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Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-293997

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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-293997

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On an extension of norms from a subspace to the whole Banach space keeping their rotundity

by

M. FABIAN (Praha)

Abstract. Let $\mathcal{R}$ denote some kind of rotundity, e.g., the uniform rotundity. Let X admit an $\mathcal{R}$-norm and let Y be a reflexive subspace of X with some $\mathcal{R}$-norm $\|\cdot\|$. Then we are able to extend $\|\cdot\|$ from Y to an $\mathcal{R}$-norm on X.

Introduction. In [5] and [4, Section II.8] it is shown that if Y is a subspace of a separable Banach space X and if Y has a LUR norm $\|\cdot\|$, then there exists on X an equivalent LUR norm such that its restriction to Y coincides with $\|\cdot\|$. An extension of this result to X/Y separable can be found in [6]. In this note we present a different method of extending a norm from a reflexive subspace to the whole Banach space which conserves rotundity properties of the norm on the subspace. In particular, we do so for strict convexity, local uniform rotundity (LUR), uniform rotundity (UR) and for UR with moduli of rotundity of power type. We thus answer affirmatively a question raised in [4, p. 177].

Let $(X,\|\cdot\|)$ be a real Banach space. We say that $\|\cdot\|$ is *strictly convex* if $x_1=x_2$ whenever $x_1,x_2\in X$, $\|x_1\|=\|x_2\|=1$ and $\|x_1+x_2\|=2$. The norm $\|\cdot\|$ is called *LUR* if $\|x_n-x\|\to 0$ whenever $x_n,x\in X$, $\|x_n\|=\|x\|=1$, and $\|x_n+x\|\to 2$. Finally, $\|\cdot\|$ is said to be *UR* if $\|x_n^1-x_n^2\|\to 0$ whenever $\{x_n^1\},\{x_n^2\}\subset X$, $\|x_n^1\|=\|x_n^2\|=1$, and $\|x_n^1+x_n^2\|\to 2$.

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A basic construction. Let Y be a reflexive subspace of a Banach space $(X, |\cdot|)$. Let $\|\cdot\|$ be an equivalent norm on Y such that

$$2\|y\| \le |y| \quad \text{for all } y \in Y.$$

Denote by B and A the closed unit balls of $(X, |\cdot|)$ and $(Y, \|\cdot\|)$ respectively. Perhaps the most natural construction of an extension of $\|\cdot\|$ to the whole space X is to consider Minkowski's functional of the convex hull of A and B [4, p. 82], that is, of the set

$$C = \{ta + (1-t)b : a \in A,\ b \in B,\ t \in [0,1]\}.$$

However, such a body has no rotundity at some points. Fortunately, by inspecting C carefully, we can see that if we start from A and B with a certain rotundity, then the rotundity at each point of the boundary of C is violated at most in one direction.

In what follows we will cultivate this basic idea of construction. Define $\varphi : [0,1] \to [0,1]$ by

$$\varphi(t) = \sqrt{t}, \qquad t \in [0,1].$$

We note that φ is strictly concave, and $\varphi(0) = 0$, $\varphi(1) = 1$. Define then

$$D = \{\varphi(t)a + (1-t)b : a \in A,\ b \in B,\ t \in [0,1]\}.$$

LEMMA 1. *D is symmetric, convex, closed, $\operatorname{int} D \neq \emptyset$ and $D \cap Y = A$. Hence D is the unit ball of an equivalent norm $|||\cdot|||$ on X and moreover $|||\cdot|||$ extends $\|\cdot\|$, that is, $|||y||| = \|y\|$ for all $y \in Y$.*

Proof. D is clearly symmetric and has nonempty interior as $D \supset B$. Assume $x_n = \varphi(t_n)a_n + (1-t_n)b_n \in D$ converges in norm to some $x \in X$. Since Y is reflexive, A is weakly sequentially compact. So, there are subsequences $\{a_{n_i}\}$ and $\{t_{n_i}\}$ converging to some $a \in A$ and $t \in [0,1]$. If $t = 1$, then clearly $x_{n_i} \to x = \varphi(1)a = a \in D$. If $t < 1$, then $b_{n_i} = (x_{n_i} - \varphi(t_{n_i})a_{n_i})/(1-t_{n_i})$ converges weakly to $(x - \varphi(t)a)/(1-t) =: b$, which lies in B as B is closed. Hence $x = \varphi(t)a + (1-t)b \in D$ and the closedness of D is verified. Clearly $A = \varphi(1)A \subset D \cap Y$. Conversely, take $y \in D \cap Y$. Thus $y = \varphi(t)a + (1-t)b$ with some $a \in A$, $b \in B$ and $t \in [0,1]$. Then $(1-t)b \in Y$ and so

$$\begin{aligned}\|\varphi(t)a + (1-t)b\| &\le \varphi(t)\|a\| + (1-t)\|b\| \le \varphi(t) + (1-t)/2 \\ &\le \max\{\varphi(\tau) + (1-\tau)/2 : \tau \in [0,1]\} = 1.\end{aligned}$$

Hence $y \in A$.

It remains to show that D is convex. Take $x_1, x_2 \in D$ and $\alpha_1 > 0$, $\alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$. Then $x_i = \varphi(t_i)a_i + (1-t_i)b_i$, with $a_i \in A$, $b_i \in B$ and $t_i \in [0,1]$, $i = 1, 2$. If $t_1 = t_2 = 0$, we have $\alpha_1 x_1 + \alpha_2 x_2 = \alpha_1 b_1 + \alpha_2 b_2 \in B \subset D$. Similarly, if $t_1 = t_2 = 1$, then $\alpha_1 x_1 + \alpha_2 x_2 = \alpha_1 a_1 + \alpha_2 a_2 \in A \subset D$. Finally, assume $(t_1, t_2) \neq (0,0), (1,1)$. Then $\alpha_1\varphi(t_1) + \alpha_2\varphi(t_2) > 0$, $\alpha_1 t_1 + \alpha_2 t_2 > 0$, and $1 - \alpha_1 t_1 - \alpha_2 t_2 > 0$ and we can write

$$\begin{aligned}\alpha_1 x_1 + \alpha_2 x_2 &= \varphi(\alpha_1 t_1 + \alpha_2 t_2)\left[\frac{\alpha_1\varphi(t_1) + \alpha_2\varphi(t_2)}{\varphi(\alpha_1 t_1 + \alpha_2 t_2)} \sum_{i=1}^{2} \frac{\alpha_i \varphi(t_i)}{\alpha_1\varphi(t_1) + \alpha_2\varphi(t_2)} a_i\right] \\ &\quad + (1 - \alpha_1 t_1 - \alpha_2 t_2)\left[\sum_{i=1}^{2} \frac{\alpha_i - \alpha_i t_i}{1 - \alpha_1 t_1 - \alpha_2 t_2} b_i\right].\end{aligned}$$

Hence, using the concavity of φ and the convexity of A and B, we get $\alpha_1 x_1 + \alpha_2 x_2 \in D$. ∎

LEMMA 2. *For $a \in A$, $b \in B$ and $0 \le t \le 1$ we have*

$$1 - |||\varphi(t)a + (1-t)b||| \ge \tfrac{1}{2}t(1 - \|a\|) + \tfrac{1}{2}(1-t)(1 - |b|).$$

Proof. Put

$$\gamma = \tfrac{1}{2}(1-t)|b| + \tfrac{1}{2}\sqrt{(1-t)^2|b|^2 + 4t\|a\|^2}.$$

If $\gamma = 0$, then $(1-t)b = 0$, $\varphi(t)a = 0$ and so the inequality is satisfied. Further assume that $\gamma > 0$. Put

$$\tau = 1 - \frac{1-t}{\gamma}|b|.$$

We can easily verify that $(1-t)|b| \le \gamma \le 1$, $0 \le \tau \le 1$ and that

$$\varphi(\tau) = \frac{\varphi(t)\|a\|}{\gamma}, \qquad 1 - \tau = \frac{1-t}{\gamma}|b|.$$

Hence

$$\begin{aligned}|||\varphi(t)a + (1-t)b||| &= \left|\left|\left|\varphi(t)\|a\|\frac{a}{\|a\|} + (1-t)|b|\frac{b}{|b|}\right|\right|\right| \\ &= \gamma\left|\left|\left|\varphi(\tau)\frac{a}{\|a\|} + (1-\tau)\frac{b}{|b|}\right|\right|\right| \le \gamma.\end{aligned}$$

Thus

$$\begin{aligned}1 - |||\varphi(t)a + (1-t)b||| &\ge 1 - \gamma = \frac{1-\gamma^2}{1+\gamma} \ge \frac{1}{2}(1-\gamma^2) \\ &= \tfrac{1}{2}\big(1 - \tfrac{1}{4}(1-t)^2|b|^2 \\ &\quad - \tfrac{1}{2}(1-t)|b|\sqrt{(1-t)^2|b|^2 + 4t\|a\|^2} - \tfrac{1}{4}(1-t)^2|b|^2 - t\|a\|^2\big) \\ &= \tfrac{1}{2}\big(1 - t\|a\|^2 - \tfrac{1}{2}(1-t)^2|b|^2 - \tfrac{1}{2}(1-t)|b|\sqrt{(1-t)^2|b|^2 + 4t\|a\|^2}\big) \\ &\ge \tfrac{1}{2}\big(1 - t\|a\| - \tfrac{1}{2}(1-t)^2|b| - \tfrac{1}{2}(1-t)|b|(1+t)\big) \\ &= \tfrac{1}{2}t(1 - \|a\|) + \tfrac{1}{2}(1-t)(1 - |b|).\ ∎\end{aligned}$$

For $\varepsilon \in [0,2]$, we define the following moduli of rotundity on $(Y, \|\cdot\|)$ and $(X, |\cdot|)$:

$$\delta_{\|\cdot\|}(\varepsilon) = \inf\{1 - \tfrac{1}{2}\|a_1 + a_2\| : a_1, a_2 \in A,\ |||a_1 - a_2||| \ge \varepsilon\},$$
$$\delta_{|\cdot|}(\varepsilon) = \inf\{1 - \tfrac{1}{2}|b_1 + b_2| : b_1, b_2 \in B,\ |||b_1 - b_2||| \ge \varepsilon\}.$$

Clearly, it does not almost matter if $|||\cdot|||$ is replaced by $\|\cdot\|$ or $|\cdot|$. So our $\delta_{\|\cdot\|}$ and $\delta_{|\cdot|}$ do not differ too much from the usual moduli of rotundity for $\|\cdot\|$ and $|\cdot|$ respectively [4, p. 130].

LEMMA 3. *Consider $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $t_1, t_2 \in [0,1]$. Then, by putting $0/0 = 1$ if necessary, we have*

$$\begin{aligned}
1 - \frac{1}{2}|||\varphi(t_1)a_1 + (1-t_1)b_1 + \varphi(t_2)a_2 + (1-t_2)b_2|||
&\ge \frac{1}{16}(\sqrt{t_1} - \sqrt{t_2})^2 \\
&\quad + \frac{1}{4}(t_1+t_2)\frac{\varphi(t_1)+\varphi(t_2)}{\varphi(2t_1+2t_2)}\left[1 - \frac{\|\varphi(t_1)a_1 + \varphi(t_2)a_2\|}{\varphi(t_1)+\varphi(t_2)}\right] \\
&\quad + \frac{1}{2}\left(1 - \frac{t_1+t_2}{2}\right)\left[1 - \frac{|(1-t_1)b_1 + (1-t_2)b_2|}{2-t_1-t_2}\right] \\
&\ge \frac{1}{16}(\sqrt{t_1} - \sqrt{t_2})^2 + \frac{1}{2}\min(t_1,t_2)\delta_{\|\cdot\|}(|||a_1 - a_2|||) \\
&\quad + \frac{1}{2}\min(1-t_1, 1-t_2)\delta_{|\cdot|}(|||b_1 - b_2|||).
\end{aligned}$$

Proof. Set $x_i = \varphi(t_i)a_i + (1-t_i)b_i$, $i = 1,2$. Assume first $0 < t_1 + t_2 < 2$. We have

$$\begin{aligned}
\frac{1}{2}(x_1 + x_2) &= \varphi\left(\frac{t_1+t_2}{2}\right)\left[\frac{\varphi(t_1)+\varphi(t_2)}{2\varphi((t_1+t_2)/2)}\left(\frac{\varphi(t_1)a_1+\varphi(t_2)a_2}{\varphi(t_1)+\varphi(t_2)}\right)\right] \\
&\quad + \left(1 - \frac{t_1+t_2}{2}\right)\left[\frac{(1-t_1)b_1 + (1-t_2)b_2}{2-t_1-t_2}\right].
\end{aligned}$$

Hence, from the convexity of A and B and from the concavity of φ we deduce that the expressions in square brackets belong to A and B respectively. Thus, by Lemma 2,

$$\begin{aligned}
1 - \frac{1}{2}|||x_1 + x_2||| &\ge \frac{1}{2}\frac{t_1+t_2}{2}\left[1 - \frac{\varphi(t_1)+\varphi(t_2)}{2\varphi((t_1+t_2)/2)}\left\|\frac{\varphi(t_1)a_1+\varphi(t_2)a_2}{\varphi(t_1)+\varphi(t_2)}\right\|\right] \\
&\quad + \frac{1}{2}\left(1 - \frac{t_1+t_2}{2}\right)\left[1 - \left|\frac{(1-t_1)b_1 + (1-t_2)b_2}{2-t_1-t_2}\right|\right] \\
&= \frac{1}{4}(t_1+t_2)\left[1 - \frac{\varphi(t_1)+\varphi(t_2)}{2\varphi((t_1+t_2)/2)}\right] \\
&\quad + \frac{1}{4}(t_1+t_2)\frac{\varphi(t_1)+\varphi(t_2)}{2\varphi((t_1+t_2)/2)}\left[1 - \left\|\frac{\varphi(t_1)a_1+\varphi(t_2)a_2}{\varphi(t_1)+\varphi(t_2)}\right\|\right] \\
&\quad + \frac{1}{4}(2-t_1-t_2)\left[1 - \left|\frac{(1-t_1)b_1 + (1-t_2)b_2}{2-t_1-t_2}\right|\right] \\
&=: U + V + W.
\end{aligned}$$

Let us estimate the first term U:

$$\begin{aligned}
U &= \frac{1}{4}(t_1+t_2)\left[1 - \frac{\sqrt{t_1}+\sqrt{t_2}}{\sqrt{2t_1+2t_2}}\right] \\
&= \frac{1}{4}(t_1+t_2)\left[1 - \frac{t_1 + 2\sqrt{t_1t_2} + t_2}{2t_1+2t_2}\right] \Big/ \left[1 + \frac{\sqrt{t_1}+\sqrt{t_2}}{\sqrt{2t_1+2t_2}}\right] \\
&\ge \frac{1}{8}(t_1+t_2)\left[1 - \frac{t_1 + 2\sqrt{t_1t_2} + t_2}{2t_1+2t_2}\right] = \frac{1}{16}(\sqrt{t_1} - \sqrt{t_2})^2.
\end{aligned}$$

Thus

$$1 - \tfrac{1}{2}|||x_1 + x_2||| \ge \tfrac{1}{16}(\sqrt{t_1} - \sqrt{t_2})^2 + V + W,$$

which is the first inequality in our lemma.

In order to estimate V, consider $u_1, u_2 \in A$, and $\alpha \in [1/2, 1]$. Then

$$\begin{aligned}
1 - \|\alpha a_1 + (1-\alpha)a_2\| &= 1 - \left\|(2\alpha - 1)a_1 + (2-2\alpha)\tfrac{1}{2}(a_1+a_2)\right\| \\
&\ge 1 - [2\alpha - 1 + (2-2\alpha)(1 - \delta_{\|\cdot\|}(|||a_1 - a_2|||))] \\
&= 2(1-\alpha)\delta_{\|\cdot\|}(|||a_1 - a_2|||).
\end{aligned}$$

Using this estimate we have

$$\begin{aligned}
V &= \frac{1}{4}(t_1+t_2)\frac{\varphi(t_1)+\varphi(t_2)}{\varphi(2(t_1+t_2))}\left[1 - \left\|\frac{\varphi(t_1)a_1+\varphi(t_2)a_2}{\varphi(t_1)+\varphi(t_2)}\right\|\right] \\
&\ge \frac{1}{4}(t_1+t_2)\frac{\varphi(t_1)+\varphi(t_2)}{\varphi(2(t_1+t_2))}\cdot\frac{2\min\{\varphi(t_1),\varphi(t_2)\}}{\varphi(t_1)+\varphi(t_2)}\delta_{\|\cdot\|}(|||a_1 - a_2|||) \\
&\ge \frac{1}{2\sqrt{2}}\sqrt{t_1+t_2}\min\{\sqrt{t_1},\sqrt{t_2}\}\delta_{\|\cdot\|}(|||a_1 - a_2|||) \\
&\ge \frac{1}{2}\min\{t_1,t_2\}\delta_{\|\cdot\|}(|||a_1 - a_2|||).
\end{aligned}$$

Finally, proceeding in a similar way, we have

$$\begin{aligned}
W &\ge \frac{1}{4}(2-t_1-t_2)\cdot\frac{2\min\{1-t_1, 1-t_2\}}{2-t_1-t_2}\delta_{|\cdot|}(|||a_1 - a_2|||) \\
&= \frac{1}{2}\min\{1-t_1, 1-t_2\}\delta_{|\cdot|}(|||a_1 - a_2|||).
\end{aligned}$$

The cases when $t_1 = t_2 = 0$ or $t_1 = t_2 = 1$ can be obtained as limit cases of the above. Thus our inequalities also hold in these cases. ■

Extension statements

THEOREM 1. *Assume a Banach space X has a strictly convex (LUR) [UR] norm $|\cdot|$ and let Y be a reflexive subspace of X with an equivalent strictly convex (LUR) [UR] norm $\|\cdot\|$. Then there exists on X an equivalent strictly convex (LUR) [UR] norm $|||\cdot|||$ extending $\|\cdot\|$, that is, $|||y||| = \|y\|$ for all $y \in Y$.*

Proof. By multiplying $|\cdot|$ by a constant factor if necessary, we may and do assume that $2\|y\| \le |y|$ for all $y \in Y$. We will perform the basic construction, that is, we put

$$A = \{y \in Y : \|y\| \le 1\}, \quad B = \{x \in X : |x| \le 1\},$$
$$\varphi(t) = \sqrt{t} \quad \text{and} \quad D = \{\varphi(t)a + (1-t)b : a \in A,\ b \in B,\ t \in [0,1]\}.$$

Let $|||\cdot|||$ be Minkowski's functional of D. According to Lemma 1, $|||\cdot|||$ will be an equivalent norm on X such that $|||y||| = \|y\|$. We will consider the three rotundity notions separately.

Strict convexity. Consider $x_1, x_2 \in X$ with $|||x_1||| = |||x_2||| = \frac{1}{2}|||x_1 + x_2||| = 1$. Write $x_i = \varphi(t_i)a_i + (1-t_i)b_i$, where $t_i \in [0,1]$, $a_i \in A$, $\|a_i\| = 1$, $b_i \in B$, $|b_i| = 1$, $i = 1,2$. From Lemma 3 we immediately get $t_1 = t_2 =: \tau$. If $\tau = 0$, then Lemma 3 yields $|b_1 + b_2| = 2$ and hence, by the strict convexity of $|\cdot|$, we conclude $x_1 = b_1 = b_2 = x_2$. If $\tau = 1$, Lemma 3 gives $\|\varphi(1)a_1 + \varphi(1)a_2\| = \varphi(1) + \varphi(1)$ and the strict convexity of $\|\cdot\|$ yields $x_1 = a_1 = a_2 = x_2$. Finally, assume $0 < \tau < 1$; then from Lemma 3 we have

$$\|\varphi(\tau)a_1 + \varphi(\tau)a_2\| = 2\varphi(\tau), \qquad |(1-\tau)b_1 + (1-\tau)b_2| = 2 - 2\tau.$$

So, by the strict convexity of $\|\cdot\|$ and $|\cdot|$ we have $a_1 = a_2$, $b_1 = b_2$ and therefore $x_1 = x_2$.

LUR. Assume we have any x and x_n in X, with $|||x||| = |||x_n||| = 1$, such that $|||x + x_n||| \to 2$. We will be done when we show that a subsequence of $\{x_n\}$ converges to x in norm. Write $x = \varphi(t)a + (1-t)b$ and $x_n = \varphi(t_n)a_n + (1-t_n)b_n$, with $t, t_n \in [0,1]$, $a, a_n \in A$, $\|a\| = \|a_n\| = 1$, $b, b_n \in B$, $|b| = |b_n| = 1$. Lemma 3 immediately gives $t_n \to t$. Then, setting $\widetilde{x}_n = \varphi(t)a_n + (1-t)b_n$, we have $x_n - \widetilde{x}_n \to 0$, $|||\widetilde{x}_n||| \to 1$ and $|||x + \widetilde{x}_n||| \to 2$. If now $t = 0$, then from Lemma 3 we have $|b + b_n| \to 2$; so, by the LUR of $|\cdot|$, $b_n \to b$ and hence

$$x_n = x_n - \widetilde{x}_n + \widetilde{x}_n = x_n - \widetilde{x}_n + b_n \to b = x.$$

If $t = 1$, then, again by Lemma 3, we have $\|a + a_n\| \to 2$ and the LUR of $\|\cdot\|$ ensures $a_n \to a$; thus

$$x_n = x_n - \widetilde{x}_n + \widetilde{x}_n = x_n - \widetilde{x}_n + a_n \to a = x.$$

Finally, assume $0 < t < 1$. Then it is easy to check that $\|a_n + a\| \to 2$ and $|b_n + b| \to 2$. Hence, by Lemma 3 and the LUR of $\|\cdot\|$ and $|\cdot|$, we have $a_n \to a$ and $b_n \to b$. Therefore

$$x_n = x_n - \widetilde{x}_n + \varphi(t)a_n + (1-t_n)b_n \to \varphi(t)a + (1-t)b = x.$$

UR. Assume we have any x_n^1, x_n^2 in X, with $|||x_n^1||| = |||x_n^2||| = 1$, such that $|||x_n^1 + x_n^2||| \to 2$; then by Lemma 3, $t_n^1 - t_n^2 \to 0$. Write $x_n^i = \varphi(t_n^i)a_n^i + (1-t_n^i)b_n^i$, with $t_n^i \in [0,1]$, $a_n^i \in A$, $\|a_n^i\| = 1$, $b_n^i \in B$, $|b_n^i| = 1$; $i = 1,2$. Assume, for simplicity, that $t_n^1 \to \tau$, $t_n^2 \to \tau$. If now $\tau < 1$, from Lemma 3 we have $|b_n^1 + b_n^2| \to 2$, while, when $\tau > 0$, we get $\|a_n^1 + a_n^2\| \to 2$. Hence, as in the case of LUR, we can conclude from the UR of $|\cdot|$ and $\|\cdot\|$ that $x_n^1 - x_n^2 \to 0$. ∎

The case of power type moduli of rotundity deserves a separate statement. Here $\delta_{\|\cdot\|}(\varepsilon)$ and $\delta_{|\cdot|}(\varepsilon)$ are defined as before.

THEOREM 2. *Let $(X, |\cdot|)$ be a superreflexive Banach space and Y be a subspace of X with an equivalent norm $\|\cdot\|$. Assume there exist $c > 0$ and $q \in [2,\infty)$ such that*

$$\delta_{\|\cdot\|}(\varepsilon) \ge c\varepsilon^q \quad \text{and} \quad \delta_{|\cdot|}(\varepsilon) \ge c\varepsilon^q, \quad \varepsilon \in [0,2].$$

Then there exists an equivalent norm $|||\cdot|||$ on X and $c' > 0$ such that $|||y||| = \|y\|$ for all $y \in Y$ and

$$\begin{aligned}\delta_{|||\cdot|||}(\varepsilon) := \inf\{1 - \tfrac{1}{2}|||x_1 + x_2||| &: x_1, x_2 \in X,\\ &|||x_1||| \le 1,\ |||x_2||| \le 1,\ |||x_1 - x_2||| \ge \varepsilon\}\\ \ge c'\varepsilon^q \quad &\text{for all } \varepsilon \in [0,2].\end{aligned}$$

Proof. Let $|||\cdot|||$ be the norm constructed above. Fix an arbitrary $\varepsilon \in [0,2]$ and arbitrary $x_1, x_2 \in X$, with $|||x_1||| \le 1$, $|||x_2||| \le 1$ and $|||x_1 - x_2||| \ge \varepsilon$. Then we can write $x_i = \varphi(t_i)a_i + (1-t_i)b_i$, where $a_i \in A$, $b_i \in B$ and $t_i \in [0,1]$, $i = 1,2$. If $(|\varphi(t_1) - \varphi(t_2)| =)\ |\sqrt{t_1} - \sqrt{t_2}| \ge \varepsilon/8$, then Lemma 3 gives

$$1 - \tfrac{1}{2}|||x_1 + x_2||| \ge \varepsilon^2/64 \ge \varepsilon^q/64.$$

Further assume $|\sqrt{t_1} - \sqrt{t_2}| < \varepsilon/8$. Then

$$|t_1 - t_2| = |\sqrt{t_1} - \sqrt{t_2}|(\sqrt{t_1} + \sqrt{t_2}) < (\varepsilon/8)\cdot 2 = \varepsilon/4.$$

Assume moreover that, say, $t_1 \le t_2$. Then by Lemma 3 we have

$$\begin{aligned}1 - \tfrac{1}{2}|||x_1 + x_2||| &\ge \tfrac{1}{2}t_1\delta_{\|\cdot\|}(|||a_1 - a_2|||) + \tfrac{1}{2}(1-t_2)\delta_{|\cdot|}(|||b_1 - b_2|||)\\ &\ge \tfrac{1}{2}ct_1|||a_1 - a_2|||^q + \tfrac{1}{2}c(1-t_2)|||b_1 - b_2|||^q\\ &\ge \tfrac{1}{2}c\varphi(t_1)^q|||a_1 - a_2|||^q + \tfrac{1}{2}c(1-t_2)^q|||b_1 - b_2|||^q\\ &\ge \tfrac{1}{2}cc_q|||\varphi(t_1)(a_1 - a_2) + (1-t_2)(b_1 - b_2)|||^q\end{aligned}$$

(where c_q is a constant such that $\alpha^q + \beta^q \ge c_q(\alpha+\beta)^q$ for all $\alpha, \beta \ge 0$)

$$\geq \tfrac{1}{2}cc_q(|||x_1 - x_2||| - (\varphi(t_2) - \varphi(t_1)) - (t_2 - t_1))^q$$
$$> \tfrac{1}{2}cc_q(\varepsilon - \varepsilon/8 - \varepsilon/4)^q > 2^{-1-q}cc_q\varepsilon^q.$$

It follows that $\delta_{|||\cdot|||}(\varepsilon) \geq \min\{2^{-6}, 2^{-1-q}cc_q\}\varepsilon^q$. ■

Remarks. 1. The above theorems can be restated as: *If X admits an equivalent $\mathcal{R}$-norm and Y is a reflexive subspace of X, then all equivalent $\mathcal{R}$-norms on Y can be obtained as restrictions of $\mathcal{R}$-norms on X.*

2. We are convinced that our construction is also suitable for other kinds of rotundity.

3. In what follows we translate our geometric constructions into analytic forms. Let $(X, |\cdot|)$, $(Y, \|\cdot\|)$, A, B be the same as above. Then it is an easy excercise to check that the convex hull of A and B is the set

$$C = \{x \in X : \inf\{\|y\| + |x - y| : y \in Y\} \leq 1\},$$

that is, the unit ball of a norm defined as the inf-convolution of $\|\cdot\|$ and $|\cdot|$. Similarly, we can check that our basic body

$$D = \bigcup\{\sqrt{t}A \cup (1-t)B : t \in [0,1]\}$$

can be rewritten as

$$\{x \in X : \inf\{\|y\|^2 + |x - y| : y \in Y\} \leq 1\}.$$

It should be noted that this construction is not homogeneous and so the inf-convolution involved here is no longer a norm. It would be quite natural to consider the norm

$$x \to \inf\{(\|y\|^2 + |x - y|^2)^{1/2} : y \in Y\},$$

whose unit ball is $\bigcup\{\sqrt{t}A \cup \sqrt{1-t}B : t \in [0,1]\}$. This norm will have the same rotundity as $\|\cdot\|$ and $|\cdot|$ do (and the same smoothness as $|\cdot|$ does). However, its restriction to Y is different from $\|\cdot\|$.

4. It seems that the reflexivity of Y, that is, the weak compactness of its unit ball is substantial in our construction. Thus our result does not cover that of [5], [4, Section II.8], where X is separable and Y is any subspace of X.

5. We set as an open problem the dual question: *How to extend smooth norms keeping their smoothness?* It seems that this question is more delicate. Indeed, if we replace rotundity by Gateaux smoothness, then a corresponding extension may not exist. In fact, under some circumstances such an extension implies the complementability of Y in X [7], [4, Section II.8]; or [2, Example] shows a Gateaux smooth norm $\|\cdot\|$ on c_0 and $0 \neq y \in c_0$ such that $\|\cdot\|$ cannot be extended to l_∞ keeping its Gateaux smoothness at y. And as regards our basic construction we are affraid that the norm so constructed will not be smooth at the points of Y. Yet we do hope for a positive result, at least for uniform smoothness.

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INSTITUTE OF MATHEMATICS
ACADEMY OF SCIENCES OF THE CZECH REPUBLIC
ŽITNÁ 25
115 67 PRAHA 1, CZECH REPUBLIC

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