx_i||| < |||Tx_i||| + \delta < \varepsilon_n^p \tag{3.1}$$

for all $1 \le i \le m$. By using condition (4) and the fact that $|||\cdot|||$ is a super p-norm, we obtain

$$|||Sa_n - Ta_n||| = \left|\left|\left|\frac{1}{m}\sum_{i=1}^{m}(Sx_i - Tx_i)\right|\right|\right| \le \frac{1}{m^p}\sum_{i=1}^{m}|||Sx_i - Tx_i||| < \delta m^{1-p}.$$

Now $\delta m^{1-p} < \varepsilon_n^p$ by the choice of δ, so we have

$$|||Sa_n - Ta_n||| < \varepsilon_n^p. \tag{3.2}$$

For each i, (3.1) implies the existence of $x_1^i, \ldots, x_k^i$, where $x_j^i \in V_1 + \ldots + V_j$, $Sx_i = x_1^i + \ldots + x_k^i$, and

$$\sum_{j=1}^{k} |||x_j^i|||_j < \varepsilon_n^p.$$

We can assume that the same upper index k holds for all i by allowing some of the x_j^i's to be zero, if necessary. Note that k might very well be larger than the fixed integer n. For the sake of argument, we may assume that this is the case (by again throwing in zero terms if necessary). Also, for any fixed i and for each integer j in the range from 1 to k, there exists ${}_1x_j^i, \ldots, {}_jx_j^i$ such that ${}_lx_j^i \in V_l$, $x_j^i = {}_1x_j^i + \ldots + {}_jx_j^i$, and

$$\|x_j^i\|_j = |{}_1x_j^i|_1 + \ldots + |{}_jx_j^i|_j.$$

Equality is attainable here since $V_1 + \ldots + V_j$ is a finite-dimensional space, and so a compactness argument as in Lemma 1.2 can be applied. Recall that $|||x_j^i|||_j = 4^{j-1}\|x_j^i\|_j$, so that

$$|||x_j^i|||_j = 4^{j-1}\sum_{l=1}^{j}|{}_lx_j^i|_l$$

for each $1 \le j \le k$. We now have, for each $i = 1, \ldots, m$,

$$Sx_i = \sum_{j=1}^{k}\sum_{l=1}^{j}{}_lx_j^i$$

and

$$\sum_{j=1}^{k}4^{j-1}\sum_{l=1}^{j}|{}_lx_j^i|_l < \varepsilon_n^p. \tag{3.3}$$

For the moment, fix i. Consider now the element of $V_1 + \ldots + V_{n-1}$ obtained by adding up all of the ${}_lx_j^i$'s for which $l \le n-1$:

$$\sum_{l \le n-1}{}_lx_j^i = \sum_{j=1}^{k}\sum_{l=1}^{\min\{j,n-1\}}{}_lx_j^i.$$

By the definition of inf-norms, and using subadditivity and (3.3), we obtain

$$\left\|\sum_{l\le n-1}{}_lx_j^i\right\|_{n-1} \le \sum_{j=1}^{k}\sum_{l=1}^{\min\{j,n-1\}}|{}_lx_j^i|_l < \sum_{j=1}^{k}4^{j-1}\sum_{l=1}^{j}|{}_lx_j^i|_l < \varepsilon_n^p.$$

As this holds for all $1 \le i \le m$, condition (3) implies

$$\left\|\frac{1}{m}\sum_{i=1}^{m}\sum_{l\le n-1}{}_lx_j^i\right\|_{n-1} < \frac{\varepsilon_{n-1}}{4^{n-2}}.$$

By definition then,

$$\left|\left|\left|\frac{1}{m}\sum_{i=1}^{m}\sum_{l\le n-1} {}_l x_j^i\right|\right|\right| < \left|\left|\left|\frac{1}{m}\sum_{i=1}^{m}\sum_{l\le n-1} {}_l x_j^i\right|\right|\right|_{n-1} < \varepsilon_{n-1}. \tag{3.4}$$

We also need to find a bound for the convex combination

$$\left|\left|\left|\frac{1}{m}\sum_{i=1}^{m}\sum_{l\ge n+1} {}_l x_j^i\right|\right|\right|.$$

Again by the definition of inf-norms,

$$\begin{aligned}\left|\left|\left|\frac{1}{m}\sum_{i=1}^{m}\sum_{l\ge n+1} {}_l x_j^i\right|\right|\right| \le \left|\left|\left|\frac{1}{m}\sum_{i=1}^{m} {}_{n+1}x_{n+1}^i\right|\right|\right|_{n+1} \\ + \left|\left|\left|\frac{1}{m}\sum_{i=1}^{m}({}_{n+1}x_{n+2}^i + {}_{n+2}x_{n+2}^i)\right|\right|\right|_{n+2} + \ldots \\ + \left|\left|\left|\frac{1}{m}\sum_{i=1}^{m}({}_{n+1}x_k^i + {}_{n+2}x_k^i + \ldots + {}_k x_k^i)\right|\right|\right|_k,\end{aligned}$$

which in turn is less than or equal to

$$\begin{aligned}&4^n\left|\frac{1}{m}\sum_{i=1}^{m} {}_{n+1}x_{n+1}^i\right|_{n+1} \\ &+ 4^{n+1}\left(\left|\frac{1}{m}\sum_{i=1}^{m} {}_{n+1}x_{n+2}^i\right|_{n+1} + \left|\frac{1}{m}\sum_{i=1}^{m} {}_{n+2}x_{n+2}^i\right|_{n+2}\right) + \ldots \\ &+ 4^{k-1}\left(\left|\frac{1}{m}\sum_{i=1}^{m} {}_{n+1}x_k^i\right|_{n+1} + \left|\frac{1}{m}\sum_{i=1}^{m} {}_{n+2}x_k^i\right|_{n+2} + \ldots + \left|\frac{1}{m}\sum_{i=1}^{m} {}_k x_k^i\right|_k\right).\end{aligned}$$

We are now in a position to apply condition (5). The constant m above corresponds to $|F_n|$, and as j in the above sum takes on integer values from $n+1$ to k, certainly $m \le M_{j-1}$ for all such j. Hence, for all j in this range,

$$m^{1-p_j} \le M_{j-1}^{1-p_j} < 2.$$

We will use this fact by rewriting it as

$$\frac{1}{m^{p_j}} < \frac{2}{m}$$

for all $n+1 \le j \le k$. Recalling that each $|\cdot|_j$ is a super p_j-norm and sub-additive, the last expression is less than

$$\begin{aligned}&4^n\left(\frac{2}{m}\sum_{i=1}^{m} |{}_{n+1}x_{n+1}^i|_{n+1}\right) \\ &+ 4^{n+1}\left(\frac{2}{m}\sum_{i=1}^{m} |{}_{n+1}x_{n+2}^i|_{n+1} + \frac{2}{m}\sum_{i=1}^{m} |{}_{n+2}x_{n+2}^i|_{n+2}\right) + \ldots \\ &+ 4^{k-1}\left(\frac{2}{m}\sum_{i=1}^{m} |{}_{n+1}x_k^i|_{n+1} + \frac{2}{m}\sum_{i=1}^{m} |{}_{n+2}x_k^i|_{n+2} + \ldots + \frac{2}{m}\sum_{i=1}^{m} |{}_k x_k^i|_k\right),\end{aligned}$$

which, when rewritten, is

$$\frac{2}{m}\sum_{i=1}^{m}\sum_{j=n+1}^{k} 4^{j-1}\sum_{l=n+1}^{j} |{}_l x_j^i|_l.$$

Applying (3.3) we then have the desired bound:

$$\left|\left|\left|\frac{1}{m}\sum_{i=1}^{m}\sum_{l\ge n+1} {}_l x_j^i\right|\right|\right| < 2\varepsilon_n^p. \tag{3.5}$$

At this point, the only ${}_l x_j^i$'s which have not been considered are those for which $l = n$. Accordingly, let

$$y_n = \frac{1}{m}\sum_{i=1}^{m}\sum_{l=n} {}_l x_j^i.$$

Note that

$$\begin{aligned}Sa_n - y_n &= \frac{1}{m}\sum_{i=1}^{m}\left(\left(\sum_{j=1}^{k}\sum_{l=1}^{j} {}_l x_j^i\right) - \sum_{l=n} {}_l x_j^i\right) \\ &= \frac{1}{m}\sum_{i=1}^{m}\left(\sum_{l\le n-1} {}_l x_j^i + \sum_{l\ge n+1} {}_l x_j^i\right).\end{aligned}$$

The two preceding bounds, (3.4) and (3.5), have both been found in order to show then that

$$|||Sa_n - y_n||| \le \left|\left|\left|\frac{1}{m}\sum_{i=1}^{m}\sum_{l\le n-1} {}_l x_j^i\right|\right|\right| + \left|\left|\left|\frac{1}{m}\sum_{i=1}^{m}\sum_{l\ge n+1} {}_l x_j^i\right|\right|\right| < \varepsilon_{n-1} + 2\varepsilon_n^p.$$

Finally, we use (3.2) to obtain

$$\begin{aligned}|||Ta_n - y_n||| &\le |||Ta_n - Sa_n||| + |||Sa_n - y_n||| \\ &< \varepsilon_n^p + \varepsilon_{n-1} + 2\varepsilon_n^p < 4\varepsilon_{n-1}.\end{aligned} \tag{3.6}$$

Recall now that for each $a \in A$, $a_n = a$ for infinitely many n. Choose a particular $a \in A$ and consider integers m and n for which $2 \le m < n$ and $a = a_m = a_n$. Then from (3.6) we have

$$|||y_m - y_n||| \le |||y_m - Ta_m||| + |||Ta_n - y_n||| < 4\varepsilon_{m-1} + 4\varepsilon_{n-1} < 8\varepsilon_{m-1}.$$

Since the sequence $\langle (X_n, |||\cdot|||_n)\rangle$ satisfies the hypothesis of Lemma 1.2, the fact that $y_m - y_n \in V_1 + \ldots + V_n$ implies the existence of $v_j \in V_1 + \ldots + V_j$ for $1 \le j \le n$ so that $y_m - y_n = v_1 + \ldots + v_n$ and

$$\sum_{j=1}^{n} |||v_j|||_j < 8\varepsilon_{m-1}.$$

Using the same argument as that preceding (3.3), for each fixed j and for $1 \le l \le j$ there exists ${}_l v_j \in V_l$ so that $v_j = {}_1v_j + \ldots + {}_jv_j$ and

$$\sum_{j=1}^{n} 4^{j-1} \sum_{l=1}^{j} |{}_lv_j|_l < 8\varepsilon_{m-1}. \tag{3.7}$$

Note now that

$$y_m - [({}_1v_1) + ({}_1v_2 + {}_2v_2) + \ldots + ({}_1v_n + \ldots + {}_{n-1}v_n)] = y_n + {}_nv_n \in (V_1 + \ldots + V_{n-1}) \cap V_n$$

and so

$$y_m - [({}_1v_1) + ({}_1v_2 + {}_2v_2) + \ldots + ({}_1v_n + \ldots + {}_{n-1}v_n)] = \lambda a$$

for some scalar λ. Then

$$|||y_m - \lambda a||| \le |||{}_1v_1|||_1 + |||{}_1v_2 + {}_2v_2|||_2 + \ldots + |||{}_1v_n + {}_2v_n + \ldots + {}_{n-1}v_n|||_{n-1}.$$

When we apply the definitions of the various inf-norms and (3.7) we obtain

$$|||y_m - \lambda a||| \le \sum_{j=1}^{n} 4^{j-1} \sum_{l=1}^{j} |{}_lv_j|_l < 8\varepsilon_{m-1}.$$

Finally, using this and (3.6) we obtain

$$|||Ta - \lambda a||| \le |||Ta_m - y_m||| + |||y_m - \lambda a||| < 4\varepsilon_{m-1} + 8\varepsilon_{m-1} = 12\varepsilon_{m-1}.$$

As we consider arbitrarily large m for which $a = a_m$, $\varepsilon_{m-1} \to 0$ and so we have shown that $Ta \in \mathbb{R}a$, i.e. $Ta = \lambda a$ for some scalar λ.

The end is now in sight. Since $e_1 \in A$ we know that $Te_1 = \lambda e_1$ for some λ. The claim is that this same λ works for all $a \in A$. The existence of scalars λ_n^+ and λ_n^- for which $T(e_1+e_n) = \lambda_n^+(e_1+e_n)$ and $T(e_1-e_n) = \lambda_n^-(e_1-e_n)$ has been shown. Now,

$$\begin{aligned}2\lambda e_1 = T(2e_1) &= T(e_1+e_n) + T(e_1-e_n)\\ &= \lambda_n^+(e_1+e_n) + \lambda_n^-(e_1-e_n) = (\lambda_n^+ + \lambda_n^-)e_1 + (\lambda_n^+ - \lambda_n^-)e_n\end{aligned}$$

and so it must be that $\lambda_n^+ = \lambda_n^- = \lambda$. As span A is dense in X, $T = \lambda I$ on X.

The only remaining task is to demonstrate that $(X, |||\cdot|||)$ admits compact operators. The existence of such operators is based on the following theorem from [5].

THEOREM 3.1. *Let $0 < p < 1$ be fixed. Let $(X, \|\cdot\|_0)$ be a separable p-Banach space and let $X_1 \subset X_2 \subset \ldots$ be any sequence of finite-dimensional subspaces so that $\bigcup X_n$ is $\|\cdot\|_0$-dense in X. Then $(X, \|\cdot\|_0)$ admits compact operators (to other p-Banach spaces) if and only if for some subsequence $\langle X_{n_k}\rangle$ of $\langle X_n\rangle$ there is a corresponding sequence of p-seminorms $\langle \|\cdot\|_k\rangle$ (with $\|\cdot\|_1$ non-trivial) and a constant $\eta > 0$ so that*

(1) $\|\cdot\|_{k+1} \ge (1+\eta)\|\cdot\|_k$ *on* X_{n_k}, *and*
(2) inf-norm$\langle (X_{n_k}, \|\cdot\|_k)\rangle$ *is weaker than* $\|\cdot\|_0$ *on* $\bigcup X_{n_k}$.

The proof of this theorem will not be presented here, as the following argument is sufficient for the task at hand.

In an arbitrary F-space $(X, \|\cdot\|)$, let $\mathcal{H}_{\|\cdot\|}(A, B)$ denote the Hausdorff distance between two sets A and B with respect to the metric generated by $\|\cdot\|$. The goal is to show the existence of another F-norm $|||\cdot|||^*$ defined on X for which

$$\sum_{n=1}^{\infty} \mathcal{H}_{|||\cdot|||^*}(B_n, B_{n+1}) < \infty, \tag{3.8}$$

where B_n is the $|||\cdot|||$-unit ball of X_n. Since each B_n is of course $|||\cdot|||^*$-relatively compact, it is easy to see that (3.8) implies $\bigcup B_n$ is also $|||\cdot|||^*$-relatively compact, and hence the $|||\cdot|||$- unit ball B of X is $|||\cdot|||^*$-relatively compact. This makes the identity map from $(X, |||\cdot|||)$ to $(X, |||\cdot|||^*)$ a compact operator. Of course, X will not be complete with respect to $|||\cdot|||^*$.

To construct $|||\cdot|||^*$, begin by letting $\langle \delta_n\rangle$ be any decreasing sequence of positive numbers such that $3/2 \le \delta_n/\delta_{n+1} \le 2$. Now define $|||\cdot|||_n^*$ to be $\delta_n|||\cdot|||_n$ and let $|||\cdot|||^* = \text{inf-norm}\langle (X_n, |||\cdot|||_n^*)\rangle$. Then given $x \in X_{n+1}$ with $|||x||| < 1$, Lemma 1.2 implies the existence of $x_i \in X_i$ for $1 \le i \le n+1$ so that

$$x = \sum_{i=1}^{n+1} x_i \quad \text{and} \quad |||x||| = \sum_{i=1}^{n+1} |||x_i|||_i.$$

So $\widetilde{x} \equiv \sum_{i=1}^{n} x_i \in X_n$, $|||\widetilde{x}||| < 1$ and

$$|||x - \widetilde{x}|||^* = |||x_{n+1}|||^* \le |||x_{n+1}|||_{n+1}^* = \delta_{n+1}|||x_{n+1}|||_{n+1} < \delta_{n+1}.$$

This implies $\mathcal{H}_{|||\cdot|||^*}(B_n, B_{n+1}) < \delta_{n+1}$ and by the choice of $\langle \delta_n\rangle$, $\sum \delta_n < \infty$.

We also need to verify that the super p-norm $|||\cdot|||^*$ is non-trivial, for otherwise the compact operator just constructed would be identically zero. This follows from the second restriction on the sequence $\langle \delta_n\rangle$, since on the space X_{n-1},

$$|||\cdot|||_n^* = \delta_n|||\cdot|||_n \ge 2\delta_n|||\cdot|||_{n-1} \ge \delta_{n-1}|||\cdot|||_{n-1} = |||\cdot|||_{n-1}^*,$$

and so the hypotheses of Lemma 1.2 are met once more.

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Construction of standard exact sequences of power series spaces

by

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Abstract. The following result is proved: Let $\Lambda_R^p(\alpha)$ denote a power series space of infinite or of finite type, and equip $\Lambda_R^p(\alpha)$ with its canonical fundamental system of norms, $R \in \{0,\infty\}$, $1 \le p < \infty$. Then a tamely exact sequence

$$(*) \qquad 0 \to \Lambda_R^p(\alpha) \to \Lambda_R^p(\alpha) \to \Lambda_R^p(\alpha)^{\mathbb{N}} \to 0$$

exists iff α is strongly stable, i.e. $\lim_n \alpha_{2n}/\alpha_n = 1$, and a linear-tamely exact sequence $(*)$ exists iff α is uniformly stable, i.e. there is A such that $\limsup_n \alpha_{Kn}/\alpha_n \le A < \infty$ for all K. This result extends a theorem of Vogt and Wagner which states that a topologically exact sequence $(*)$ exists iff α is stable, i.e. $\sup_n \alpha_{2n}/\alpha_n < \infty$.

An important tool in structure theory of power series spaces is the existence of exact sequences of the form

$$(*) \qquad 0 \to \Lambda_R^p(\alpha) \to \Lambda_R^p(\alpha) \to \Lambda_R^p(\alpha)^{\mathbb{N}} \to 0.$$

Here $\Lambda_R^p(\alpha)$ denotes a power series space of infinite type if $R = \infty$ and of finite type if $R = 0$, respectively, $1 \le p \le \infty$. A topologically exact sequence $(*)$ exists if and only if α is stable, i.e. $\sup_n \alpha_{2n}/\alpha_n < \infty$; this result has been proved for the nuclear case in [11] and in [6] for the general case. The existence of such sequences has been used to characterize the subspaces, quotient spaces and complemented subspaces of stable power series spaces of infinite type (cf. [11]) and of finite type (cf. [7], [8], [6]).

The purpose of this note is the investigation of the existence of tamely and linear-tamely exact sequences of the form $(*)$ (for the concept of tameness see below, or [1], [9], [4], [5]). We shall prove the following main result: Let $\Lambda_R^p(\alpha)$ be equipped with its canonical fundamental system of norms, $R \in \{0,\infty\}$, $1 \le p < \infty$. Then a tamely exact sequence $(*)$ exists if and only if α is strongly stable, i.e. $\lim_n \alpha_{2n}/\alpha_n = 1$, and a linear-tamely exact sequence $(*)$ exists iff α is uniformly stable, i.e. there is A such that $\limsup_n \alpha_{Kn}/\alpha_n \le A < \infty$ for all K. We notice that we do not need any nu-

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