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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
TORONTO, ONTARIO
CANADA M5S 1A1
E-mail: DELJUNCO@MATH.TORONTO.EDU

DEPARTMENT OF MATHEMATICS AND INFORMATICS
NICHOLAS COPERNICUS UNIVERSITY
CHOPINA 12/18
87-100 TORUŃ, POLAND
E-mail: MLEM@MAT.UNI.TORUN.PL
MENTZEN@MAT.UNI.TORUN.PL

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Derivability, variation and range of a vector measure

by

L. RODRÍGUEZ-PIAZZA (Sevilla)

Abstract. We prove that the range of a vector measure determines the σ-finiteness of its variation and the derivability of the measure. Let F and G be two countably additive measures with values in a Banach space such that the closed convex hull of the range of F is a translate of the closed convex hull of the range of G; then F has a σ-finite variation if and only if G does, and F has a Bochner derivative with respect to its variation if and only if G does. This complements a result of [Ro] where we proved that the range of a measure determines its total variation. We also give a new proof of this fact.

Answering a question of Anantharaman and Diestel [AD], we proved in [Ro] that if the ranges of two measures with values in a Banach space have the same closed convex hull, then they have the same total variation. So we can say that the range of a vector measure determines its total variation. The purpose of this paper is to show two other properties of a vector measure which are determined by its range: the σ-finiteness of its variation, and the Bochner derivability.

In Section 1 we introduce the notation and collect some known results we will use throughout the paper. We first establish some properties of the Bartle integral and vector measures with scalar density with respect to another vector measure; and we finish with a result about the determination of real-valued symmetric measures defined on the euclidean unit sphere (Theorem 1.3).

The fact that the range determines the total variation does not imply directly that the range determines the σ-finiteness of the variation. If we know that Z, the closed convex hull of the range of a vector measure F, is also the closed convex hull of the range of another vector measure of σ-finite variation, what we know is that Z can be decomposed as $Z = \sum_{n\in\mathbb{N}} Z_n$, where each Z_n is the closed convex hull of the range of a measure of finite

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variation. We need a good representation of each Z_n, in terms of F, to allow us to conclude that F also has σ-finite variation. This representation is given in Theorem 2.1, and particularly Corollary 2.2, of Section 2. We will refer to them as the decomposition theorem(s).

In Section 3, these decomposition theorems are used to give a new proof that the range of a vector measure determines its total variation, and to prove that it also determines the σ-finiteness of the variation (Theorem 3.1). We also show that there is no infinite-dimensional Banach space in which the condition that the range of F is included in the range of a measure of σ-finite variation implies that F has σ-finite variation.

In Section 4 we study the average range of a vector measure in order to prove, using a result of Rieffel and the decomposition theorem, that the range determines the Bochner derivability of a vector measure. In Section 5 we give a different proof of this fact. We obtain it as a consequence of a factorization theorem (Theorem 5.2), and the fact that the range determines the total variation.

1. Preliminary results. Notation. Throughout this paper X is a real Banach space, X^* is its topological dual and B_X its closed unit ball. Given a subset K of X, $\overline{\mathrm{co}}\,K$ (resp. $\overline{\mathrm{aco}}\,K$) denotes the closed convex (resp. absolutely convex) hull of K.

Let $(\Omega, \mathcal{A})$ be a measurable space (that is, $\mathcal{A}$ is a σ-field of subsets of Ω). A function $F : \mathcal{A} \to X$ is a *measure* if it is countably additive. The range of F will be denoted by $\mathrm{rg}\, F$, that is, $\mathrm{rg}\, F = \{F(A) : A \in \mathcal{A}\}$. It is a relatively weakly compact set in X with $\frac{1}{2}F(\Omega)$ as centre of symmetry.

A subset Z of X will be called a *zonoid* if it is the closed convex hull of the range of an X-valued measure. If $Z = \overline{\mathrm{co}}(\mathrm{rg}\, F)$, we will say that the measure F *generates* the zonoid Z. This definition agrees with the usual finite-dimensional one [B]. Actually, thanks to a construction of Kluvánek and Knowles [KK, p. 128], [DU, p. 274], every zonoid is the range of an X-valued measure. As each zonoid is a convex set having a centre of symmetry it is obvious that a zonoid Z_1 is a translate of another zonoid Z_2 if and only if $Z_1 - Z_1 = Z_2 - Z_2$.

We will denote by $|F|$ the *variation* of F, which is the (extended) positive measure defined for every A in $\mathcal{A}$ by

$$|F|(A) = \sup \Big\{ \sum_{C \in \mathcal{P}} \|F(C)\| : \mathcal{P} \text{ is a finite partition of } A \text{ in } \mathcal{A} \Big\},$$

where we allow the supremum to be ∞. The *total variation* of F is $|F|(\Omega)$. The measure F *has finite variation* if $|F|(\Omega) < \infty$, and *has σ-finite variation* if there exists a sequence (A_n) of measurable sets covering Ω such that $|F|(A_n) < \infty$ for every n.

Despite the fact that a measure may have infinite variation, thanks to a result of Bartle, Dunford and Schwartz [BDS], [DU, p. 14], there always exists a finite positive measure which *controls* it, that is, a finite positive measure μ defined on $\mathcal{A}$ satisfying

$$\lim_{\mu(A) \to 0} F(A) = 0.$$

Such a μ is called a *control measure* for F. A finite positive measure μ defined on $\mathcal{A}$ is a control measure for F if and only if $\mu(A) = 0$ implies $F(A) = 0$, for every A in $\mathcal{A}$ [DU, p. 10].

A useful tool for studying the properties of a vector measure, and, in particular, for describing the zonoid it generates is the Bartle integral. It is defined in a straightforward fashion [DU, p. 6]. If $f : \Omega \to \mathbb{R}$ is a measurable simple function and $f = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$, we define

$$\int f\, dF = \sum_{j=1}^{n} \alpha_j F(A_j),$$

which depends only on f and not on the representation as a linear combination of characteristic functions thanks to the additivity of F. Since the range of F is bounded, and every simple function with values in $[0,1]$ is a convex combination of characteristic functions, it is easy to see that there is a constant $K > 0$ such that $\|\int f\, dF\| \le K\|f\|_\infty$ for every simple function f. This inequality allows us to define by density the integral $\int f\, dF$ for every bounded measurable function f.

The Bartle integral is linear in f and F, and if $T : X \to Y$ is a bounded linear operator, then

$$\int f\, dT\circ F = T\Big(\int f\, dF\Big).$$

If F has a Bochner density with respect to a positive measure ν defined on $\mathcal{A}$, that is, if there exists $\varrho \in \mathrm{L}^1(\nu; X)$ such that $F(A) = \int_A \varrho\, d\nu$ for every A in $\mathcal{A}$, then obviously we have

$$\int f\, dF = \int f \varrho\, d\nu$$

for every real-valued bounded measurable function f.

If μ is a control measure for F, then clearly $\int f\, dF$ depends only of the class of f in $\mathrm{L}^\infty(\mu)$, and so the integral defines a bounded linear operator I_F from $\mathrm{L}^\infty(\mu)$ to X. The following proposition states several known results about I_F, the Bartle integral and the range.

PROPOSITION 1.1. *Let F be an X-valued measure defined on $(\Omega, \mathcal{A})$, and μ a control measure for F. Define $I_F : \mathrm{L}^\infty(\mu) \to X$ by $I_F(f) = \int f\, dF$. Then:*

(a) I_F *is a continuous operator from the weak* topology of* $L^\infty(\mu)$ *(considered as the dual of* $L^1(\mu)$*) to the weak topology of* X.

(b) $\overline{\mathrm{co}}(\mathrm{rg}\,F) = \{\int f\,dF : f : \Omega \to [0,1]$ *measurable*$\}$.

(c) $\overline{\mathrm{co}}(\mathrm{rg}\,F - \mathrm{rg}\,F) = \overline{\mathrm{co}}(\mathrm{rg}\,F) - \overline{\mathrm{co}}(\mathrm{rg}\,F) = \{\int f\,dF : f : \Omega \to [-1,1]$ *measurable*$\}$.

Proof. A detailed proof of (a) and (b) can be found in [DU, Lemma IX.1.3]: (a) follows from the fact that $I_F^* x^* = dx^*\circ F/d\mu \in L^1(\mu)$ for every $x^* \in X^*$, and (b) follows from (a) and the fact that the convex hull of the characteristic functions is norm dense in $\{f \in L^\infty(\mu) : 0 \le f \le 1\}$, a weak* compact convex set. The first equality in (c) is a consequence of the weak compactness of $\overline{\mathrm{co}}(\mathrm{rg}\,F)$, and the second is immediate from (b). ∎

The integral allows us to define new vector measures from F. If $f : \Omega \to \mathbb{R}$ is measurable and bounded, it is proved in [BDS] that the map

$$A \mapsto \int_A f\,dF = \int \chi_A f\,dF, \qquad A \in \mathcal{A},$$

is a measure. We will denote it by fF. So, for instance, if A_0 is a measurable set then $\chi_{A_0}F$ is the measure restriction of F to A_0, that is, the map $A \mapsto F(A\cap A_0)$. From the definition of fF it follows that for any simple function g,

$$\int g\,dfF = \int gf\,dF;$$

and, by density, this equality remains true for every bounded measurable g. Therefore $g(fF) = (fg)F$.

The following proposition relates the variations of F and fF. It is due to Lewis [L, Theorem 4.2] in the setting of a more general theory of integration.

PROPOSITION 1.2. *A measurable set A has fF-finite variation if and only if $\chi_A f \in L^1(|F|)$. In that case we have*

$$|fF|(A) = \int_A |f|d|F|.$$

As we said our aim is to prove that the range determines the total variation, the σ-finiteness of the variation and the derivability of a vector measure. The key will be the following theorem on determination of symmetric measures on the euclidean unit sphere. If x, y are two vectors in $\mathbb{R}^n$, then $\langle x, y\rangle$ denotes their scalar product, and $\mathbb{S}_{n-1}$ is the euclidean unit sphere of $\mathbb{R}^n$, that is, $\mathbb{S}_{n-1} = \{x \in \mathbb{R}^n : \langle x, x\rangle = 1\}$. A measure σ defined on the Borel subsets of $\mathbb{S}_{n-1}$ is *symmetric* if $\sigma(A) = \sigma(-A)$ for every Borel set A.

THEOREM 1.3. *Let σ and τ be two real-valued symmetric measures defined on the Borel subsets of $\mathbb{S}_{n-1}$. If*

$$\int_{\mathbb{S}_{n-1}} |\langle e, x\rangle|\,d\sigma(x) = \int_{\mathbb{S}_{n-1}} |\langle e, x\rangle|\,d\tau(x)$$

for every $e \in \mathbb{R}^n$, then $\sigma = \tau$.

This theorem goes back to Aleksandrov [A]. Several other authors have reproved it: for instance, Petty [Pe] and Rickert [R1] using spherical harmonics as Aleksandrov did; Choquet [C, p. 53] using an elementary argument of differentiation of positive quadratic forms; and Matheron [M] via a theorem on integral representation of functions of negative type. This result has been used extensively in the study of zonoids in $\mathbb{R}^n$ (see [B], [R2], [N] or [SW]). Here we give a proof that we have not found in the literature.

Proof of Theorem 1.3. The result is trivial if $n = 1$, so let $n \ge 2$. We can assume that $\tau = 0$ and we have to prove that $\sigma = 0$ if, for every $e \in \mathbb{R}^n$,

$$\int_{\mathbb{S}_{n-1}} |\langle e, x\rangle|\,d\sigma(x) = 0. \tag{1}$$

The variation $|\sigma|$ is a positive finite measure. If $V \subset \mathbb{R}^n$ is a linear subspace of dimension $\le n-2$ such that $|\sigma|(V \cap \mathbb{S}_{n-1}) = 0$, it is easy to see that there exists a subspace W with $\dim W = \dim V + 1$ such that $|\sigma|(W \cap \mathbb{S}_{n-1}) = 0$: pick two linearly independent vectors x, y in $V^\perp$ and, for every $t \in \mathbb{R}$, let W_t be the subspace generated by $x + ty$ and V. We have $W_t \cap W_s = V$ for $t \ne s$. Since $|\sigma|$ is finite, we can find the desired W in the uncountable family $\{W_t\}_{t\in\mathbb{R}}$. Beginning with $V = \{0\}$, and iterating this procedure, we conclude that there exists a hyperplane H (a linear subspace of dimension $n-1$) such that $|\sigma|(H \cap \mathbb{S}_{n-1}) = 0$. We can and will assume that $H = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$.

Let $G = \{(x_1, \ldots, x_n) \in \mathbb{S}_{n-1} : x_n > 0\}$; we have $\mathbb{S}_{n-1} = G \sqcup -G \sqcup (H \cap \mathbb{S}^n)$, and by the symmetry of σ, we only have to prove that the restriction $\sigma|_G$ is null. From (1), taking into account that σ and the function $x \mapsto |\langle e, x\rangle|$ are symmetric, we know that, for every $e \in \mathbb{R}^n$,

$$0 = \int_{\mathbb{S}_{n-1}} |\langle e, x\rangle|\,d\sigma(x) = \int_G + \int_{-G} = 2\int_G |\langle e, x\rangle|\,d\sigma(x). \tag{2}$$

Consider the map $\psi : G \to \mathbb{R}^{n-1}$ defined by

$$\psi(x_1, \ldots, x_{n-1}, x_n) = \left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right).$$

Then ψ is a homeomorphism with inverse

$$\psi^{-1}(y) = \left(\frac{y_1}{\sqrt{1+\langle y, y\rangle}}, \ldots, \frac{y_{n-1}}{\sqrt{1+\langle y, y\rangle}}, \frac{1}{\sqrt{1+\langle y, y\rangle}}\right)$$

for $y = (y_1, \ldots, y_{n-1})$. Let μ be the measure in $\mathbb{R}^{n-1}$ which is the image of $\sigma|_G$ by ψ. Since ψ is a homeomorphism, $\mu = 0$ if and only $\sigma|_G = 0$, thus if and only if $\sigma = 0$. The condition (2) becomes

$$\int_{\mathbb{R}^{n-1}} |\langle e, \psi^{-1}y\rangle|\, d\mu(y) = 0, \quad \forall e \in \mathbb{R}^n,$$

that is,

$$\int_{\mathbb{R}^{n-1}} \frac{|e_n + \sum_{k=1}^{n-1} e_k y_k|}{\sqrt{1+\langle y, y\rangle}}\, d\mu(y) = 0, \quad \forall (e_1, \ldots, e_n) \in \mathbb{R}^n. \tag{3}$$

As $1/\sqrt{1+\langle y,y\rangle} \neq 0$ for every $y \in \mathbb{R}^{n-1}$, the measure ν defined by

$$d\nu(y) = (1/\sqrt{1+\langle y,y\rangle})\, d\mu(y)$$

is null if and only if μ is. By (3),

$$\int_{\mathbb{R}^{n-1}} |\alpha\langle c, y\rangle + \beta|\, d\nu(y) = 0, \quad \forall c \in \mathbb{R}^{n-1},\ \forall \alpha, \beta \in \mathbb{R}. \tag{4}$$

We now prove that (4) implies that the Fourier transform of ν is null. For $c \in \mathbb{R}^{n-1}$, let ν_c be the measure on the real line which is the image of ν under the projection $y \mapsto \langle c, y\rangle$. By (4), we have

$$\int_{\mathbb{R}} |\alpha t + \beta|\, d\nu_c(t) = 0, \quad \forall \alpha, \beta \in \mathbb{R}. \tag{5}$$

For $a < b$, let

$$f_{a,b}(t) = 1/2 + (|t-b| - |t-a|)/(2(b-a)), \quad t \in \mathbb{R}.$$

From (5), $\int f_{a,b}\, d\nu_c = 0$. It is easy to see that, for every $t \in \mathbb{R}$, $\lim_{b\to a} f_{a,b}(t) = \chi_{(-\infty,a]}(t)$, and the convergence is dominated by 1. We see that $\nu_c((-\infty, a]) = 0$ for every real a, so $\nu_c = 0$. This, in particular, yields

$$\int_{\mathbb{R}^{n-1}} \exp(i\langle c, y\rangle)\, d\nu(y) = \int_{\mathbb{R}} \exp(it)\, d\nu_c(t) = 0, \quad \forall c \in \mathbb{R}^{n-1}.$$

Therefore the Fourier transform of ν is null, and then $\nu = 0$. The theorem follows from the previous considerations. ■

2. Decomposition of zonoids. In this section we deal with the following problem: suppose that a zonoid Z of the Banach space X is generated by the measure F, and that Z is the sum of two other zonoids $Z = Z_1 + Z_2$. How can the measure F reflect this decomposition?

A first type of decomposition of Z in terms of F which naturally comes to mind is the following: if F is defined on the measurable space (Ω, Σ),

then for every measurable set A we have the decomposition

$$(*) \qquad Z = \overline{\mathrm{co}}(\mathrm{rg}\, F) = \overline{\mathrm{co}}\{F(B) : B \in \Sigma, B \subset A\} + \overline{\mathrm{co}}\{F(B) : B \in \Sigma, B \subset \Omega \setminus A\}.$$

More generally, if $\phi : \Omega \to [0,1]$ is a measurable function we also have

$$(**) \qquad Z = \overline{\mathrm{co}}(\mathrm{rg}\, \phi F) + \overline{\mathrm{co}}(\mathrm{rg}(1-\phi)F).$$

One can wonder whether every decomposition of Z can be represented as in $(*)$, or, at least, as in $(**)$. Examples in $\mathbb{R}^n$ immediately refute such a conjecture. Neyman, in [N], characterizes geometrically the zonoids Z in $\mathbb{R}^n$ for which there exists some measure generating Z such that every decomposition can be represented as in $(*)$, or equivalently, for any measure generating Z every decomposition can be represented as in $(**)$. Let us examine an example for which this is not possible.

EXAMPLE (A decomposition of the hexagon). Identify the plane with $\mathbb{C}$, the complex numbers. For $\alpha = \exp(2\pi i/3)$, let $f : [0,3] \to \mathbb{C}$ be the function

$$f = \chi_{[0,1]} + \alpha\chi_{(1,2]} + \alpha^2\chi_{(2,3]}$$

and consider the measure F on $[0,3]$ with density f with respect to the Lebesgue measure. The range of F is the regular hexagon H determined by the sixth roots of unity. We can decompose H as the sum of the two hexagons

$$H_1 = \frac{\sqrt{3}i}{4} + \frac{1}{2}H \quad \text{and} \quad H_2 = \frac{-\sqrt{3}i}{4} + \frac{1}{2}H.$$

H_1 and H_2 are zonoids. For instance, H_1 is the range of the measure with density

$$f_1 = \frac{1}{2}(\chi_{[0,1/2]} - \chi_{(1/2,1]} + \alpha\chi_{(1,2]} - \alpha^2\chi_{(2,3]}).$$

The decomposition $H = H_1 + H_2$ cannot be represented as in $(**)$. If $\phi : [0,3] \to [0,1]$ is a measurable function, it is easy to see that there exist $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$ such that

$$\mathrm{rg}\, \phi F = [0, \lambda_1] + [0, \lambda_2]\alpha + [0, \lambda_3]\alpha^2,$$

a compact convex set. H_1 is included in the upper closed half plane, so if H_1 were $\mathrm{rg}(\phi F)$, then λ_3 should be 0, and in this case $\mathrm{rg}(\phi F)$ would be a segment or quadrilateral, but never a hexagon.

However, there exists a real bounded function $\phi_1 = f_1/f$ such that H_1 is the range of $\phi_1 F$. This is a general fact for non-atomic $\mathbb{R}^n$-valued measures (see the remark after the proof of Theorem 2.1). This is not the case if we consider the measure G defined on the subsets of $\{0,1,2\}$ as $G(\{j\}) = \alpha^j$, $j = 0, 1, 2$. This measure also generates H. The measures of type ϕG generate

the zonoids
$$[0,1]\lambda_1 + [0,1]\lambda_2\alpha + [0,1]\lambda_3\alpha^2 \quad \text{with } \lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}.$$
All these zonoids contain 0 either as an interior point or as an extreme point. H_1 is none of them because it contains 0 in the middle of the edge $[-1/4, 1/4]$.

In the following theorem we prove that, up to translations, every decomposition of Z is represented as in (**); that is, if Z_1, Z_2 are zonoids such that Z is a translate of Z_1+Z_2, then there exists $\phi : \Omega \to [0,1]$ measurable such that Z_1 is a translate of $\overline{\mathrm{co}}(\mathrm{rg}\,\phi F)$ and Z_2 is a translate of $\overline{\mathrm{co}}(\mathrm{rg}(1-\phi)F)$. It is easy to check that in the previous examples $\phi = 1/2$ has this property. Note that Z_1+Z_2 can always be realized as a decomposition of type (*) for some G: if, for $j=1,2$, Z_j is generated by the measure G_j defined on the measurable space $(\Delta_j, \mathcal{B}_j)$, consider the disjoint union $\Delta = \Delta_1 \sqcup \Delta_2$, and define G as $G(A_1 \sqcup A_2) = G_1(A_1) + G_2(A_2)$; then G generates Z_1+Z_2, and we can take Δ_1 for A in (*), but now with respect to G. This explains the statement of the following theorem, which is more analytic than geometric, and which we will need in Sections 3 and 4.

THEOREM 2.1. *Let $(\Omega,\mathcal{A})$ and $(\Delta,\mathcal{B})$ be two measurable spaces, and X be a Banach space. If $F:\mathcal{A}\to X$ and $G:\mathcal{B}\to X$ are two measures such that*
$$\overline{\mathrm{co}}(\mathrm{rg}\,F - \mathrm{rg}\,F) = \overline{\mathrm{co}}(\mathrm{rg}\,G - \mathrm{rg}\,G),$$
then, for every measurable function $\psi:\Delta\to[0,1]$, there exists a measurable function $\phi:\Omega\to[0,1]$ such that
$$\overline{\mathrm{co}}(\mathrm{rg}\,\phi F - \mathrm{rg}\,\phi F) = \overline{\mathrm{co}}(\mathrm{rg}\,\psi G - \mathrm{rg}\,\psi G) \tag{6}$$
and
$$\overline{\mathrm{co}}(\mathrm{rg}(1-\phi)F - \mathrm{rg}(1-\phi)F) = \overline{\mathrm{co}}(\mathrm{rg}(1-\psi)G - \mathrm{rg}(1-\psi)G). \tag{7}$$

Proof. First observe that, as we are dealing with closed convex sets, $\overline{\mathrm{co}}(\mathrm{rg}\,F-\mathrm{rg}\,F)$ and $\overline{\mathrm{co}}(\mathrm{rg}\,G-\mathrm{rg}\,G)$ are equal if and only if every linear functional $x^*\in X^*$ achieves the same supremum in them. By Proposition 1.1, if μ is any control measure for F, we have
$$\begin{aligned}\sup\{x^*(a) : a\in \overline{\mathrm{co}}(\mathrm{rg}\,F-\mathrm{rg}\,F)\} &= \sup\Big\{x^*\Big(\int_\Omega h\,dF\Big) : h:\Omega\to[-1,1] \text{ measurable}\Big\}\\ &= \sup\Big\{\int h\,dx^*\circ F : h:\Omega\to[-1,1]\text{ measurable}\Big\}\\ &= \sup\Big\{\int h\frac{dx^*\circ F}{d\mu}\,d\mu : h\in \mathrm{L}^\infty(\mu),\ \|h\|_\infty\le 1\Big\} = \Big\|\frac{dx^*\circ F}{d\mu}\Big\|_{\mathrm{L}^1(\mu)}\end{aligned}$$

by the duality between $\mathrm{L}^1(\mu)$ and $\mathrm{L}^\infty(\mu)$. Therefore the hypothesis of the theorem yields, for any control measure ν for G,
$$\Big\|\frac{dx^*\circ F}{d\mu}\Big\|_{\mathrm{L}^1(\mu)} = \Big\|\frac{dx^*\circ G}{d\nu}\Big\|_{\mathrm{L}^1(\nu)} \quad \text{for every } x^*\in X^*. \tag{8}$$

For the same reasons, since $dx^*\circ\psi G/d\nu = \psi\,dx^*\circ G/d\nu$, we have to prove the existence of a measurable function $\phi:\Omega\to[0,1]$ such that
$$\Big\|\phi\frac{dx^*\circ F}{d\mu}\Big\|_{\mathrm{L}^1(\mu)} = \Big\|\psi\frac{dx^*\circ G}{d\nu}\Big\|_{\mathrm{L}^1(\nu)} \quad \text{for every } x^*\in X^*. \tag{9}$$
If (9) is checked for some control measures μ and ν, we will have (6). The analogue of (9) for $1-\psi$ and $1-\phi$ will be a consequence of (8) and (9), and then we will also obtain (7).

We first establish (9) for $X=\mathbb{R}^n$. We use a standard technique in the study of zonoids (see [R2] or [B]). Recall that the linear functionals on X are the maps $x\mapsto\langle e,x\rangle$, for $e\in\mathbb{R}^n$. In this case we can use as control measures the variations of F and G with respect to the euclidean norm in $\mathbb{R}^n$. Let $\mathbf{f}$ be the Radon–Nikodym derivative of F with respect to its euclidean variation $|F|$; then $\mathbf{f}:\Omega\to\mathbb{S}_{n-1}$ almost everywhere, and we can assume this is true everywhere. Similarly, let $\mathbf{g}=dG/d|G|$, and let σ (resp. τ) be the image measure on $\mathbb{S}_{n-1}$ of $|F|$ (resp. $|G|$) under the map $\mathbf{f}$ (resp. $\mathbf{g}$); that is, $\sigma(A)=|F|(\mathbf{f}^{-1}(A))$ for every Borel subset A of $\mathbb{S}_{n-1}$, and similarly for τ.

In this situation, since $d\langle e,F\rangle/d|F| = \langle e,\mathbf{f}\rangle$, (8) yields
$$\begin{aligned}\int_{\mathbb{S}_{n-1}} |\langle e,x\rangle|\,d\sigma(x) &= \int_\Omega |\langle e,\mathbf{f}(\omega)\rangle|\,d|F|(\omega)\\ &= \Big\|\frac{d\langle e,F\rangle}{d|F|}\Big\|_{\mathrm{L}^1(|F|)} = \Big\|\frac{d\langle e,G\rangle}{d|G|}\Big\|_{\mathrm{L}^1(|G|)}\\ &= \int_{\mathbb{S}_{n-1}}|\langle e,x\rangle|\,d\sigma(x) \quad \text{for every } e\in\mathbb{R}^n.\end{aligned} \tag{10}$$
If $\widetilde\sigma$ is the symmetrization of σ ($\widetilde\sigma(A)=\frac12(\sigma(A)+\sigma(-A))$ for every Borel subset A of $\mathbb{S}_{n-1}$), then for any measurable symmetric function $h:\mathbb{S}_{n-1}\to\mathbb{R}$, we have $\int h\,d\widetilde\sigma = \int h\,d\sigma$, and so, by (10), if $\widetilde\tau$ is the symmetrization of τ,
$$\int_{\mathbb{S}_{n-1}}|\langle e,x\rangle|\,d\widetilde\sigma(x) = \int_{\mathbb{S}_{n-1}}|\langle e,x\rangle|\,d\widetilde\tau(x) \quad \text{for every } e\in\mathbb{R}^n,$$
and by Theorem 1.3, we have $\widetilde\sigma=\widetilde\tau$.

Now consider the measure τ_1 on the sphere defined by
$$\tau_1(A) = \int_{\mathbf{g}^{-1}(A)} \psi\,d|G| \quad \text{for every Borel subset } A \text{ of } \mathbb{S}_{n-1}.$$

Since $0 \leq \tau_1 \leq \tau$, we also have $0 \leq \widetilde{\tau}_1 \leq \widetilde{\tau} = \widetilde{\sigma}$. Then, by the Radon–Nikodym theorem, $\widetilde{\tau}_1$ has a derivative ϱ with respect to $\widetilde{\sigma}$. Clearly ϱ takes values in $[0,1]$ and is symmetric $\widetilde{\sigma}$-almost everywhere on $\mathbb{S}_{n-1}$; assume this is true everywhere. The following equalities prove that if we define $\phi(\omega) = \varrho(\mathbf{f}(\omega))$ for $\omega \in \Omega$, then ϕ satisfies (9). As a measure and its symmetrization give the same integral for a symmetric function, we have, for every $e \in \mathbb{R}^n$,

$$\begin{aligned}\left\|\phi\frac{d\langle e,F\rangle}{d|F|}\right\|_{\mathrm{L}^1(|F|)} &= \int_\Omega \varrho(\mathbf{f}(\omega))|\langle e,\mathbf{f}(\omega)\rangle|\,d|F|(\omega)\\ &= \int_{\mathbb{S}_{n-1}} \varrho(x)|\langle e,x\rangle|\,d\sigma(x) = \int_{\mathbb{S}_{n-1}} \varrho(x)|\langle e,x\rangle|\,d\widetilde{\sigma}(x)\\ &= \int_{\mathbb{S}_{n-1}} |\langle e,x\rangle|\,d\widetilde{\tau}_1(x) = \int_{\mathbb{S}_{n-1}} |\langle e,x\rangle|\,d\tau_1(x)\\ &= \int_\Delta \psi(\delta)|\langle e,\mathbf{g}(\delta)\rangle|\,d|G|(\delta) = \left\|\psi\frac{d\langle e,G\rangle}{d|G|}\right\|_{\mathrm{L}^1(|G|)}.\end{aligned}$$

This finishes the case $X = \mathbb{R}^n$.

For a general Banach space X we will use a compactness argument to establish the existence of ϕ satisfying (9) for some (any) control measure μ. We will look for ϕ in $\mathrm{L}^\infty(\mu)$, identifying functions with their classes. For every $x^* \in X^*$, let

$$H_{x^*} = \left\{\phi \in \mathrm{L}^\infty(\mu) : 0 \leq \phi \leq 1,\ \int \phi\left|\frac{dx^*\circ F}{d\mu}\right| d\mu = \left\|\psi\frac{dx^*\circ G}{d\nu}\right\|_{\mathrm{L}^1(\nu)}\right\}.$$

Any function ϕ in $\bigcap_{x^*\in X^*} H_{x^*}$ will satisfy (9), and we only have to see that this intersection is not void. Every H_{x^*} is weak* compact in $\mathrm{L}^\infty(\mu)$, so it is enough to check that the intersection of every finite family of the H_{x^*} is not void.

Let $x_1^*, \ldots, x_n^*$ be in X^*, and consider the operator $T : X \to \mathbb{R}^n$ defined for $x \in X$ by $Tx = (x_1^*(x), \ldots, x_n^*(x))$. Evidently the measures $T\circ F$ and $T\circ G$ satisfy the hypothesis of the theorem. As they are $\mathbb{R}^n$-valued, we have proved that there exists a measurable function $\phi_T : \Omega \to [0,1]$ such that

$$\overline{\mathrm{co}}(\mathrm{rg}\,\phi_T T\circ F - \mathrm{rg}\,\phi_T T\circ F) = \overline{\mathrm{co}}(\mathrm{rg}\,\psi T\circ G - \mathrm{rg}\,\psi T\circ G),$$

or equivalently,

$$(11)\qquad \left\|\phi_T\frac{d\langle e,T\circ F\rangle}{d\mu}\right\|_{\mathrm{L}^1(\mu)} = \left\|\phi\frac{d\langle e,T\circ G\rangle}{d\nu}\right\|_{\mathrm{L}^1(\nu)} \quad \text{for every } e \in \mathbb{R}^n.$$

If $\{e_1, \ldots, e_n\}$ is the canonical basis in $\mathbb{R}^n$, clearly we have $\langle e_k, Tx\rangle = x_k^*(x)$ for $x \in X$ and $k = 1, \ldots, n$; so $\langle e_k, T\circ F\rangle = x_k^*\circ F$ and $\langle e_k, T\circ G\rangle = x_k^*\circ G$. Thus, by (11), $\phi_T \in H_{x_1^*} \cap \ldots \cap H_{x_n^*}$. ∎

Remark. Let F be a non-atomic $\mathbb{R}^n$-valued measure. By Lyapunov's theorem, its range is compact and convex, and so is the range of the measures of type ϕF, since they are also non-atomic. If Z_1, Z_2 are two zonoids such that $\mathrm{rg}\,F = Z_1 + Z_2$, then by Theorem 2.1, there exists $\phi : \Omega \to [0,1]$ such that Z_1 is a translate of $\mathrm{rg}\,\phi F$ and Z_2 is a translate of $\mathrm{rg}(1-\phi)F$. But for a non-atomic $\mathbb{R}^n$-valued measure F, every zonoid translate of $\mathrm{rg}\,F$ is the range of a measure of type $(\chi_A + \chi_{\Omega\setminus A})F$ for a measurable set A (see [B, Lemma 1.3]). Using this, we conclude that there exist two measurable functions $\varphi_1, \varphi_2 : \Omega \to \mathbb{R}$ such that $|\varphi_1| + |\varphi_2| = 1$ and $Z_j = \mathrm{rg}\,\varphi_j F$, $j = 1, 2$. The same is true if $Z_1 + Z_2$ is a translate of $\mathrm{rg}\,F$. This can be generalized, for an arbitrary Banach space X, to measures satisfying the "Lyapunov convexity theorem in the weak topology" [DU, p. 263].

Theorem 2.1 concerns decomposition of a zonoid into the sum of two zonoids. An iterative procedure allows us to extend it to sums of a finite number of zonoids, and of a sequence of them. This is done in the following corollary.

COROLLARY 2.2. *Let $(\Omega, \mathcal{A})$ and $(\Delta, \mathcal{B})$ be two measurable spaces, and X be a Banach space. If $F : \mathcal{A} \to X$ and $G : \mathcal{B} \to X$ are two measures such that*

$$\overline{\mathrm{co}}(\mathrm{rg}\,F - \mathrm{rg}\,F) = \overline{\mathrm{co}}(\mathrm{rg}\,G - \mathrm{rg}\,G),$$

then, for every sequence of measurable functions $\psi_n : \Delta \to [0,1]$ such that $\sum_{n=1}^\infty \psi_n = 1$ pointwise, there exists a sequence of measurable functions $\phi_n : \Omega \to [0,1]$ such that $\sum_{n=1}^\infty \phi_n = 1$ pointwise, and

$$\overline{\mathrm{co}}(\mathrm{rg}\,\phi_n F - \mathrm{rg}\,\phi_n F) = \overline{\mathrm{co}}(\mathrm{rg}\,\psi_n G - \mathrm{rg}\,\psi_n G) \quad \text{for every } n = 1, 2, \ldots$$

Proof. First we use Theorem 2.1 for F, G, and ψ_1 to obtain $\phi_1 : \Omega \to [0,1]$ satisfying

$$\overline{\mathrm{co}}(\mathrm{rg}\,\phi_1 F - \mathrm{rg}\,\phi_1 F) = \overline{\mathrm{co}}(\mathrm{rg}\,\psi_1 G - \mathrm{rg}\,\psi_1 G).$$

Suppose that, for some m, we have found $\phi_1, \ldots, \phi_m$ satisfying

$$(12)\qquad \overline{\mathrm{co}}(\mathrm{rg}\,\phi_n F - \mathrm{rg}\,\phi_n F) = \overline{\mathrm{co}}(\mathrm{rg}\,\psi_n G - \mathrm{rg}\,\psi_n G)$$

for $n = 1, \ldots, m$, and $\sum_{n=1}^m \phi_n \leq 1$ pointwise. Then we apply Theorem 2.1 to the measures $(1 - \sum_{n=1}^m \phi_n)F$ and $(\sum_{n=m+1}^\infty \psi_n)G$, and the function $\psi = \psi_{m+1}/\sum_{n=m+1}^\infty \psi_n$ (with the convention $0/0 = 0$). We obtain $\phi : \Omega \to [0,1]$ such that

$$\overline{\mathrm{co}}\left(\mathrm{rg}\,\phi\Big(1 - \sum_{n=1}^m \phi_n\Big)F - \mathrm{rg}\,\phi\Big(1 - \sum_{n=1}^m \phi_n\Big)F\right) = \overline{\mathrm{co}}(\mathrm{rg}\,\psi_{m+1}G - \mathrm{rg}\,\psi_{m+1}G).$$

Taking $\phi_{m+1} = \phi(1 - \sum_{n=1}^m \phi_n)$ we get (12) for $n = m+1$, and we still have $\sum_{n=1}^{m+1} \phi_n \leq 1$.

So inductively we have found a sequence ϕ_n of functions satisfying (12) for every $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \phi_n \le 1$. We now show that, if $\varphi = 1 - \sum_{n=1}^{\infty} \phi_n$, then the measure φF is null, and so we can add φ to one of the functions ϕ_n without changing the measure $\phi_n F$, which will finish the proof.

Suppose that there exists $A \in \mathcal{A}$ such that $\varphi F(A) \neq 0$, and take $z^* \in X^*$ such that $z^*(\varphi F(A)) > 0$. Since $\overline{\mathrm{co}}(\mathrm{rg}\, G - \mathrm{rg}\, G)$ is weakly compact, there exists $x \in \overline{\mathrm{co}}(\mathrm{rg}\, G - \mathrm{rg}\, G)$ such that $z^*(x)$ is the maximum of z^* in $\overline{\mathrm{co}}(\mathrm{rg}\, G - \mathrm{rg}\, G)$. By Proposition 1.1, there exists $h : \Delta \to [-1,1]$ measurable such that

$$x = \int h\, dG = \sum_{n=1}^{\infty} \int h\psi_n\, dG.$$

The series converges in the weak topology of X by the dominated convergence theorem applied to the measures $x^* \circ G$, $x^* \in X^*$ (by the Orlicz–Pettis theorem, it is also convergent in norm; but we do not need this).

Using again Proposition 1.1 and (12) we have, for every n, a measurable function $f_n : \Omega \to [-1,1]$ such that $\int h\psi_n\, dG = \int f_n \phi_n\, dF$. Then $f = \chi_A \varphi + \sum_{n=1}^{\infty} f_n \phi_n$ satisfies $-1 \le f \le 1$, and so $\int f\, dF$ is in $\overline{\mathrm{co}}(\mathrm{rg}\, F - \mathrm{rg}\, F)$, and it is easy to see that $\int f\, dF = \varphi F(A) + x$. This contradicts the fact that $\overline{\mathrm{co}}(\mathrm{rg}\, F - \mathrm{rg}\, F)$ is equal to $\overline{\mathrm{co}}(\mathrm{rg}\, G - \mathrm{rg}\, G)$, since z^* achieves different supremums in each. ■

3. Range and variation. We begin by showing that the range of a vector measure determines its total variation and the σ-finiteness of its variation. We have done almost all the work in the previous section; the following theorem is a consequence of Corollary 2.2. Part (a) was proved in [Ro]; here we include a different proof.

THEOREM 3.1. *Let X be a Banach space, and let F and G be two X-valued measures such that*

$$\overline{\mathrm{co}}(\mathrm{rg}\, F - \mathrm{rg}\, F) = \overline{\mathrm{co}}(\mathrm{rg}\, G - \mathrm{rg}\, G)$$

(*that is, the zonoid generated by F is a translate of that generated by G*). *Then*:

(a) *F and G have the same total variation.*
(b) *F has σ-finite variation if and only if G does.*

Proof. Suppose F is defined on the measurable space $(\Omega, \mathcal{A})$, and G on $(\Delta, \mathcal{B})$. To prove (a), that is, $|F|(\Omega) = |G|(\Delta)$, by symmetry we only need to check the inequality $|F|(\Omega) \ge |G|(\Delta)$. If $|F|(\Omega) = \infty$ there is nothing to prove, so we assume that F has finite variation. Take a partition $\{B_1, \ldots, B_m\}$ of Δ in $\mathcal{B}$, and apply Corollary 2.2 to the sequence (ψ_n) where $\psi_n = \chi_{B_n}$ for $n \le m$, and $\psi_n = 0$ for $n > m$. We obtain a sequence of measurable functions $\phi_n : \Omega \to [0,1]$ such that $\sum_{n\ge1} \phi_n = 1$ and

$$\overline{\mathrm{co}}(\mathrm{rg}\, \phi_n F - \mathrm{rg}\, \phi_n F) = \overline{\mathrm{co}}(\mathrm{rg}\, \psi_n G - \mathrm{rg}\, \psi_n G) \quad \text{for every } n = 1, 2, \ldots$$

In particular, $G(B_n) \in \overline{\mathrm{co}}(\mathrm{rg}\, \phi_n F - \mathrm{rg}\, \phi_n F)$ for $n = 1, \ldots, m$, and, by Proposition 1.2, there exists $h_n : \Omega \to [-1,1]$ measurable such that

$$G(B_n) = \int h_n\, d\phi_n F = \int h_n \phi_n\, dF = h_n \phi_n F(\Omega).$$

By Proposition 1.3, we obtain

$$\|G(B_n)\| \le |h_n \phi_n F|(\Omega) = \int_\Omega |h_n \phi_n|\, d|F| \le \int_\Omega \phi_n\, d|F|,$$

and thus,

$$\sum_{n=1}^{m} \|G(B_n)\| \le \sum_{n=1}^{m} \int \phi_n\, d|F| \le \int_\Omega d|F| = |F|(\Omega).$$

Taking supremum over all partitions we get $|G|(\Delta) \le |F|(\Omega)$ as desired.

Again to prove (b) we only need to show that F has σ-finite variation if G does. Suppose that (B_n) is a sequence of pairwise disjoint sets in $\mathcal{B}$ with union Δ. Applying Corollary 2.2 to (χ_{B_n}), we obtain measurable functions $\phi_n : \Omega \to [0,1]$ such that $\sum_{n\ge1} \phi_n = 1$ and

$$\overline{\mathrm{co}}(\mathrm{rg}\, \phi_n F - \mathrm{rg}\, \phi_n F) = \overline{\mathrm{co}}(\mathrm{rg}\, \psi_n G - \mathrm{rg}\, \psi_n G) \quad \text{for every } n = 1, 2, \ldots$$

By (a) this implies $\phi_n F$ has finite variation, and, by Proposition 1.3, $\phi_n \in \mathrm{L}^1(|F|)$. Thus $|F|(\{\phi_n \ge 1/k\}) < \infty$ for every $k \in \mathbb{N}$. Therefore $|F|$ is σ-finite since $\Omega = \bigcup_{k,n} \{\phi_n \ge 1/k\}$. ■

Remark. Actually in [Ro] it is proved that F and G have the same total variation if $\overline{\mathrm{co}}(\mathrm{rg}\, F) = \overline{\mathrm{co}}(\mathrm{rg}\, G)$. But (a) of the theorem can be deduced from that since $\mathrm{rg}\, F - \mathrm{rg}\, F$ (resp. $\mathrm{rg}\, G - \mathrm{rg}\, G$) is the range of a measure $\widetilde{F}$ (resp. $\widetilde{G}$) whose total variation is $2|F|(\Omega)$ (resp. $2|G|(\Delta)$). Take as $\widetilde{F}$ the measure defined on the disjoint union $\Omega \sqcup \Omega$ by $\widetilde{F}(A_1 \sqcup A_2) = F(A_1) - F(A_2)$.

Some Banach spaces have a remarkable property of monotonicity of total variation with respect to the range. Anantharaman and Diestel [AD] observed that, thanks to a result of Grothendieck, if F and G are two measures with values in (a subspace of) an L^1 space such that F has finite variation and $\mathrm{rg}\, G \subset \mathrm{rg}\, F$, then G has finite variation. We proved in [Ro] that, in fact, this property characterizes the Banach spaces which are isomorphic to a subspace of an L^1 space.

There is nothing similar for the σ-finiteness of variation, except in the trivial case of finite-dimensional spaces. For every infinite-dimensional Banach space X, there exist two X-valued measures F, G such that F has σ-finite variation, G does not have σ-finite variation, and $\mathrm{rg}\, G \subset \mathrm{rg}\, F$. This will be shown in Theorem 3.3. For the proof we will need the following lemma based on known results about p-summing operators; we refer to the first chapter of Pisier's book [P]. Recall that for an operator $T: X \to Y$, its *p-summing norm* is defined by

$$\pi_p(T) = \sup \Big\{ \Big(\sum_{k=1}^{n} \|Tx_k\|^p \Big)^{1/p} \Big\},$$

where the supremum, which may be ∞, is taken over all finite sequences $x_1, \ldots, x_n$ in X satisfying $\sum_{k=1}^{n} |x^*(x_k)|^p \le 1$ for all x^* in the unit ball of X^*.

LEMMA 3.2. *Let n be a natural number, and X a Banach space of dimension $> n^6$. There exists a measurable function $f : [0,1] \to X$ such that:*

(a) *$\|f(t)\| = n$ for all $t \in [0,1]$.*
(b) *$\| \int_M f(t)\,dt\| \le 1/n^2$ for all measurable sets $M \subseteq [0,1]$.*

Proof. As the 2-summing norm of the identity in X is $\sqrt{\dim X}$, and the 1-summing norm is greater than the 2-summing norm, we see that the identity in our Banach space X has a 1-summing norm strictly greater than n^3. By the definition of this norm we easily get the existence of a finite subset $\{x_1, \ldots, x_m\}$ of $X \setminus \{0\}$ such that $\sum_{k=1}^{m} \|x_k\| = 1$, and

$$\sum_{k=1}^{m} |x^*(x_k)| \le \frac{1}{n^3} \quad \text{for every } x^* \text{ in the unit ball of } X^*. \tag{13}$$

For $k = 0, 1, \ldots, m$ let $t_k = \sum_{j=1}^{k} \|x_j\|$. Then $t_0 = 0$, $t_m = 1$, and the finite sequence (t_k) is increasing. Define f at 1 as any vector of norm n, and in $[0,1)$ by

$$f = \sum_{k=1}^{m-1} \frac{n x_k}{\|x_k\|} \chi_{[t_{k-1}, t_k)}.$$

It is clear that f so defined satisfies (a). To check (b), take a measurable set M and pick x^* in the unit ball of X^* such that $x^*(\int_M f) = \|\int_M f\|$. By (13) we have, for λ being the Lebesgue measure,

$$\Big\| \int_M f(t)\,dt \Big\| = \sum_{k=1}^{m} \frac{n}{\|x_k\|} x^*(x_k) \lambda(M \cap [t_{k-1}, t_k))$$
$$\le \sum_{k=1}^{m} \frac{n|x^*(x_k)|}{\|x_k\|}(t_k - t_{k-1}) = n \sum_{k=1}^{m} |x^*(x_k)| \le \frac{1}{n^2}. \ \blacksquare$$

THEOREM 3.3. *Given an infinite-dimensional Banach space X, there exist two X-valued measures, F and G, such that:*

(a) *$\operatorname{rg} G \subseteq \operatorname{rg} F$.*
(b) *F has σ-finite variation.*
(c) *G does not have σ-finite variation.*

Proof. Take a basic sequence (e_n) in X, and let Y be the closed subspace of X spanned by it. Take an increasing sequence (m_n) of natural numbers satisfying $m_{n+1} - m_n > n^6$ for every $n \in \mathbb{N}$, and let Y_n be the subspace of Y spanned by the basic vectors $\{e_j : m_n + 1 \le j \le m_{n+1}\}$. There exists a constant $C > 0$ such that the natural projections $P_n : Y \to Y_n$ have norm $\|P_n\| \le C$ for every n.

By the previous lemma, for each $n \in \mathbb{N}$, there exists a measurable function $f_n : [0,1] \to Y_n$ such that

$$\|f_n(t)\| = n \quad \text{for all } t \in [0,1], \tag{14}$$

$$\Big\| \int_M f_n(t)\,dt \Big\| \le \frac{1}{n^2} \quad \text{for every measurable } M \subseteq [0,1]. \tag{15}$$

Let H_n be the measure defined on $[0,1]$, with density f_n with respect to the Lebesgue measure. By the Lyapunov theorem, the range of H_n is compact and convex and, by Proposition 1.1(b), $\int h(t) f_n(t)\,dt$ belongs to $\operatorname{rg} H_n$ for every measurable function $h : [0,1] \to [0,1]$.

Consider the product space $\Delta = [0,1]^{\mathbb{N}}$, with the product probability $\mathbb{P}$ obtained by taking the Lebesgue measure in each factor. Let $\mathcal{B}$ be the σ-algebra where $\mathbb{P}$ is defined. We define $G : \mathcal{B} \to Y$ by

$$G(B) = \sum_{n \ge 1} \int_B f_n(t_n)\,d\mathbb{P}(t_1, t_2, \ldots), \quad B \in \mathcal{B}. \tag{16}$$

For every $B \in \mathcal{B}$, using the conditional expectation of χ_B with respect to the nth coordinate, there exists a measurable function $h_n : [0,1] \to [0,1]$ such that

$$\int_B f_n(t_n)\,d\mathbb{P}(t_1, t_2, \ldots) = \int h_n(t_n) f_n(t_n)\,d\mathbb{P}(t_1, t_2, \ldots) \tag{17}$$
$$= \int h_n(t) f_n(t)\,dt \in \operatorname{rg} H_n.$$

This implies, by (15), that the series in (16) converges absolutely, and that G is a well defined measure.

To see that G does not have σ-finite variation it is enough to check that $|G|(B) = \infty$ for every B in $\mathcal{B}$ with $\mathbb{P}(B) > 0$. This a consequence of (14), since for every $n \in \mathbb{N}$, we have

$$|G|(B) \ge \frac{|P_n \circ G|(B)}{C} = \frac{1}{C} \int_B \|f_n(t_n)\|\,d\mathbb{P} = \frac{n\mathbb{P}(B)}{C}.$$

The measure F will be the disjoint sum of the measures H_n, which we can define on the Lebesgue σ-algebra of $[1, \infty)$ by

$$F(B) = \sum_{n \ge 1} \int_{B \cap [n, n+1)} f_n(t - n)\,dt \quad \text{for every measurable } B \subseteq [1, \infty).$$

As above, (15) yields the convergence of the series and that F is a measure. This measure has σ-finite variation since $|F|([n,n+1]) = n$. To check $\operatorname{rg} G \subseteq \operatorname{rg} F$, take B in $\mathcal{B}$, and observe that, by (17), for every n, there exists a measurable set $A_n \subset [0,1]$ such that

$$\int_B f_n(t_n)\, d\mathbb{P}(t_1,t_2,\ldots) = \int_{A_n} f_n(t)\, dt.$$

Then the set $A = \bigcup_{n\ge 1} n + A_n$ satisfies $F(A) = G(B)$, and so $G(B) \in \operatorname{rg} F$. ∎

4. Range and derivability. Average range. In this section we prove that the range determines the Bochner derivability of a vector measure. This will be done through the study of the average range. In the next section we give a different proof which involves factorization of linear operators.

Let F be a vector measure defined on $(\Omega,\mathcal{A})$, and let μ be a control measure for F; we will say that F *is derivable with respect to* μ if there exists $\varrho = dF/d\mu \in L^1(\mu, X)$ such that

$$F(A) = \int_A \varrho\, d\mu \quad \text{for every } A \in \mathcal{A}.$$

In that case, F has finite variation and also has a derivative with respect to $|F|$, namely $\varrho/\|\varrho\|$. Conversely, if F has a derivative with respect to its variation, then it has one with respect to any control measure. Thus we only need to deal with derivability with respect to variation.

If F is a measure of finite variation, the *average range of F with respect to its variation*, or simply the *average range* of F, denoted by $\operatorname{Ave}(F)$, is the subset of X defined by

$$\operatorname{Ave}(F) = \{F(A)/|F|(A) : A \in \mathcal{A}\},$$

where we adopt the convention $0/0 = 0$. If $C \in \mathcal{A}$, $\operatorname{Ave}_C(F)$ denotes the average range of the measure restriction of F to C, the measure $\chi_C F$; thus

$$\operatorname{Ave}_C(F) = \operatorname{Ave}(\chi_C F) = \{F(A)/|F|(A) : A \in \mathcal{A},\ A \subseteq C\}.$$

The relation between these average ranges and the derivability of F is given by the following theorem due to Rieffel [Ri], [DU, Th. III.2.6, III.2.7]; we state it in the context of derivability with respect to variation.

THEOREM 4.1. *Let F be a vector measure of finite variation. Then F is derivable with respect to its variation if and only if for every measurable set A with $|F|(A) > 0$, there exists a measurable set $B \subset A$ such that $|F|(B) > 0$ and $\operatorname{Ave}_B(F)$ is relatively compact.*

We will need two lemmas. The second one states that the range of a vector measure determines the absolutely convex closed hull of its average range. Recall our convention $0/0 = 0$.

LEMMA 4.2. *Let $(\Omega,\mathcal{A})$ be a measurable space, $F : \mathcal{A} \to X$ a measure of finite variation, and $h : \Omega \to \mathbb{R}$ a measurable bounded function. We have:*

(a) $\overline{\operatorname{aco}}(\operatorname{Ave}(F)) = \overline{\operatorname{co}}\left\{ \dfrac{\int f\, dF}{\int |f|\, d|F|} \,\middle|\, f : \Omega \to [-1,1] \text{ measurable} \right\}$.

(b) $\overline{\operatorname{aco}}(\operatorname{Ave}(hF)) \subseteq \overline{\operatorname{aco}}(\operatorname{Ave}(F))$.

Proof. The right hand side of (a) is the closed convex hull of a symmetric set, and so it is absolutely convex and closed. As it contains $F(A)/|F|(A)$ for every A in $\mathcal{A}$, it contains $\overline{\operatorname{aco}}(\operatorname{Ave}(F))$.

For the reverse inclusion, by density, it is enough to check

$$\frac{\int f\, dF}{\int |f|\, d|F|} \in \operatorname{aco}(\operatorname{Ave}(F)) \tag{18}$$

for every measurable simple function $f : \Omega \to [-1,1]$. If $f = \sum_{k=1}^n \alpha_k \chi_{A_k}$ with $\alpha_k \in [-1,1]$ for $k = 1,\ldots,n$, and $\{A_1,\ldots,A_n\}$ being pairwise disjoint, then

$$\frac{\int f\, dF}{\int |f|\, d|F|} = \frac{\sum_{k=1}^n \alpha_k F(A_k)}{\sum_{k=1}^n |\alpha_k|\, |F|(A_k)} = \sum_{k=1}^n \beta_k \frac{F(A_k)}{|F|(A_k)},$$

where $\beta_k = \alpha_k |F|(A_k)/\sum_{j=1}^n |\alpha_j|\, |F|(A_j)$. Since $\sum |\beta_k| \le 1$, we get (18), and (a) follows.

Now (b) is easy from (a) and Proposition 1.3. If $K > 0$ is such that $|h(\omega)| \le K$ and $g = h/K$, we have, for every measurable $f : \Omega \to [-1,1]$,

$$\frac{\int f\, dhF}{\int |f|\, d|hF|} = \frac{\int fh\, dF}{\int |fh|\, d|F|} = \frac{\int fg\, dF}{\int |fg|\, d|F|},$$

which is in $\overline{\operatorname{aco}}(\operatorname{Ave}(F))$, as fg has values in $[-1,1]$. ∎

LEMMA 4.3. *Let F and G be two vector measures of finite variation such that*

$$\overline{\operatorname{co}}(\operatorname{rg} F - \operatorname{rg} F) = \overline{\operatorname{co}}(\operatorname{rg} G - \operatorname{rg} G).$$

Then $\overline{\operatorname{aco}}(\operatorname{Ave}(F)) = \overline{\operatorname{aco}}(\operatorname{Ave}(G))$.

Proof. Suppose F is defined on $(\Omega,\mathcal{A})$ and G on $(\Delta,\mathcal{B})$, and take $f : \Delta \to [-1,1]$ measurable. We apply Theorem 2.1 to $\psi = |f|$. This produces a measurable function $\phi : \Omega \to [-1,1]$ such that

$$\overline{\operatorname{co}}(\operatorname{rg} \phi F - \operatorname{rg} \phi F) = \overline{\operatorname{co}}(\operatorname{rg} |f|G - \operatorname{rg} |f|G). \tag{19}$$

This implies, by Theorem 3.1, that the total variations of ϕF and $|f|G$ are the same, and by Proposition 1.2, we have $\int \phi\, d|F| = \int |f|\, d|G|$.

From Proposition 1.1 we see that $\int f\, dG$ is in $\overline{\operatorname{co}}(\operatorname{rg}|f|G - \operatorname{rg}|f|G)$. Again by this proposition and (19), there exists $h : \Omega \to [-1,1]$ measurable such

that $\int f\,dG = \int h\phi\,dF$. Thus
$$\frac{\int f\,dG}{\int |f|\,d|G|} = \frac{\int h\phi\,dF}{\int \phi\,d|F|} = \frac{\int |h\phi|\,d|F|}{\int \phi\,d|F|}\cdot\frac{\int h\phi\,dF}{\int |h\phi|\,d|F|},$$
which is in $\overline{\mathrm{aco}}(\mathrm{Ave}(F))$, by the previous lemma, since $\int |h\phi|\,d|F|/\int \phi\,d|F|$ is in $[-1,1]$. We have proved that $\overline{\mathrm{aco}}(\mathrm{Ave}(G)) \subset \overline{\mathrm{aco}}(\mathrm{Ave}(F))$. The other inclusion is proved in the same way. ■

Now we are ready to prove that the range of a vector measure determines its derivability.

THEOREM 4.4. *Let X be a Banach space, and let F and G be two X-valued measures such that*
$$\overline{\mathrm{co}}(\mathrm{rg}\,F - \mathrm{rg}\,F) = \overline{\mathrm{co}}(\mathrm{rg}\,G - \mathrm{rg}\,G).$$
Then F is derivable with respect to $|F|$ if and only if G is derivable with respect to $|G|$.

Proof. We only have to prove that G has a derivative if F does. In that case F has finite variation, and so does G by Theorem 3.1. Then we can apply Rieffel's characterization (Theorem 4.1) to establish the derivability of G. Let A be in $\mathcal{B}$ with $|G|(A) > 0$. Theorem 2.1 gives us $\phi_A : \Omega \to [0,1]$ measurable such that
$$\overline{\mathrm{co}}(\mathrm{rg}\,\phi_A F - \mathrm{rg}\,\phi_A F) = \overline{\mathrm{co}}(\mathrm{rg}\,\chi_A G - \mathrm{rg}\,\chi_A G).$$
Obviously $\phi_A F$ has a derivative with respect to $|F|$, and so too with respect to $|\phi_A F|$. As, by Theorem 3.1, $|\phi_A F|(\Omega) = |G|(A) > 0$, Theorem 4.1 provides us with a set $C \in \mathcal{A}$, with $|\phi_A F|(C) > 0$, such that $\mathrm{Ave}_C(\phi_A F)$ is relatively compact in X. Then $\overline{\mathrm{aco}}(\mathrm{Ave}(\chi_C\phi_A F))$ is compact.

Applying Theorem 2.1 again, but now to $\phi_A F$ and $\chi_A G$, we obtain $\psi_C : \Delta \to [0,1]$ measurable such that
$$\overline{\mathrm{co}}(\mathrm{rg}\,\chi_C\phi_A F - \mathrm{rg}\,\chi_C\phi_A F) = \overline{\mathrm{co}}(\mathrm{rg}\,\psi_C\chi_A G - \mathrm{rg}\,\psi_C\chi_A G).$$
So, by Lemma 4.3, $\overline{\mathrm{aco}}(\mathrm{Ave}(\psi_C\chi_A G))$ is compact, and, by Theorem 3.1, $|\psi_C\chi_A G|(\Delta) = |\phi_A F|(C) > 0$.

By Proposition 1.3, $\int \psi_C\chi_A\,d|G| > 0$; so for some $r > 0$, the set $B = \{t \in \Delta : \psi_C(t)\chi_A(t) \ge r\}$ satisfies $|G|(B) > 0$. Let h be defined by $h(t) = 1/\psi_C(t)$ for $t \in B$ and zero elsewhere. The function h is measurable, bounded, and $h\psi_C\chi_A = \chi_B$. Thus Lemma 4.2(b) yields
$$\overline{\mathrm{aco}}(\mathrm{Ave}_B(G)) = \overline{\mathrm{aco}}(\mathrm{Ave}(h\psi_C\chi_A G)) \subseteq \overline{\mathrm{aco}}(\mathrm{Ave}(\psi_C\chi_A G)),$$
and $\mathrm{Ave}_B(G)$ is relatively compact. So G satisfies Rieffel's condition and is derivable. ■

Remark. The proof of this theorem also gives some information about the essential ranges of the derivatives. If $\varrho = dF/d|F|$, and $\xi = dG/d|G|$, let K be the (closed) essential range of ξ, that is, the closed subset of the unit sphere defined by
$$K = \{x \in X : |G|(\xi^{-1}(V)) > 0 \text{ for every open set } V \ni x\}.$$
Then the essential range of ϱ is contained in $K \cup -K$:

Let x be in the essential range of ϱ, so $\|x\| = 1$; for ε positive, consider the closed ball $D(x,\varepsilon)$ of centre x and radius ε, and the measurable set $C = \varrho^{-1}(D(x,\varepsilon))$. We know that $|F|(C) > 0$, and it is easy to see that
$$\overline{\mathrm{aco}}(\mathrm{Ave}_C(F)) \subset \overline{\mathrm{co}}(D(x,\varepsilon) \cup D(-x,\varepsilon)).$$
Similar arguments to those in the proof of the theorem produce a measurable set $B \in \mathcal{B}$ such that $|G|(B) > 0$ and
$$\overline{\mathrm{aco}}(\mathrm{Ave}_B(G)) \subset \overline{\mathrm{aco}}(\mathrm{Ave}_C(F)) \subset \overline{\mathrm{co}}(D(x,\varepsilon) \cup D(-x,\varepsilon)).$$
This implies that $\xi(t) \in \overline{\mathrm{co}}(D(x,\varepsilon) \cup D(-x,\varepsilon))$ for almost every $t \in B$. But $\|\xi\| = 1$ almost everywhere, and every $y \in \overline{\mathrm{co}}(D(x,\varepsilon)\cup D(-x,\varepsilon))$ with $\|y\| = 1$ is included in $D(x,2\varepsilon) \cup D(-x,2\varepsilon)$, so $|G|(\xi^{-1}(D(x,2\varepsilon)\cup D(-x,2\varepsilon))) > 0$. As ε was arbitrary, we conclude that x or $-x$ is in K.

5. Range and derivability. Factorization method. In this section we give a second proof of Theorem 4.4. We will not use Theorem 2.1 on decomposition of zonoids; actually, of the previous results in the paper, we will only need the fact that range determines total variation (Theorem 3.1(a)), which was already proved in [Ro].

We will exploit the relation between the properties of a vector measure and the properties of the underlying integration operator. An example of this relation is the following theorem which goes back to the works of Grothendieck [G], Diestel [D], and Tong [T] on operators defined on a $C(K)$ space. Recall that a bounded operator $T : X \to Y$ is said to be *nuclear* if there exist two sequences, (x_n^*) in X^* and (y_n) in Y, such that
$$\sum_{n=1}^{\infty} \|x_n^*\|\,\|y_n\| < \infty \quad\text{and}\quad Tx = \sum_{n=1}^{\infty} x_n^*(x) y_n \quad\text{for all } x \text{ in } X.$$
The statement of the following theorem is a particular case of Theorem VI.4.4 in [DU].

THEOREM 5.1. *Let X be a Banach space, F an X-valued mesure, and μ a control measure for F. The integration operator $I_F : \mathrm{L}^\infty(\mu) \to X$ defined by $I_F f = \int f\,dF$ for $f \in \mathrm{L}^\infty(\mu)$ is nuclear if and only if F has bounded variation and F has a Bochner derivative with respect to $|F|$.*

If two measures F, G generate the same zonoid in X, and F is derivable, then the integration operator I_F factorizes, as every nuclear operator, through a diagonal operator from ℓ_∞ to ℓ_1. But a priori this factorization has nothing to do with G. We need a kind of factorization more related to the range which, being equivalent to the fact of having a derivative, allows us to conclude that G also has a derivative. This is done in Theorem 5.2. We need to introduce some notations.

If C is a bounded absolutely convex closed set of a Banach space X, we will denote by X_C the linear subspace of X generated by C, that is,

$$X_C = \{\lambda x : x \in C,\ \lambda > 0\}.$$

Provided with the Minkowski functional $\|\cdot\|_C$ of C, X_C becomes a Banach space whose closed unit ball is C. In general the norm topology in X_C is finer than the one induced by the norm topology of X, and the same is true for the weak topologies (X_C is not necessarily closed in X). If K is a subset of X_C, we will denote by $\overline{K}^X$ its closure in X, and by $\overline{K}^{X_C}$ its closure in X_C.

If F is an X-valued measure whose range is included in X_C, we can consider F as having values in X_C; but, in general, F is only finitely additive in X_C. F will be a measure in X_C if (and only if) $\operatorname{rg} F$ is relatively weakly compact in X_C: in that case the weak topology of X and the weak topology of X_C coincide on $\operatorname{rg} F$, and then F is countably additive for the weak topology of X_C and, by the Orlicz–Pettis theorem, for the norm topology of X_C. In the following theorem we characterize the measures with a Bochner derivative as those having a good factorization through the injection $X_C \to X$ for a suitable C.

THEOREM 5.2. *Let X be a Banach space, and F an X-valued measure. The following properties are equivalent*:

(a) *F has finite variation and a Bochner derivative with respect to $|F|$.*

(b) *There exists an absolutely convex compact subset C of X such that $\overline{\mathrm{co}}^X(\operatorname{rg} F - \operatorname{rg} F)$ is compact in X_C, and F has finite variation in X_C.*

(c) *There exists an absolutely convex weakly compact subset C of X such that $\operatorname{rg} F$ is relatively weakly compact in X_C, and F has finite variation in X_C.*

Proof. It is obvious that (b) implies (c). Let us show the implication (c)⇒(a). Suppose F satisfies (c). As $\operatorname{rg} F$ is relatively weakly compact in X_C, F is a measure (countably additive) when considered as having values in X_C. We will denote this measure by F_C. If μ is a control measure for F, it is still a control measure for F_C since $\mu(A) = 0$ still implies $F_C(A) = 0$, for every measurable set A.

Since C is weakly compact in X, the injection $X_C \to X$ is a weakly compact operator. Thanks to the Davis–Figiel–Johnson–Pełczyński factorization lemma [DFJP], [DU, p. 250], this injection factorizes through a reflexive Banach space; in particular, there exist a reflexive Banach space R and two bounded operators $S : X_C \to R$ and $T : R \to X$ such that $F = T\circ S\circ F_C$.

Let H be the measure defined in R by $H = S\circ F_C$. As F_C has finite variation, so does H; but H has values in a space with the Radon–Nikodym property, therefore H has a derivative with respect to μ, which is a control measure for H. This implies that $F = T\circ H$ also has a derivative with respect to μ, thus F has finite variation, and a Bochner derivative with respect to $|F|$, and (a) follows.

(a)⇒(b). Let μ be a control measure for F. If F satisfies (a), then by Theorem 5.1, the integration operator $I_F : \mathrm{L}^\infty(\mu) \to X$ is nuclear, so it factors through a diagonal operator from ℓ_∞ to ℓ_1. Thus we have a factorization

$$\begin{tikzcd} \mathrm{L}^\infty(\mu) \arrow[r, "I_F"] \arrow[d, "S"'] & X \\ \ell_\infty \arrow[r, "D"] & \ell_1 \arrow[u, "T"'] \end{tikzcd} \tag{20}$$

where the diagram is commutative, S and T are bounded operators, and D is a diagonal operator, that is, there exists a sequence (d_n) of real numbers such that

$$\sum_{n=1}^{\infty} |d_n| < \infty \quad \text{and} \quad D((\alpha_n)_{n\ge1}) = (d_n\alpha_n)_{n\ge1} \quad \text{for } (\alpha_n)_{n\ge1} \in \ell_\infty.$$

We can assume that T in (20) is compact. For, take a sequence (λ_n) of positive real numbers tending to infinity such that we still have

$$\sum_{n=1}^{\infty} \lambda_n |d_n| < \infty.$$

If D_1 is the diagonal operator associated with $(\lambda_n d_n)$, and $T_1 : \ell_1 \to X$ is the operator defined in the canonical basis (e_n) of ℓ_1 by $T_1 e_n = Te_n/\lambda_n$ for every $n \in \mathbb{N}$, then $T_1\circ D_1 = T\circ D$, and T_1 is compact.

Since T is a compact operator, $C = \overline{T(B_{\ell_1})}$ is absolutely convex and compact in X. Let us show that C satisfies (b). The range of T is contained in X_C; write T_C for the operator T considered as having values in X_C. Then $T_C : \ell_1 \to X_C$ is bounded since $T_C(B_{\ell_1})$ is included in C, the unit ball of X_C. Note that as D is a nuclear operator, it is compact; then $K = \overline{D\circ S(B_{\mathrm{L}^\infty})}$ is compact in ℓ_1. By Proposition 1.1(c) we have

$$\begin{aligned} \overline{\mathrm{co}}^X(\operatorname{rg} F - \operatorname{rg} F) &= I_F(B_{\mathrm{L}^\infty(\mu)}) \\ &= T\circ D\circ S(B_{\mathrm{L}^\infty}) = T_C\circ D\circ S(B_{\mathrm{L}^\infty}) \subset T_C(K). \end{aligned}$$

Thus $\overline{\mathrm{co}}^X(\operatorname{rg} F - \operatorname{rg} F)$ is compact in X_C since it is closed in X, hence in X_C, and it is contained in $T_C(K)$, a compact subset of X_C.

Write F_C for F considered as having values in X_C. Then μ is also a control measure for F_C. It is easy to see that the integration operator $I_{F_C} : \mathrm{L}^\infty(\mu) \to X_C$ factorizes as $I_{F_C} = T_C\circ D\circ S$, thus I_{F_C} is nuclear since so is D.

By Theorem 5.1, F_C has bounded variation (actually it has a derivative with respect to $|F_C|$). The theorem follows. ∎

Using the equivalence (a)⇔(b) in the last theorem we can give a new proof of the fact that range determines derivability.

Second proof of Theorem 4.4. It is enough to prove that G has a derivative if F does and $\overline{\mathrm{co}}(\mathrm{rg}\, F - \mathrm{rg}\, F) = \overline{\mathrm{co}}(\mathrm{rg}\, G - \mathrm{rg}\, G)$. Then F satisfies (b) of Theorem 5.2: there exists an absolutely convex compact set C in X such that $\overline{\mathrm{co}}^X(\mathrm{rg}\, F - \mathrm{rg}\, F)$ is compact in X_C, and F has finite variation in X_C. Now, G also satisfies these conditions for the same C. The first one is obvious since we know that

$$\overline{\mathrm{co}}^X(\mathrm{rg}\, F - \mathrm{rg}\, F) = \overline{\mathrm{co}}^X(\mathrm{rg}\, G - \mathrm{rg}\, G).$$

To check that G has finite variation in X_C first note that, as $\overline{\mathrm{co}}^X(\mathrm{rg}\, F - \mathrm{rg}\, F)$ is compact in X_C, on this set the norm topology of X and the norm topology of X_C coincide. So the convex hull of $\mathrm{rg}\, F - \mathrm{rg}\, F$ is also dense in $\overline{\mathrm{co}}^X(\mathrm{rg}\, F - \mathrm{rg}\, F)$ for the topology of X_C, and the same is true for the convex hull of $\mathrm{rg}\, G - \mathrm{rg}\, G$. Therefore,

$$\begin{aligned}\overline{\mathrm{co}}^{X_C}(\mathrm{rg}\, F - \mathrm{rg}\, F) &= \overline{\mathrm{co}}^X(\mathrm{rg}\, F - \mathrm{rg}\, F)\\ &= \overline{\mathrm{co}}^X(\mathrm{rg}\, G - \mathrm{rg}\, G) = \overline{\mathrm{co}}^{X_C}(\mathrm{rg}\, G - \mathrm{rg}\, G),\end{aligned}$$

and, by Theorem 3.1(a), the variation of G in X_C is finite since F and G have the same total variation in X_C. Now we see that G has a derivative using again Theorem 5.2. ∎

We have shown that the total variation, the σ-finiteness of the variation, and the Bochner derivability of a vector measure are determined by the zonoid it generates. It is a natural question to look for more properties that might also be determined by the range. It would also be interesting to find geometrical properties characterizing those zonoids for which any measure generating them has bounded variation, σ-finite variation, or a derivative.

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DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
FACULTAD DE MATEMÁTICAS
UNIVERSIDAD DE SEVILLA
P.O. BOX 1160
41080 SEVILLA, SPAIN
E-mail: PIAZZA@CICA.ES

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