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# On vector spaces and algebras with maximal locally pseudoconvex topologies

by

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**Abstract.** Let $X$ be a real or complex vector space. We show that the maximal $p$-convex topology makes $X$ a complete Hausdorff topological vector space. If $X$ has an uncountable dimension, then different $p$ give different topologies. However, if the dimension of $X$ is at most countable, then all these topologies coincide. This leads to an example of a complete locally pseudoconvex space $X$ that is not locally convex, but all of whose separable subspaces are locally convex. We apply these results to topological algebras, considering the problem of uniqueness of a complete topology for semitopological algebras and giving an example of a complete locally convex commutative semitopological algebra without multiplicative linear functionals, but with every separable subalgebra having a total family of such functionals.

Let $X$ be a real or complex vector space. A *$p$-homogeneous seminorm* on $X$ $(0 < p \le 1)$ is a non-negative function $x \to \|x\|$, $x \in X$, such that

(i) $\|0\| = 0$,
(ii) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$, and
(iii) $\|\lambda x\| = |\lambda|^p \|x\|$ for all $x$ in $X$ and all scalars $\lambda$.

The inequality $(u+v)^p \le u^p + v^p$, $0 < p \le 1$, $u, v \ge 0$, implies that if $\|x\|$ is a $p$-homogeneous seminorm on $X$, and $0 < r \le 1$, then $\|x\|^r$ is a $pr$-homogeneous seminorm on $X$.

A topological vector space $X$ is said to be *locally pseudoconvex* if its topology is given by means of a family $(\|\cdot\|_\alpha)$ of $p(\alpha)$-homogeneous seminorms, $0 < p(\alpha) \le 1$. For more details on locally pseudoconvex spaces the reader is referred to [2] and [4].

Let $X$ be a vector space and $0 < p \le 1$. The *maximal locally $p$-convex topology* $\tau^p_{\max}$ on $X$ is the topology given by means of all $p$-homogeneous seminorms. It is a Hausdorff vector space topology. For $p = 1$ it is the maximal locally convex topology on $X$. In this case we denote it by $\tau^{\mathrm{LC}}_{\max}$

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instead of $\tau^1_{\max}$. Note that if $p < q \le 1$, then all $q$-homogeneous seminorms on $X$ are continuous in the topology $\tau^p_{\max}$. This follows from the fact that if $\|\cdot\|$ is a $q$-homogeneous seminorm on $X$, then $x \to \|x\|^{p/q}$ is a $p$-homogeneous seminorm on $X$, since $p/q < 1$. Another type of maximal topology is the topology $\tau^{q+}_{\max}$, where $0 \le q < 1$, given by means of all $p$-homogeneous seminorms on with $q < p \le 1$. Thus for $0 \le q < p < r \le 1$ the topology $\tau^{q+}_{\max}$ is stronger than $\tau^p_{\max}$, and $\tau^p_{\max}$ is stronger than $\tau^{p+}_{\max}$ and $\tau^r_{\max}$. It is easy to verify that if a vector space $X$ is provided with one of the above maximal topologies, then all of its linear functionals and endomorphisms are continuous, and all of its linear subspaces are closed.

Extending a result in ([3], example on p. 59) we prove the following

THEOREM 1. *Let $X$ be a real or complex vector space and let $0 < p \le 1$ (resp. $0 \le q < 1$). Then $(X, \tau^p_{\max})$ (resp. $(X, \tau^{q+}_{\max})$) is a complete Hausdorff topological vector space.*

Proof. If $0 \ne x^0 \in X$, then there is a linear functional $f$ on $X$ with $f(x^0) \ne 0$. Then for any $r$ with $0 < r \le 1$ the formula $x \to |f(x)|^r$ gives an $r$-homogeneous seminorm $\|\cdot\|$ on $X$ with $\|x^0\| \ne 0$. Thus both $\tau^p_{\max}$ and $\tau^{q+}_{\max}$ are Hausdorff. It remains to show that $X$ is complete in both topologies. Fix a Hamel basis $(h_\alpha)_{\alpha\in\mathfrak{a}}$ for $X$, so that every $x$ in $X$ has a unique representation of the form

$$x = \sum_\alpha g_\alpha(x) h_\alpha, \tag{1}$$

where all but a finite number of the coefficients $g_\alpha(x)$, which are linear functionals in $x$, are zero. Let $(x_\gamma)$ be a Cauchy net in $(X, \tau^p_{\max})$ (resp. $(X, \tau^{q+}_{\max})$). By the continuity of $g_\alpha$ the limits

$$C_\alpha = \lim_\gamma g_\alpha(x_\gamma)$$

exist for all $\alpha$ in $\mathfrak{a}$.

First we show that all but finitely many $C_\alpha$ are zero. If not, choose $C_{\alpha_1}, C_{\alpha_2}, \ldots$ so that $C_{\alpha_i} \ne 0$ for all $i$, and put

$$|x| = \Big(\sum_{k=1}^\infty \frac{k|g_{\alpha_k}(x)|}{C_{\alpha_k}}\Big)^p$$

with $q < p \le 1$ in the case of the topology $\tau^{q+}_{\max}$. This is a continuous $p$-homogeneous seminorm on the considered space. For any fixed $k$ we have $|x_\gamma| > k$ for sufficiently large $\gamma$. But this is impossible, since for every continuous seminorm $|x|$ on $X$, the finite limit $\lim_\gamma |x_\gamma|$ exists. Thus only finitely many of the considered numbers, say $C_{\alpha_1}, \ldots, C_{\alpha_k}$, are different from zero.

Put $x_0 = C_{\alpha_1} h_{\alpha_1} + \ldots + C_{\alpha_k} h_{\alpha_k}$. We have to show that $y_\gamma = x_\gamma - x_0$ tends to zero. If not, then there is a $p$-homogeneous seminorm $|\cdot|$ on $X$ ($p > q$ in the case of $\tau^{q+}_{\max}$) with $\lim_\gamma |y_\gamma| > 0$. Put $r_\alpha = |h_\alpha|$, $\alpha \in \mathfrak{a}$, and

$$\|x\| = \sum_\alpha |g_\alpha(x)|^p r_\alpha. \tag{2}$$



This is again a well defined $p$-homogeneous seminorm on $X$. Since

$$|x| = \Big|\sum_\alpha g_\alpha(x) h_\alpha\Big| \le \sum_\alpha |g_\alpha(x)|^p |h_\alpha| = \|x\|,$$

we also have $M = \lim_\gamma \|x_\gamma\| > 0$.

Define the *support* of a non-zero element $x$ of $X$ setting

$$\operatorname{supp} x = \{\alpha \in \mathfrak{a} : g_\alpha(x) \ne 0\};$$

it is a finite subset of $\mathfrak{a}$. We put $\operatorname{supp} 0 = \emptyset$. It is clear that $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$ implies $\|x + y\| = \|x\| + \|y\|$ for the seminorm given by (2). Since $(y_\gamma)$ is a Cauchy net, there is an index $\gamma_0$ such that $\|y_\gamma - y_{\gamma_0}\| < M/2$ for all $\gamma > \gamma_0$. Let $P$ be the projection on $X$ given by the formula

$$Px = \sum_{\alpha\in\operatorname{supp} y_{\gamma_0}} g_\alpha(x) h_\alpha.$$

Clearly, $\operatorname{supp} Px \cap \operatorname{supp}(I - P)x = \emptyset$ and $\operatorname{supp} y_{\gamma_0} \cap \operatorname{supp}(I - P)x = \emptyset$ for all $x$ in $X$, where $I$ is the identity operator. Thus

$$\|y_\gamma - y_{\gamma_0}\| = \|Py_\gamma - y_{\gamma_0} + (I - P)y_\gamma\| = \|Py_\gamma - y_{\gamma_0}\| + \|(I - P)y_\gamma\|,$$

which implies $\|(I - P)y_\gamma\| < M/2$ for all $\gamma > \gamma_0$. Since $\lim_\gamma g_\alpha(y_\gamma) = 0$ for all $\alpha$ and $\operatorname{supp} y_{\gamma_0}$ is finite, we have $\lim_\gamma Py_\gamma = 0$. Thus

$$M = \lim_\gamma \|y_\gamma\| = \lim_\gamma \|Py_\gamma\| + \lim_\gamma \|(I - P)y_\gamma\| < M/2,$$

a contradiction proving $\lim_\gamma y_\gamma = 0$. The conclusion follows.

We shall need the following notation. Let $X$ be as above. Let $a(\alpha)$ be a non-negative function on the index set $\mathfrak{a}$ for a fixed Hamel basis for $X$. Writing an element $x$ in the form (1) we put

$$\|x\|_{(p,a)} = \sum_\alpha |g_\alpha(x)|^p a(\alpha), \quad 0 < p \le 1.$$

For $a(\alpha) \equiv 1$ we simply write $\|x\|_p$ instead of $\|x\|_{(p,a)}$, and if $p = 1$ and $a \not\equiv 1$ we write $\|x\|_a$ for $\|x\|_{(p,a)}$.

We say that $X$ is *at most countably dimensional* (resp. *uncountably dimensional*) if it has an at most countable (resp. uncountable) Hamel basis.

PROPOSITION 2. *Let $X$ be a real or complex uncountably dimensional vector space. Then all topologies $\tau^p_{\max}$ $(0 < p \le 1)$ and $\tau^{q+}_{\max}$ are pairwise different on $X$.*

Proof. It is sufficient to show that if $0 < p < q \le 1$, then there is a $p$-homogeneous seminorm on $X$ which is not continuous in the topology

$\tau^q_{\max}$ (this proves both $\tau^p_{\max} \neq \tau^q_{\max}$ and $\tau^p_{\max} \neq \tau^{q+}_{\max}$ as well as $\tau^{r+}_{\max} \neq \tau^p_{\max}$ for any $0 < r < p$). Indeed, we simply take $\|\cdot\|_p$. Suppose, to the contrary, that it is continuous in the topology $\tau^q_{\max}$, i.e. that there is a $q$-homogeneous seminorm $\|\cdot\|$ on $X$ and a $C > 0$ such that for all $x$ in $X$ we have

$$\|x\|_p \le C\|x\|^{p/q}. \tag{3}$$

We have

$$\|x\| = \Big\|\sum_\alpha g_\alpha(x)h_\alpha\Big\| \le \sum_\alpha |g_\alpha(x)|^q\|h_\alpha\|$$

and setting $a(\alpha) = \|h_\alpha\|$ we obtain

$$\|x\| \le \|x\|_{(q,a)}. \tag{4}$$

Now, (3) and (4) imply

$$\|x\|_p \le C\|x\|^{p/q}_{(q,a)} \tag{5}$$

for all $x$ in $X$. Since the dimension of $X$ is uncountable, we find an integer $n > 0$ such that the set $\mathfrak{a}_n = \{\alpha \in \mathfrak{a} : a(\alpha) \le n\}$ is infinite. Take an element $x_0$ so that $g_\alpha(x_0) = 1/k$ for $\alpha$ in a subset of $\mathfrak{a}_n$ of cardinality $k$ and $g_\alpha(x_0) = 0$ for all remaining $\alpha$. Setting this element in (5) we obtain

$$\frac{k}{k^p} = k^{1-p} \le Cn^{p/q}(k^{1-q})^{p/q},$$

giving

$$k^{1-p/q} \le Cn^{p/q}$$

for all natural $k$, which is the desired contradiction. The conclusion follows.

We now prove a somewhat surprising fact that the above fails to be true if the dimension of $X$ is at most countable.

PROPOSITION 3. *Let $X$ be a real or complex at most countably dimensional vector space. Then all topologies $\tau^p_{\max}$ $(0 < p \le 1)$ and $\tau^{q+}_{\max}$ $(0 \le q < 1)$ coincide. In particular, $(X, \tau^p_{\max})$ or $(X, \tau^{q+}_{\max})$ is a locally convex space.*

Proof. It is sufficient to prove the proposition assuming that $X$ has a countable Hamel basis $(h_i)_{i=1}^\infty$. We shall be done if we show that for a given $p$-homogeneous seminorm $\|\cdot\|$ on $X$, $0 < p < 1$, there is a sequence $a = (a_i)_{i=1}^\infty$ of positive numbers such that

$$\|x\| \le \|x\|^p_a \tag{6}$$

for all $x$ in $X$. We have

$$\|x\| = \Big\|\sum_i g_i(x)h_i\Big\| \le \sum_i |g_i(x)|^p\|h_i\|.$$

Thus it is sufficient to prove (6) for $\|x\| = \|x\|_{(p,b)}$, where $b = (b_i)_{i=1}^\infty$ with $b_i = \max(1, \|h_i\|)$, so that all $b_i$ are positive. Therefore in order to prove (6) we have to show that there is a sequence $a = (a_i)_{i=1}^\infty$ of non-negative numbers such that

$$\sum_{i=1}^n b_i t_i^p \le \Big(\sum_{i=1}^n a_i t_i\Big)^p \tag{7}$$

for all finite sequences $(t_1, \ldots, t_n)$ of non-negative numbers.

To this end we use the following Hölder inequality (it follows immediately from the inequality D1 in Chapter 16 of [1]). Let $\mu$ be a probability measure on a space $\Omega$. Let $0 < p \le 1$. Then

$$\int_\Omega f^p d\mu \le \Big(\int_\Omega f d\mu\Big)^p \tag{8}$$

for any non-negative measurable function $f$. Setting here $\Omega = \mathbb{N}$, and $\mu(k) = C_k$, where $C_k > 0$ and $\sum_{k=1}^\infty C_k = 1$, we can rewrite (8) as

$$\sum_{i=1}^\infty C_i r_i^p \le \Big(\sum_{i=1}^\infty C_i r_i\Big)^p \tag{9}$$

for all sequences $(r_i)_{i=1}^\infty$ with non-negative entries. Setting $r_i = b_i^{1/p}C_i^{-1/p}t_i$ in (9), we rewrite it as

$$\sum_i b_i t_i^p \le \Big(\sum_i C_i^{1-1/p} b_i^{1/p} t_i\Big)^p$$

and this is exactly (7) with $a_i = C_i^{1-1/p}b_i^{1/p}$. The conclusion follows.

We can now prove the existence of an example announced in the abstract.

THEOREM 4. *There exists a complete pseudoconvex space $X$ that is not locally convex, but all of whose separable subspaces are locally convex.*

Proof. Let $X$ be any uncountably dimensional space equipped with the topology $\tau^p_{\max}$ with $0 < p < 1$. By Theorem 1 it is a complete locally pseudoconvex space, which by Proposition 2 is not locally convex. Let $X_0$ be a separable subspace of $X$. One can easily verify that the topology of $X$ restricted to $X_0$ coincides again with the topology $\tau^p_{\max}$. Let $S$ be a countable dense subset of $X_0$ and put $Y = \operatorname{span}(S)$. Since all subspaces of $X$ are closed, we have $X_0 = Y$. Thus $Y$ is at most countably dimensional, and so, by Proposition 3, it is a locally convex space. The conclusion follows.

The results of Propositions 2 and 3 can be formulated as follows.

THEOREM 5. *Let $X$ be a real or complex vector space. Then either*

(i) *all topologies $\tau^p_{\max}$, $0 < p \le 1$, and $\tau^{q+}_{\max}$, $0 \le q < 1$, are pairwise different and this happens exactly when $X$ is uncountably dimensional, or*

(ii) *all topologies $\tau^p_{\max}$, $0 < p \le 1$, and $\tau^{q+}_{\max}$, $0 \le q < 1$, coincide, and this happens exactly when the dimension of $X$ is at most countable.*

We now apply the maximal pseudoconvex topologies to topological algebras. A real or complex algebra $A$ provided with a topological vector space topology is said to be a *semitopological algebra* if multiplication is separately continuous, i.e. for fixed $y$ both maps $x \to xy$ and $x \to yx$ are continuous. Since under the considered maximal topologies all endomorphisms are continuous, we obtain

PROPOSITION 6. *Let $A$ be a real or complex algebra. Then for $0 < p \le q$ and $0 \le q < 1$, the algebras $(A, \tau_{\max}^p)$ and $(A, \tau_{\max}^{q+})$ are complete semitopological algebras.*

We say that an algebra $A$ is *at most countably generated* if there is an at most countable subset $S$ such that $A$ coincides with the smallest subalgebra of $A$ containing $S$. Otherwise we say that $A$ is *uncountably generated*. It is easy to see that an uncountably generated algebra has an uncountable Hamel basis. Propositions 2 and 6 immediately imply the following result about non-uniqueness of a complete topology for uncountably generated semitopological algebras.

PROPOSITION 7. *Let $A$ be an uncountably generated algebra. Then there are at least a continuum of different topologies making $A$ a complete semitopological algebra.*

In [6] it was shown that if a real or complex algebra $A$ is at most countably generated, then $(A, \tau_{\max}^{\mathrm{LC}})$ is a topological algebra, i.e. multiplication is jointly continuous. This result, together with Proposition 7, implies

COROLLARY 8. *Suppose that an algebra $A$ has a unique topology making it a complete semitopological algebra. Then this topology makes $A$ a topological algebra.*

We see that the question of uniqueness of a complete topology making an algebra $A$ a complete semitopological algebra makes sense only for at most countably generated algebras. We now ask a particular version of this question.

PROBLEM. Is $\tau_{\max}^{\mathrm{LC}}$ the only topology making the algebra of all polynomials in one variable a complete topological (resp. locally convex) algebra? (Added in proof. This question has a negative answer.)

We close this paper with an example concerning multiplicative linear functionals in semitopological algebras. A family $\mathcal{F}$ of linear functionals on a vector space $X$ is said to be *total* if $f(x) = 0$ for all $f$ in $\mathcal{F}$ implies $x = 0$.

THEOREM 9. *There exists a complete locally convex commutative semitopological algebra $A$ without multiplicative linear functionals such that every separable subalgebra of $A$ has a total family of such (continuous) functionals.*



Proof. Denote by $Q(t)$ the real or complex algebra of all rational functions in one variable $t$ and put $A = (Q(t), \tau_{\max}^{\mathrm{LC}})$. This is a complete semitopological algebra without multiplicative linear functionals. Let $\mathcal{A}$ be a separable subalgebra of $A$ with a countable dense subset $(x_i)_{i=1}^{\infty}$. Every $x_i$ has a finite number of poles, so that the set $P$ of all poles of the elements $x_i$ is at most countable. The smallest subalgebra of $A$ containing all the $x_i$ must coincide with $\mathcal{A}$, since all subalgebras of $A$ are closed. It follows that the set of all poles of the elements of $\mathcal{A}$ coincides with $P$. Now every point $t$ in $\mathbb{C}\backslash P$ (resp. $\mathbb{R}\backslash P$) gives the evaluation functional $f_t(x) = x(t)$ on $\mathcal{A}$, which is a continuous multiplicative linear functional. Clearly the set of all these functionals is total in $\mathcal{A}$. The conclusion follows.

Remark. It is known that the algebra $A$ in the above proof is not a topological algebra (see [7]), while, by a result of [6], all of its separable subalgebras, being at most countably generated algebras, are topological. Thus we also have an example of a commutative semitopological algebra that is not topological, but all of whose separable subalgebras are topological.

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