

Semisimplicity, joinings and group extensions

by

A. DEL JUNCO (Toronto, Ont.), M. LEMAŃCZYK (Toruń)
and M. K. MENTZEN (Toruń)

Abstract. We present a theory of self-joinings for semisimple maps and their group extensions which is a unification of the following three cases studied so far:

- (i) Gaussian-Kronecker automorphisms: [Th], [Ju-Th].
- (ii) MSJ and simple automorphisms: [Ru], [Ve], [Ju-Ru].
- (iii) Group extension of discrete spectrum automorphisms: [Le-Me], [Le], [Me].

0. Introduction. In [Ve], Veech proved a theorem describing factors of ergodic 2-fold simple automorphisms in terms of subgroups of the centralizer. The property of “2-fold simplicity” is defined by 2-joinings—invariant measures on the Cartesian square of the given system, projecting onto the system as the original measures. In particular, each system is a factor of any of its joinings.

In the 2-fold simple case, each ergodic 2-self-joining is a graph measure or the product measure and this property is sufficient to describe all factors. But a graph measure, as a dynamical system, is isomorphic to the original system and the natural projection factor map is one-to-one a.s. with respect to the joining measure. In other words, a graph measure λ is a one-point extension of the base system X . In particular, the relative product $\lambda \times_X \lambda$ is ergodic.

We will use this observation to define a new class of ergodic automorphisms, called semisimple automorphisms. Formally, an automorphism $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is semisimple if for each ergodic 2-self-joining λ , the relative product $\lambda \times_X \lambda$ is ergodic. It turns out that many classes of automorphisms previously studied are semisimple. Indeed, all discrete spectrum, 2-fold simple, direct products of minimal self-joinings, and Gaussian-Kronecker automorphisms are semisimple. We exhibit the structure of fac-

1991 *Mathematics Subject Classification*: Primary 28D15.

Research of the first author supported by an NSERC grant.

Research of the second and third authors supported by KBN grant 512/2/91.

tors of semisimple automorphisms; in particular, we prove that one can decompose a given factor map $X \rightarrow Y$ of a semisimple X into $X \rightarrow \tilde{Y} \rightarrow Y$, where the extension $X \rightarrow \tilde{Y}$ is relatively weakly mixing and $\tilde{Y} \rightarrow Y$ is a group extension.

In order to study the structure of factors of a given automorphism, we introduce the notion of a natural family of factors. A general factorization theorem for an automorphism X with a natural family of factors says that if Y is a factor of X then there exists a decomposition $X \rightarrow \tilde{Y} \rightarrow Y$ for some natural factor \tilde{Y} with the remaining properties as above.

We also explore ergodic group extensions of semisimple automorphisms. In Section 6, we describe ergodic joinings of such extensions. In Section 8, we apply the concept of a natural family of factors to give a description of factors of group extensions of 2-fold simple automorphisms, generalizing earlier results from [Le-Me] and [Me].

Finally, we consider the conjecture that if, for an automorphism with a natural family of factors, all natural factors are coalescent then so are all factors. We give an affirmative answer for group extensions of rotations.

The authors would like to thank B. Host and D. Rudolph for some discussions on the subject.

1. Group extensions, isometric extensions and some facts about joinings. Let $T : (X, \mathcal{B}_0) \rightarrow (X, \mathcal{B}_0)$ be an automorphism of a standard compact Borel space (X, \mathcal{B}_0) , i.e. T is a bijective map such that $T^{-1}\mathcal{B}_0 = \mathcal{B}_0$. Let μ be a probability T -invariant measure on (X, \mathcal{B}_0) . Denote by \mathcal{B} the (T -invariant) σ -algebra of all μ -measurable subsets of X . Then (X, \mathcal{B}, μ) is a probability Lebesgue space with T being an automorphism of it. In what follows all σ -algebras under consideration will be complete with respect to the corresponding measure.

Let G be a metric compact group equipped with the probability Haar measure ν . For a Borel map $\varphi : X \rightarrow G$ we define a (Borel) automorphism $T_\varphi : X \times G \rightarrow X \times G$ by

$$T_\varphi(x, g) = (Tx, \varphi(x)g).$$

Then T_φ preserves the measure $\mu \times \nu$. We call T_φ a *group extension* of T , or, indicating the group, a *G-extension* of T . For each $g \in G$, let $\sigma_g(x, h) = (x, hg)$. For this right action of G on $X \times G$ we have $T_\varphi \sigma_g = \sigma_g T_\varphi$.

If $H \subset G$ is a closed subgroup then we define $T_{\varphi, H} : X \times G/H \rightarrow X \times G/H$ by

$$T_{\varphi, H}(x, gH) = (Tx, \varphi(x)gH).$$

If no confusion can arise then we will denote the measure ν restricted to the sets of the form $BH = \bigcup_{b \in B} bH$ with $B \subset G$, i.e. to the sets invariant

with respect to the right action of H on $X \times G$, again by ν . Let $\tilde{\mu} = \mu \times \nu$. If \mathcal{D} denotes the σ -algebra of ν -measurable subsets of G then the product σ -algebra $\mathcal{B} \otimes \mathcal{D}$ will be denoted by $\tilde{\mathcal{B}}$. If $p : G \rightarrow G/H$ is the natural projection then we set $\tilde{\mathcal{B}}_H = \mathcal{B} \otimes p(\mathcal{D})$. The factor $T_{\varphi, H}$ of T_φ will be called a *natural factor* of T_φ , and $T_{\varphi, H}$ an *isometric extension* of T . If H is normal in G , then we call $T_{\varphi, H}$ a *normal natural factor* of T_φ . For an integrable function $f : X \times G \rightarrow \mathbb{C}$, denote the conditional expectation $E(f | \tilde{\mathcal{B}}_H) : X \times G/H \rightarrow \mathbb{C}$ by $E(f | H)$. If $f : X \times G \rightarrow \mathbb{C}$ and $g \in G$ then define $f \circ g : X \times G \rightarrow \mathbb{C}$ by $(f \circ g)(x, h) = f(x, hg)$ and $g \circ f : X \times G \rightarrow \mathbb{C}$ by $(g \circ f)(x, h) = f(x, gh)$.

If λ is a measure on $X \times G$ and $g \in G$ then we denote by λg (resp. $g\lambda$) the measure on $X \times G$ given on rectangles by

$$\lambda g(A \times B) = \lambda(A \times Bg^{-1}) \quad (\text{resp. } g\lambda(A \times B) = \lambda(A \times g^{-1}B))$$

for $A \subset X$ and $B \subset G$.

If $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ and $S : (Y, \mathcal{C}, m) \rightarrow (Y, \mathcal{C}, m)$ are ergodic automorphisms then by a *joining* of T and S we mean any $T \times S$ -invariant measure λ on $X \times Y$ such that for $A \in \mathcal{B}$ and $B \in \mathcal{C}$,

$$\lambda(A \times Y) = \mu(A), \quad \lambda(X \times B) = m(B).$$

The set of all joinings of T and S will be denoted by $J(T, S)$ or $J(X, Y)$, while the subset of $J(T, S)$ consisting of all $T \times S$ -ergodic joinings by $J^e(T, S)$ or $J^e(X, Y)$. It is well known that if $\lambda \in J(T, S)$ and if $\lambda = \int_{E(T, S)} \gamma d\tau(\gamma)$ is its ergodic decomposition ($E(T, S)$ stands for all $T \times S$ -ergodic measures on $X \times Y$), then $\tau(J^e(T, S)) = 1$. Obviously the product measure $\mu \times \nu$ is a joining of T and S . Therefore $J^e(T, S) \neq \emptyset$.

If $f : X \rightarrow Y$ is a measurable map then we define the *graph measure* μ_f on $X \times Y$ by

$$\mu_f(A \times B) = \mu(A \cap f^{-1}(B)).$$

It is easy to observe that the μ_f -measure of the graph of f is 1. Moreover, if $\lambda \in J^e(T, S)$ then

$$(1) \quad \lambda = \mu_f \text{ iff } \forall B \subset Y \exists A \subset X \quad \lambda(A \times B^c \cup A^c \times B) = 0.$$

If $Y = X$ and $f = \text{id}$, the identity function, then the graph measure μ_{id} will be called the *diagonal measure*.

Let $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be an ergodic automorphism. By $C(T)$ we denote the *centralizer* of T , i.e.

$$C(T) = \{S : X \rightarrow X : S \text{ preserves } \mu \text{ and } ST = TS\}.$$

We will say that T is *coalescent* ([Ne]) if $C(T)$ is a group. It is easy to prove that if $f : X \rightarrow X$ is a measurable map then

$$\mu_f \in J^e(T, T) \text{ iff } f \in C(T).$$

If $S : (Y, \mathcal{C}, \nu) \rightarrow (Y, \mathcal{C}, \nu)$ is a common factor of $T_i : (X_i, \mathcal{B}_i, \mu) \rightarrow (X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, and $\lambda \in J(Y, Y)$, then the relatively independent extension $\hat{\lambda} \in J(X_1, X_2)$ of λ is the measure

$$\hat{\lambda}(A_1 \times A_2) = \int_{Y \times Y} E(A_1 | Y)(y_1)E(A_2 | Y)(y_2) d\lambda(y_1, y_2).$$

The relative product $T_1 \times_S T_2$ of T_1 and T_2 with respect to S is the relatively independent extension of the diagonal measure on Y . If $S : (Y, \mathcal{C}, \nu) \rightarrow (Y, \mathcal{C}, \nu)$ is a factor of $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$, and if no confusion can arise, we will use the following abbreviations: $T \rightarrow S$ or $\mathcal{B} \rightarrow \mathcal{C}$ or even $X \rightarrow Y$.

The fact that an extension $X \rightarrow Y$ is a group extension can be expressed in terms of joinings.

THEOREM ([Ve]). *Suppose that $X \rightarrow Y$ is ergodic. Then $X \rightarrow Y$ is a group extension iff each ergodic $\lambda \in J^e(X, X)$ projecting onto the diagonal measure on $Y \times Y$ is a graph joining. ■*

Remark. It follows from this theorem and the relative version of the main result of [Ve] that if $T_\varphi : (X \times G, \tilde{\mathcal{B}}, \tilde{\mu}) \rightarrow (X \times G, \tilde{\mathcal{B}}, \tilde{\mu})$ is an ergodic group extension of $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$, then each factor of T_φ containing \mathcal{B} is determined by a compact subgroup of G , hence it is an isometric extension of \mathcal{B} .

Suppose now that $\mathcal{B}_1 \subset \mathcal{B}$ is a T -invariant sub- σ -algebra (factor), hence giving rise to a factor $\bar{T} : (\bar{X}, \mathcal{B}_1, \bar{\mu}) \rightarrow (\bar{X}, \mathcal{B}_1, \bar{\mu})$ of T . Note that if we take the family of all factors of T , say $\mathcal{B}_\kappa, \kappa \in \Lambda$, containing \mathcal{B}_1 with the property that each $\lambda \in J^e(\mathcal{B}_\kappa, \mathcal{B}_\kappa)$ that projects onto the diagonal measure on $\mathcal{B}_1 \otimes \mathcal{B}_1$ is a graph joining, then the smallest factor of T containing all $\mathcal{B}_\kappa, \kappa \in \Lambda$, enjoys the same property. Hence there exists a maximal factor $\tilde{\mathcal{B}} \subset \mathcal{B}$ such that $\tilde{\mathcal{B}} \rightarrow \mathcal{B}_1$ is a group extension. Note also that if $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B}$ are factors then the smallest factor of \mathcal{B} containing \mathcal{B}_1 and \mathcal{B}_2 can be naturally identified with an ergodic joining of \mathcal{B}_1 and \mathcal{B}_2 .

Set $H = L^2(X, \mu)$. If \mathcal{B}_1 is a factor of \mathcal{B} then we say that the extension $\mathcal{B} \rightarrow \mathcal{B}_1$ is a compact extension if the set of AP functions is dense in H . To be precise, let

$$\mu = \int_{\bar{X}} \mu_{\bar{x}} d\bar{\mu}(\bar{x})$$

be the disintegration of μ over $\bar{\mu}$. We have $T = \bar{T}_{\Theta}$, where

$$\bar{T}_{\Theta}(\bar{x}, z) = (\bar{T}\bar{x}, \Theta_{\bar{x}}(z))$$

with $X = \bar{X} \times Z$ and $\mu = \bar{\mu} \times \nu$ (see [Fu]). Then $\mu_{\bar{x}}$ can be viewed as a measure on \mathcal{B} just concentrated on the fibers of the natural map $\pi : X \rightarrow \bar{X}$

(i.e. $\mu_{\bar{x}} = \delta_{\bar{x}} \times \nu$). We say that a function $f \in H$ is AP (almost periodic) if for each $\varepsilon > 0$ there are $g_1, \dots, g_k \in H$ such that for each $p \in \mathbb{Z}$,

$$(2) \quad \min_{1 \leq j \leq k} \|fT^p - g_j\|_{L^2(\mu_{\bar{x}})} < \varepsilon.$$

for a.a. $\bar{x} \in \bar{X}$.

THEOREM ([Zi]). *$X \rightarrow \bar{X}$ is compact if and only if there is a compact group G and a closed subgroup H of G such that $Z = G/H$ and $\Theta_{\bar{x}} = \varphi(\bar{x})H$ for a cocycle $\varphi : \bar{X} \rightarrow G$, i.e. the extension $X \rightarrow \bar{X}$ is an isometric extension. ■*

PROPOSITION 1.1. *Suppose that $(X, \mathcal{B}, T) \rightarrow (\bar{X}, \mathcal{B}_1, \bar{T})$ is an ergodic isometric extension. Then there exists an ergodic extension (Y, \mathcal{C}, S) of X such that $Y \rightarrow \bar{X}$ is a group extension and moreover for each ergodic extension (Y', \mathcal{B}', S') of X with $Y' \rightarrow \bar{X}$ a group extension we have*

$$\begin{array}{ccccc} Y' & \longrightarrow & X & \longrightarrow & \bar{X} \\ & & \searrow & \uparrow & \\ & & & Y & \end{array}$$

Proof. Let $\tilde{S} : (\tilde{Y}, \tilde{\mathcal{C}}, \tilde{\nu}) \rightarrow (\tilde{Y}, \tilde{\mathcal{C}}, \tilde{\nu})$ be any ergodic extension of X which is a group extension of \bar{X} . Then take the family of all factors $\tilde{\mathcal{C}}_\kappa \subset \tilde{\mathcal{C}}, \kappa \in \Lambda$, which are group extensions of \mathcal{B}_1 and set

$$\mathcal{C} = \bigcap_{\kappa \in \Lambda} \tilde{\mathcal{C}}_\kappa.$$

Note that if $\lambda \in J^e(\mathcal{C}, \mathcal{C})$ projects onto the diagonal measure on $\mathcal{B}_1 \otimes \mathcal{B}_1$ then any ergodic extension $\hat{\lambda}$ of λ on $\tilde{\mathcal{C}} \otimes \tilde{\mathcal{C}}$ is a graph measure. Hence if $A \in \mathcal{C}$ then there is a set $\tilde{B} \in \tilde{\mathcal{C}}$ such that

$$\hat{\lambda}(A \times \tilde{Y} \Delta \tilde{Y} \times \tilde{B}) = 0.$$

Thus, it is clear that $\tilde{B} \in \tilde{\mathcal{C}}_\kappa$ for each $\kappa \in \Lambda$, and consequently $\tilde{B} \in \mathcal{C}$. By Veech's Theorem, $\mathcal{C} \rightarrow \mathcal{B}_1$ is a group extension.

Take any ergodic joining of Y' and Y which is diagonal on \bar{X} ; we get a system Z . Now, in Z, Y and Y' are represented by some invariant σ -algebras, say \mathcal{A} and \mathcal{A}' . Let $\mathcal{C}_1 = \mathcal{A} \cap \mathcal{A}' \subset \mathcal{C} \otimes \mathcal{C}$. Take any ergodic self-joining λ on $\mathcal{C}_1 \otimes \mathcal{C}_1$ which is diagonal on $\bar{X} \times \bar{X}$. Then λ has an ergodic extension $\tilde{\lambda}$ to $Z \times Z$. Take any set $C \in \mathcal{C}$. Because \mathcal{A} and \mathcal{A}' are group extensions of \bar{X} , there are $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$ such that

$$\tilde{\lambda}(C \times Z \Delta Z \times A) = 0, \quad \tilde{\lambda}(C \times Z \Delta Z \times A') = 0.$$

Therefore $A = A' \in \mathcal{A} \cap \mathcal{A}' = \mathcal{C}$. Hence λ is a graph joining and consequently \mathcal{C}_1 is a group extension of \bar{X} . ■

The extension Y of X (defined up to isomorphism) will be called the *minimal group cover* of X .

An extension $X \rightarrow \bar{X}$ is called *distal* if for some ordinal η we have a family of factors \mathcal{B}_κ , $\kappa \leq \eta$, such that $\mathcal{B}_{\kappa+1} \rightarrow \mathcal{B}_\kappa$ is compact and if κ is a limit ordinal then $\mathcal{B}_\kappa = \bigcup_{\kappa' < \kappa} \mathcal{B}_{\kappa'}$. Furstenberg [Fu] proved for each factor $\mathcal{B}_1 \subset \mathcal{B}$ the existence of a maximal $\hat{\mathcal{B}} \subset \mathcal{B}$ such that $\hat{\mathcal{B}} \rightarrow \mathcal{B}_1$ is distal. Actually, this follows from

LEMMA 1.1. *If $\mathcal{B}_1 \supset \mathcal{B}$ and $\mathcal{B}_2 \supset \mathcal{B}$ are ergodic distal extensions and $\lambda \in J^e(\mathcal{B}_1, \mathcal{B}_2)$ satisfies $\lambda|_{\mathcal{B} \otimes \mathcal{B}} = \Delta$, then $(\mathcal{B}_1 \otimes \mathcal{B}_2, \lambda)$ is a distal extension of \mathcal{B} .*

Proof. Let $\lambda \in J^e(\mathcal{B}_1, \mathcal{B}_2)$ and $\lambda|_{\mathcal{B} \otimes \mathcal{B}} = \Delta$. We will see in Section 5 (Fact 5.3) that if \mathcal{B}_1 and \mathcal{B}_2 are group extensions of \mathcal{B} then λ is a group extension of μ because $(\mathcal{B} \otimes \mathcal{B}, \Delta)$ is isomorphic to (\mathcal{B}, μ) (in fact, this is well known). Consequently, if \mathcal{B}_1 and \mathcal{B}_2 are isometric extensions of \mathcal{B} , then by the Remark after Veech's Theorem, λ is also an isometric extension of \mathcal{B} .

Now we use transfinite induction. Assume that $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$ are ergodic extensions of \mathcal{B} such that each ergodic joining of $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$ which projects onto $\mathcal{B} \otimes \mathcal{B}$ as the diagonal measure is a distal extension of \mathcal{B} . Let $\mathcal{B}_1 \subset \tilde{\mathcal{B}}_1$ and $\mathcal{B}_2 \subset \tilde{\mathcal{B}}_2$ be ergodic isometric extensions. Extend λ to an ergodic joining $\hat{\lambda}$ of some ergodic group covers of \mathcal{B}_1 and \mathcal{B}_2 . Then $\hat{\lambda}$ is a group extension of \mathcal{B} . Again by the Remark after Veech's Theorem, λ is an isometric extension of \mathcal{B} .

If \mathcal{B}_1 and \mathcal{B}_2 are inverse limits of consecutive isometric extensions, then by the considerations above λ is a distal extension of \mathcal{B} as an inverse limit of isometric extensions of \mathcal{B} . ■

Let $\lambda \in J(T, S)$, where $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ and $S : (Y, \mathcal{C}, \nu) \rightarrow (Y, \mathcal{C}, \nu)$. Then there are largest σ -algebras $\mathcal{B}_1(\lambda) \subset \mathcal{B}$ and $\mathcal{B}_2(\lambda) \subset \mathcal{C}$ such that λ identifies $\mathcal{B}_1(\lambda) \times Y$ with $X \times \mathcal{B}_2(\lambda)$. Indeed, if we take the family of all pairs $(\mathcal{B}_1, \mathcal{B}_2)$, $\mathcal{B}_1 \subset \mathcal{B}$, $\mathcal{B}_2 \subset \mathcal{C}$, where λ identifies $\mathcal{B}_1 \times Y$ with $X \times \mathcal{B}_2$, then the smallest factor containing all of \mathcal{B}_1 , say $\tilde{\mathcal{B}}_1$, and the smallest one containing all of \mathcal{B}_2 , say $\tilde{\mathcal{B}}_2$, has the property that $\tilde{\mathcal{B}}_1 \times Y \stackrel{\lambda}{=} X \times \tilde{\mathcal{B}}_2$. In fact, consider $\mathcal{B} \times Y$ and $X \times \mathcal{C}$ as two sub- σ -algebras of $\mathcal{B} \times \mathcal{C}$, where equality between sets is understood mod λ . Then

$$\mathcal{B} \times Y \cap X \times \mathcal{C}$$

is, on the one hand, a sub- σ -algebra of $\mathcal{B} \times Y$, and so of the form $\mathcal{B}' \times Y$, and on the other hand, a sub- σ -algebra of $X \times \mathcal{C}$, so of the form $X \times \mathcal{C}'$. We have

$$\mathcal{B}_1(\lambda) = \mathcal{B}', \quad \mathcal{B}_2(\lambda) = \mathcal{C}'.$$

2. Some facts about weak mixing and distal extensions. Furstenberg decomposition. Let $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be an ergodic automorphism and $\mathcal{A} \subset \mathcal{B}$ be a T -invariant σ -algebra. We will often write $\mathcal{B} \rightarrow \mathcal{A}$ to say that \mathcal{A} is a factor of \mathcal{B} . We call T *relatively weakly mixing* (rel.w.m.) with respect to \mathcal{A} if the relatively independent extension of the diagonal measure on \mathcal{A} , say $\lambda = \mu \times_{\mathcal{A}} \mu$, is ergodic. For short this will be denoted by $\mathcal{B} \rightarrow \mathcal{A}$ rel.w.m. Note that if T_2 is weakly mixing and T_1 ergodic then $T_1 \times T_2 \rightarrow T_1$ rel.w.m.

Suppose that $\mathcal{B} \rightarrow \mathcal{A}_2$ rel.w.m. and $\mathcal{B} \supset \mathcal{A}_1 \supset \mathcal{A}_2$. Then we can consider the relatively independent extension of the diagonal measure on \mathcal{A}_2 in $\mathcal{B} \otimes \mathcal{B}$ as well as in $\mathcal{A}_1 \otimes \mathcal{A}_1$. The latter is a factor of the former, so obviously $\mathcal{A}_1 \rightarrow \mathcal{A}_2$ rel.w.m.

Let $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be ergodic and \mathcal{A} be a factor of it. Assume that $\mathcal{A} \subset \mathcal{A}_1 \subset \mathcal{B}$ is another factor. The decomposition

$$\mathcal{B} \rightarrow \mathcal{A}_1 \rightarrow \mathcal{A}$$

is called a *Furstenberg decomposition* of $\mathcal{B} \rightarrow \mathcal{A}$ if $\mathcal{B} \rightarrow \mathcal{A}_1$ rel.w.m. and $\mathcal{A}_1 \rightarrow \mathcal{A}$ is distal. By the method presented in [Fu] we know that for each $\mathcal{A} \subset \mathcal{B}$ there is a Furstenberg decomposition of $\mathcal{B} \rightarrow \mathcal{A}$.

PROPOSITION 2.1. *There exists only one Furstenberg decomposition of $\mathcal{B} \rightarrow \mathcal{A}$.*

Proof. Let \mathcal{C} be the maximal distal extension of \mathcal{A} such that $\mathcal{B} \rightarrow \mathcal{C} \rightarrow \mathcal{A}$. Take any Furstenberg decomposition $\tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{A}} \rightarrow \mathcal{A}$ of $\mathcal{B} \rightarrow \mathcal{A}$. Then by Lemma 1.1 each ergodic joining of \mathcal{C} and $\tilde{\mathcal{A}}$ which projects onto $\mathcal{A} \otimes \mathcal{A}$ as the diagonal measure is a distal extension of \mathcal{A} . Therefore $\tilde{\mathcal{A}} \subset \mathcal{C}$. Conversely, since $\mathcal{B} \rightarrow \tilde{\mathcal{A}}$ is rel.w.m., so is $\mathcal{C} \rightarrow \tilde{\mathcal{A}}$. Hence $\mathcal{C} = \tilde{\mathcal{A}}$. ■

PROPOSITION 2.2. *Let $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be ergodic and let $T' : (X', \mathcal{B}', \mu') \rightarrow (X', \mathcal{B}', \mu')$ and $T_1 : (X_1, \mathcal{B}_1, \mu_1) \rightarrow (X_1, \mathcal{B}_1, \mu_1)$ be two ergodic extensions of T . Suppose that $\lambda \in J^e(T', T_1)$ with $\lambda|_{X \times X} = \Delta_X$. Assume, moreover, that $(X_1, \mu_1) \rightarrow (X, \mu)$ is distal and $(X' \times X_1, \lambda) \rightarrow (X', \mu')$ rel.w.m. Then in $(X' \times X_1, \lambda)$ we have $\mathcal{B}' \times X_1 \rightarrow X' \times \mathcal{B}_1$.*

Proof. Let $(X' \times X_1, \lambda) \rightarrow (\hat{X}_1, \hat{\mu}_1) \rightarrow (X_1, \mu_1)$ and $(X', \mu') \rightarrow (\hat{X}, \hat{\mu}) \rightarrow (X, \mu)$ be Furstenberg decompositions. It is then clear that the extension $(\hat{X}_1, \hat{\mu}_1) \rightarrow (X, \mu)$ is distal. By Lemma 1.1, the maximality of \hat{X}_1 and the fact that the extension $\tilde{X} \rightarrow X$, where \tilde{X} is the smallest factor of $(X' \times \hat{X}_1, \lambda)$ containing \hat{X}_1 and \hat{X} , is distal, we must have $\hat{X} \subset \hat{X}_1$. Therefore, (X', μ') and $(\hat{X}_1, \hat{\mu}_1)$ are relatively disjoint over \hat{X} . Thus, no harm arises if we assume that λ is the relative product of X' and \hat{X}_1 over \hat{X} . To be more precise,

let

$$\begin{aligned} X' &= \widehat{X} \times Z', & T'(\widehat{x}, z') &= (\widehat{T}\widehat{x}, \Theta'_{\widehat{x}}(z')), \\ \widehat{X}_1 &= \widehat{X} \times Z_1, & T_1(\widehat{x}, z_1) &= (\widehat{T}\widehat{x}, \Theta^1_{\widehat{x}}(z_1)). \end{aligned}$$

Hence, the relative product $X' \times_{\widehat{X}} \widehat{X}_1$, say $\widetilde{T} : \widehat{X} \times Z' \times Z_1 \rightarrow \widehat{X} \times Z' \times Z_1$, is defined by $\widetilde{T}(\widehat{x}, z', z_1) = (\widehat{T}\widehat{x}, \Theta'_{\widehat{x}}(z'), \Theta^1_{\widehat{x}}(z_1))$. By our assumption, the relative product $\widetilde{T} = \widetilde{T} \times_{X'} \widetilde{T} : \widehat{X} \times Z' \times Z_1 \times Z_1 \rightarrow \widehat{X} \times Z' \times Z_1 \times Z_1$ is ergodic. It is clear that

$$\widetilde{T}(\widehat{x}, z', z_1, z_2) = (\widehat{T}\widehat{x}, \Theta'_{\widehat{x}}(\Theta^1_{\widehat{x}, z'}(z_1)), \Theta^1_{\widehat{x}, z'}(z_2)),$$

where $\Theta^1_{\widehat{x}, z'}(z_i) = \Theta^1_{\widehat{x}}(z_i)$, $i = 1, 2$. Therefore the relative product $\widehat{X}_1 \times_{\widehat{X}} \widehat{X}_1$, which is defined on $\widehat{X} \times Z_1 \times Z_1$ by

$$(\widehat{x}, z_1, z_2) \mapsto (\widehat{T}\widehat{x}, \Theta^1_{\widehat{x}}(z_1), \Theta^1_{\widehat{x}}(z_2)),$$

is a factor of \widetilde{T} , hence is ergodic. This, however, means that $\widehat{X}_1 = \widehat{X}$, which completes the proof. ■

As a consequence we have

PROPOSITION 2.3. *Let $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be ergodic and $\{\mathcal{A}_i : i \in I\}$ a family of factors of T such that $\mathcal{B} \rightarrow \mathcal{A}_i$ rel.w.m. for each $i \in I$. Then*

$$\mathcal{B} \rightarrow \mathcal{A} = \bigcap_{i \in I} \mathcal{A}_i \quad \text{rel.w.m.}$$

Proof. Let $\mathcal{A}' \subset \mathcal{A}$ be the maximal distal extension of \mathcal{A} in \mathcal{B} . Then $\mathcal{B} \rightarrow \mathcal{A}'$ rel.w.m. From Proposition 2.2, $\mathcal{A}' \subset \mathcal{A}_i$, and consequently $\mathcal{A}' \subset \mathcal{A}$, hence $\mathcal{A}' = \mathcal{A}$. ■

PROPOSITION 2.4. *Suppose that $\mathcal{B} \supset \mathcal{A}_1 \supset \mathcal{A}_2$, $\mathcal{B} \rightarrow \mathcal{A}_1$ rel.w.m. and $\mathcal{A}_1 \rightarrow \mathcal{A}_2$ rel.w.m. Then $\mathcal{B} \rightarrow \mathcal{A}_2$ rel.w.m.*

Proof. Let $\widehat{\mathcal{A}}_2$ be the maximal distal extension of \mathcal{A}_2 in \mathcal{B} . Then $\widehat{\mathcal{A}}_1 \rightarrow \mathcal{A}_2$ is distal while $\mathcal{A}_1 \rightarrow \mathcal{A}_2$ rel.w.m. Therefore $\widehat{\mathcal{A}}_2$ and \mathcal{A}_1 are disjoint relative to \mathcal{A}_2 , so $\widehat{\mathcal{A}}_2 \cap \mathcal{A}_1 = \mathcal{A}_2$. From Proposition 2.3, $\mathcal{B} \rightarrow \widehat{\mathcal{A}}_2 \cap \mathcal{A}_1$ rel.w.m. and the result follows. ■

3. Semisimplicity. Let $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be ergodic.

DEFINITION. We say that T is *semisimple* if for every self-joining $\lambda \in J^e(T, T)$ we have

$$(X \times X, \lambda) \xrightarrow{\pi_i} (X, \mu) \quad \text{rel.w.m.,}$$

where $\pi_i : X \times X \rightarrow X$, $\pi_i(x_1, x_2) = x_i$, $i = 1, 2$.

Below we present some examples.

EXAMPLE 1. T has discrete spectrum. Then each joining $\lambda \in J^e(T, T)$ is a graph joining, so T is semisimple.

EXAMPLE 2. T is 2-fold simple, i.e. if $\lambda \in J^e(T, T)$ then either λ is a graph joining or $\lambda = \mu \times \mu$. Immediately from this definition we see that T is semisimple.

EXAMPLE 3. T_1, \dots, T_k , $1 \leq k \leq \infty$, with MSJ (for the definition see [Ru]). Then $T_1 \times \dots \times T_k$ is semisimple (see [Ju-Ru]).

EXAMPLE 4. Each Gaussian–Kronecker automorphism $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is semisimple. Indeed, let $E \subset L^2(X, \mu)$ be the corresponding space of Gaussian vectors. Take $\lambda \in J^e(T, T)$. Then, as shown in [Ju-Th], $T \times T : (X \times X, \lambda) \rightarrow (X \times X, \lambda)$ is again Gaussian–Kronecker determined by $E + E \subset L^2(X \times X, \lambda)$. But E has its orthocomplement in $E + E \subset L^2(X \times X, \lambda)$ so again by [Ju-Th], T is a *direct* factor of $T \times T : (X \times X, \lambda) \rightarrow (X \times X, \lambda)$. Since all automorphisms under consideration are weakly mixing, the assertion follows.

All the examples above are in some sense pure; they are either weakly mixing or have discrete spectrum. Semisimple maps can, however, have mixed spectrum.

EXAMPLE 5. $T = T_1 \times T_2$, where T_1 has discrete spectrum and T_2 has MSJ. Then each $\lambda \in J^e(T, T)$ is either a graph joining (T can be viewed as a group extension of T_2 with a constant cocycle) or appears in the ergodic decomposition of $\mu \times \mu$ ($\mu = \mu_1 \times \mu_2$). Any such λ is isomorphic to $T_1 \times T_2 \times T_2$, so T is semisimple.

PROPOSITION 3.1. *Suppose that $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is semisimple and let \mathcal{A}_1 and $\mathcal{A}_2 \subset \mathcal{B}$ be factors. Suppose that $\mathcal{B} \rightarrow \mathcal{A}_j$ rel.w.m. ($j = 1, 2$). Then for each $\bar{\lambda} \in J^e(\mathcal{A}_1, \mathcal{A}_2)$ we have*

$$(\mathcal{A}_1 \otimes \mathcal{A}_2, \bar{\lambda}) \rightarrow (\mathcal{A}_j, \mu) \quad \text{rel.w.m.} \quad \text{for } j = 1, 2.$$

Proof. Extend $\bar{\lambda}$ to $\lambda \in J^e(T, T)$ whose projection on $\mathcal{A}_1 \otimes \mathcal{A}_2$ is $\bar{\lambda}$. Then $(X \times X, \lambda) \rightarrow (X, \mu) \rightarrow (\mathcal{A}_1, \mu)$ are rel.w.m. By Proposition 2.4, $(X \times X, \lambda) \rightarrow (\mathcal{A}_2, \mu)$ rel.w.m. But obviously, we have a sequence of factors $(X \times X, \lambda) \rightarrow (\mathcal{A}_1 \otimes \mathcal{A}_2, \bar{\lambda}) \rightarrow (\mathcal{A}_1, \mu)$ so we must have $(\mathcal{A}_1 \otimes \mathcal{A}_2, \bar{\lambda}) \rightarrow (\mathcal{A}_1, \mu)$ rel.w.m. ■

Substituting, in Proposition 3.1, $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$ we obtain

COROLLARY 3.1. *Suppose that $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is semisimple and let $\mathcal{A} \subset \mathcal{B}$ be a factor. If $\mathcal{B} \rightarrow \mathcal{A}$ rel.w.m. then $\bar{T} : (\bar{X}, \mathcal{A}, \bar{\mu}) \rightarrow (\bar{X}, \mathcal{A}, \bar{\mu})$ is semisimple. ■*

Remark. For semisimple maps on $J^e(T, T)$ there is a natural structure of a monoid (see [Gl-Ho-Ru]). Suppose that $\lambda_1, \lambda_2 \in J^e(T, T)$. We have

$$(X \times X, \lambda_1) \xrightarrow{\text{rel.w.m.}} X \xleftarrow{\text{rel.w.m.}} (X \times X, \lambda_2)$$

so the relative product over X is rel.w.m. Since $\lambda_1 \times_X \lambda_2$ is ergodic, the projection on the first and third coordinates gives an ergodic self-joining obtained by $\lambda_1 \circ \lambda_2 \in J^e(T, T)$. This multiplication is associative and has a unit, the diagonal measure on X . If T is weakly mixing then $\mu \times \mu \in J^e(T, T)$ and $(\mu \times \mu) \circ \lambda = \mu \times \mu$ for each $\lambda \in J^e(T, T)$. More generally, if \mathcal{A} is a factor and $\lambda \in J^e(T, T)$ is diagonal on \mathcal{A} then $(\mu \times_{\mathcal{A}} \mu) \circ \lambda = \mu \times_{\mathcal{A}} \mu$. In particular, the relatively independent extensions of diagonal measures give idempotents. The only invertible elements are graph joinings μ_S with $S \in C(T)$ necessarily invertible.

4. Natural factors and the structure of factors for semisimple automorphisms. Let $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be ergodic. Suppose that η is a class of factors satisfying

(*) η contains \mathcal{B} and the trivial σ -algebra \mathcal{N} , and is closed under taking intersections.

We will call η *natural* if

(i) $\forall \lambda \in J^e(T, T)$, $\mathcal{B}_i(\lambda) \in \eta$, $i = 1, 2$.

(ii) If $\mathcal{A}_1, \mathcal{A}_2 \in \eta$ and $S : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ establishes an isomorphism then $S\mathcal{A}'_1 \in \eta$ provided that $\mathcal{A}'_1 \subset \mathcal{A}_1$ and $\mathcal{A}'_1 \in \eta$.

Remark. Since η is closed under intersections, for each factor $\mathcal{A} \subset \mathcal{B}$ we have a smallest natural factor $\widehat{\mathcal{A}} \in \eta$ with $\widehat{\mathcal{A}} \subset \mathcal{A}$. Call $\widehat{\mathcal{A}}$ the *natural cover* of \mathcal{A} .

Remark. Suppose that T is an ergodic automorphism. Then directly from the definition it follows that there exists a smallest family η_0 of natural factors. Note also that if $\mathcal{B} \rightarrow \mathcal{A}$ rel.w.m. then $\mathcal{A} \in \eta_0$. Indeed, we have $\mu \times_{\mathcal{A}} \mu \in J^e(\mathcal{B}, \mathcal{B})$ and obviously $\mathcal{B}_i(\mu \times_{\mathcal{A}} \mu) = \mathcal{A}$, $i = 1, 2$.

PROPOSITION 4.1. *A family η satisfying (*) is natural iff whenever $\lambda \in J^e(T, T)$ and λ restricted to factors $\mathcal{A}_1 \otimes \mathcal{A}_2$ establishes their isomorphism then λ is an isomorphism on the natural covers.*

Proof. \Rightarrow Suppose $\lambda \in J^e(T, T)$ and $\lambda|_{\mathcal{A}_1 \otimes \mathcal{A}_2}$ is an isomorphism. By (i), $\widehat{\mathcal{A}}_j \subset \mathcal{B}_j(\lambda)$, $j = 1, 2$, and (ii) completes this part of the proof.

\Leftarrow Take $\lambda \in J^e(T, T)$; then λ establishes an isomorphism between $\mathcal{B}_1(\lambda)$ and $\mathcal{B}_2(\lambda)$. Since these two are the largest factors with this property we must have $\widehat{\mathcal{B}}_j(\lambda) = \mathcal{B}_j(\lambda)$, $j = 1, 2$, and (i) follows.

Now, let $\mathcal{A}_1, \mathcal{A}_2 \in \eta$ and S be an isomorphism between them. Lift this isomorphism to a $\lambda \in J^e(T, T)$. Take $\mathcal{A} \in \eta$ with $\mathcal{A} \subset \mathcal{A}_1$. Then λ is an isomorphism of $\mathcal{A} = \widehat{\mathcal{A}}$ with $S\mathcal{A}$ but also with $\widehat{S\mathcal{A}}$. Hence $S\mathcal{A} = \widehat{S\mathcal{A}}$ so $S\mathcal{A} \in \eta$, which completes the proof. ■

COROLLARY 4.1. *Let η be a natural family of factors for T . Then for each factor \mathcal{A} of T the extension $\widehat{\mathcal{A}} \rightarrow \mathcal{A}$ is a group extension.*

Proof. Take any ergodic self-joining λ on $\widehat{\mathcal{A}} \otimes \widehat{\mathcal{A}}$ which is diagonal on $\mathcal{A} \otimes \mathcal{A}$. Hence λ establishes an isomorphism of \mathcal{A} with itself (determined by the identity). From Proposition 4.1, λ is an isomorphism of $\widehat{\mathcal{A}}$ with itself, so λ is a graph joining on $\widehat{\mathcal{A}} \otimes \widehat{\mathcal{A}}$. By Veech's Theorem, $\widehat{\mathcal{A}} \rightarrow \mathcal{A}$ is a group extension. ■

LEMMA 4.1. *Let $\widehat{T}_i : (\widehat{X}_i, \widehat{\mathcal{B}}_i, \widehat{\mu}_i) \rightarrow (\widehat{X}_i, \widehat{\mathcal{B}}_i, \widehat{\mu}_i)$, $i = 1, 2$, be ergodic distal extensions of $T_i : (X_i, \mathcal{B}_i, \mu_i) \rightarrow (X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$. Assume that $\widehat{\lambda} \in J^e(\widehat{T}_1, \widehat{T}_2)$ has the property that its restriction λ to $\mathcal{B}_1 \otimes \mathcal{B}_2$ is a graph joining and moreover for $i = 1, 2$ the extension $(\widehat{\mathcal{B}}_1 \otimes \widehat{\mathcal{B}}_2, \widehat{\lambda}) \rightarrow (\widehat{\mathcal{B}}_i, \widehat{\mu}_i)$ is rel.w.m. Then $\widehat{\lambda}$ is also a graph joining.*

Proof. Note that in $(\widehat{\mathcal{B}}_1 \otimes \widehat{\mathcal{B}}_2, \widehat{\lambda})$ we have $\mathcal{B}_1 = \mathcal{B}_2 \pmod{\widehat{\lambda}}$. Therefore

$$\widehat{\mathcal{B}}_1 \otimes \widehat{\mathcal{B}}_2 \rightarrow \widehat{\mathcal{B}}_1 \rightarrow \mathcal{B}_1 \quad \text{and} \quad \widehat{\mathcal{B}}_1 \otimes \widehat{\mathcal{B}}_2 \rightarrow \widehat{\mathcal{B}}_2 \rightarrow \mathcal{B}_2$$

are, by assumption, two Furstenberg decompositions of $\mathcal{B}_1 = \mathcal{B}_2$. By Proposition 2.1 we have $\widehat{\mathcal{B}}_1 = \widehat{\mathcal{B}}_2 \pmod{\widehat{\lambda}}$, so $\widehat{\lambda}$ is an isomorphism of $\widehat{\mathcal{B}}_1$ and $\widehat{\mathcal{B}}_2$. ■

Below, we will consider a family of natural factors (in fact it will be equal to η_0) for semisimple maps.

Let $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be ergodic and semisimple. Put

$$\eta = \{\mathcal{A} \subset \mathcal{B} : \mathcal{B} \rightarrow \mathcal{A} \text{ rel.w.m.}\} \cup \{\mathcal{N}\}.$$

PROPOSITION 4.2. *The family η is natural.*

Proof. By Proposition 2.3, η is closed under intersections. We will prove that if $\lambda \in J^e(T, T)$ establishes an isomorphism of \mathcal{A}_1 and \mathcal{A}_2 then λ is also an isomorphism of the natural covers $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{A}}_2$. Now, $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{A}}_2$ can be described as the maximal distal extensions of \mathcal{A}_1 and \mathcal{A}_2 (respectively) in \mathcal{B} . By Proposition 3.1, if we denote by $\widehat{\lambda}$ the restriction of λ to $\widehat{\mathcal{A}}_1 \otimes \widehat{\mathcal{A}}_2$ then $(\widehat{\mathcal{A}}_1 \otimes \widehat{\mathcal{A}}_2, \widehat{\lambda}) \rightarrow (\widehat{\mathcal{A}}_i, \mu)$ rel.w.m. ($i = 1, 2$). Then the previous lemma finishes the proof.

By applying Proposition 4.2 and Corollary 4.1 we obtain the following

THEOREM 1 (Structure theorem for factors of semisimple maps). *If $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is ergodic and semisimple then for each factor \mathcal{A} there exists an $\widehat{\mathcal{A}}$ with $\mathcal{B} \rightarrow \widehat{\mathcal{A}}$ rel.w.m. such that $\widehat{\mathcal{A}}$ is a group extension of \mathcal{A} . ■*

Remark. If T is 2-fold simple then the only factors with respect to which T is rel.w.m. are the trivial ones, so applying Theorem 1 we obtain the well known Veech's Theorem on factors of 2-fold simple maps (see [Ve] and also [Ju-Ru]).

Remark. Applying Theorem 1 it is very easy to give examples of T which are not semisimple. Indeed, if there are $\mathcal{B}_2 \subset \mathcal{B}_1 \subset \mathcal{B}$ such that $\mathcal{B} \rightarrow \mathcal{B}_1$ and $\mathcal{B}_1 \rightarrow \mathcal{B}_2$ are isometric but $\mathcal{B} \rightarrow \mathcal{B}_2$ is not isometric, then \mathcal{B} is not semisimple. Since $\mathcal{B} \rightarrow \mathcal{B}_2$ is distal, we must have $\tilde{\mathcal{B}}_2 = \mathcal{B}$. If \mathcal{B} were semisimple, then, by Theorem 1, $\mathcal{B} \rightarrow \mathcal{B}_2$ would be a group extension.

COROLLARY 4.2. *If $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is ergodic and semisimple then its entropy $h(T)$ is zero.*

Proof. First, note that no Bernoulli $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is semisimple. Indeed, take any nontrivial weakly mixing compact group extension $T_\varphi : (X \times G, \tilde{\mathcal{B}}, \tilde{\mu}) \rightarrow (X \times G, \tilde{\mathcal{B}}, \tilde{\mu})$ of T . By [Ru1], T_φ is again Bernoulli with the same entropy as T . Now, in $\tilde{\mathcal{B}}$ we have two factors, namely, $\tilde{\mathcal{B}}$ and \mathcal{B} isomorphic to T . If T were semisimple, then the smallest factor containing these two factors (equal to $\tilde{\mathcal{B}}$) would have to be rel.w.m. with respect to \mathcal{B} ; a contradiction.

Suppose that $h(T) > 0$. Then there exists a Bernoulli factor \mathcal{A} with the same entropy. Take the natural cover $\hat{\mathcal{A}}$ of \mathcal{A} . Then $\hat{\mathcal{A}} \rightarrow \mathcal{A}$ is a compact group extension. If $\hat{\mathcal{A}}$ is weakly mixing then $\hat{\mathcal{A}}$ is Bernoulli, so that $\hat{\mathcal{A}}$ is semisimple. In general, $\hat{\mathcal{A}}$ can be represented as $\hat{\mathcal{A}} = \tilde{\mathcal{A}} \otimes \mathcal{K}$, where $\tilde{\mathcal{A}}$ is Bernoulli and \mathcal{K} is the maximal Kronecker factor of $\hat{\mathcal{A}}$ (see [Ru1]). Moreover, $\tilde{\mathcal{A}}$ can be represented as a nontrivial group extension of a Bernoulli factor, say of $\tilde{\mathcal{A}}_1$. Hence $\tilde{\mathcal{A}} \otimes \mathcal{K}$ is a nontrivial group extension of $\tilde{\mathcal{A}}_1 \otimes \mathcal{K}$. But these two automorphisms are isomorphic so the former is not semisimple. ■

Remark. Suppose that T is ergodic and distal. Then T is semisimple iff T has discrete spectrum. Indeed, if T is semisimple and \mathcal{K} is its maximal Kronecker factor then $\mathcal{B} \rightarrow \mathcal{K}$ rel.w.m. ($\hat{\mathcal{K}}$ is a group extension of \mathcal{K} which is a group extension of a one-point dynamical system; since $\hat{\mathcal{K}}$ must be semisimple, we have $\hat{\mathcal{K}} = \mathcal{K}$).

5. Basic facts on nonergodic extensions of ergodic automorphisms. The content of this section is rather classical and can be found e.g. in [Ke-Ne1], [Ke-Ne2], [Ke-Ne3]. We list some basic facts concerning the ergodic decomposition of a compact group extension of an ergodic automorphism and, in Section 6, apply them in our analysis of ergodic joinings for group extensions of semisimple automorphisms.

Let (X, \mathcal{B}, μ, T) be an ergodic dynamical system. Let G be a compact metric group equipped with the normalized Haar measure ν on the family

\mathcal{D} of Borel subsets of G . Assume that $\varphi : X \rightarrow G$ is a Borel map. Because the G -extension T_φ is not necessarily ergodic with respect to $\tilde{\mu}$, let

$$\tilde{\mu} = \int_{E(T_\varphi)} \lambda d\gamma(\lambda)$$

be the ergodic decomposition of $\tilde{\mu}$.

Take any $\lambda \in E(T_\varphi)$. Denote by H the stabilizer of λ in G , i.e. $H = \{g \in G : \lambda g = \lambda\}$.

FACT 5.1. (i) H is a closed subgroup of G .

(ii) If $(x, g), (x, h) \in Y$, then $hH = gH$.

Let us decompose λ over the factor (X, μ, T) :

$$\lambda = \int_X \lambda_x d\mu(x).$$

Let ν_H denote the Haar measure on H .

FACT 5.2. *For almost each $x \in X$ there exists a $g = g_x \in G$ such that*

$$\lambda_x = \delta_x \times g\nu_H.$$

Let us define a function $\tau : X \rightarrow G/H$ by

$$(3) \quad \tau(x) = g_x H,$$

where g_x is defined by Fact 5.2. By this fact, $(X \times G/H, \lambda, T_\varphi)$ is isomorphic to (X, μ, T) : the map $p : X \times G/H \rightarrow X$, $p(x, gH) = x$, is measurable and λ -a.e. one-to-one. Therefore p is invertible and $p^{-1}(x) = (x, \tau(x))$. This forces τ to be measurable. Also

$$(4) \quad \tau(Tx) = \varphi(x)\tau(x).$$

FACT 5.3. *There is a function $t : X \rightarrow G$ such that $(X \times G, \lambda, T_\varphi)$ is isomorphic to $(X \times H, \mu \times \nu_H, T_\psi)$, where $\psi(x) = t(Tx)^{-1}\varphi(x)t(x)$.*

6. Joinings of ergodic group extensions of semisimple automorphisms. Assume that $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ and $S : (Y, \mathcal{C}, m) \rightarrow (Y, \mathcal{C}, m)$ are ergodic automorphisms. Let G_1 and G_2 be compact metric groups with Haar measures ν_1 and ν_2 respectively. Let $\varphi_1 : X \rightarrow G_1$ and $\varphi_2 : Y \rightarrow G_2$ be such that T_{φ_1} and S_{φ_2} are ergodic.

Suppose that $\lambda \in J^e(T, S)$ has the property that the two extensions

$$(T \times S, \lambda) \rightarrow (T, \mu) \quad \text{and} \quad (T \times S, \lambda) \rightarrow (S, m)$$

are rel.w.m. The following theorem describes any $\tilde{\lambda} \in J^e(T_{\varphi_1}, S_{\varphi_2})$ whose projection on $\mathcal{B} \otimes \mathcal{C}$ is λ .

THEOREM 2. *There are normal closed subgroups $H_1 \subset G_1$ and $H_2 \subset G_2$, a continuous group isomorphism $v : G_1/H_1 \rightarrow G_2/H_2$ and a Borel map*

$f : X \times Y \rightarrow G_2/H_2$ such that for any Borel sets $A \subset X$, $C_1 \subset G_1$, $B \subset Y$ and $C_2 \subset G_2$ we have

$$\begin{aligned} & \tilde{\lambda}(A \times C_1 \times B \times C_2) \\ &= \int_{X \times Y \times G_1/H_1} E(\chi_{A \times Y \times C_1} | H_1)(x, y, g_1 H_1) \\ & \quad \times E(\chi_{X \times B \times C_2} | H_2)(x, y, f(x, y)v(g_1 H_1)) d(\lambda \times \nu_1)(x, y, g_1 H_1). \end{aligned}$$

The proof of Theorem 2 is long and is divided into several lemmas.

Let $\pi : X \times G_1 \times Y \times G_2 \rightarrow X \times Y$, $\pi(x, g_1, y, g_2) = (x, y)$. Then $\pi^* \tilde{\lambda} = \lambda$.

Let us decompose $\tilde{\lambda}$ over the factor $(X \times Y, \lambda, T \times S)$:

$$\tilde{\lambda} = \int_{X \times Y} \tilde{\lambda}_{(x,y)} d\lambda(x, y).$$

Let $H = \{(h_1, h_2) \in G_1 \times G_2 : \tilde{\lambda}(h_1, h_2) = \tilde{\lambda}\}$ be the stabilizer of $\tilde{\lambda}$. By Fact 5.2,

$$\tilde{\lambda} = \int_{X \times Y} \delta_{(x,y)} \times (g_1, g_2) \nu_H d\lambda(x, y),$$

where $(g_1, g_2)H = \tau(x, y)$.

Let

$$H_1 = \{g_1 \in G_1 : (g_1, e_2) \in H\}, \quad H_2 = \{g_2 \in G_2 : (e_1, g_2) \in H\},$$

where e_i denotes the unit of G_i , $i = 1, 2$. Put $\pi_i : G_1 \times G_2 \rightarrow G_i$, $\pi_i(g_1, g_2) = g_i$, $i = 1, 2$.

LEMMA 6.1. $\pi_i(H) = G_i$, $i = 1, 2$.

Proof. First, we note that $(T_{\varphi_1} \times S_{\varphi_2}, X \times G_1 \times Y \times G_2, \tilde{\lambda})$ is a group extension of $T \times S$, where the group by means of which we extend is H (see Fact 5.3). If we take the projection onto the first three coordinates, then we get a group extension of $(T \times S, X \times Y, \lambda)$ by $\pi_1(H)$. This group extension is ergodic. On the other hand, $(T_{\varphi_1}, X \times G_1, \tilde{\mu}) \rightarrow (T, X, \mu)$ is a group extension and $(T \times S, X \times Y, \lambda) \rightarrow (T, X, \mu)$ rel.w.m. So the relative product $T_{\varphi_1} \times_{(T, X, \mu)} (T \times S, \lambda)$ is ergodic. This relative product is equal to $((T \times S)_{\varphi_2}, \tilde{\lambda})_{\varphi_1}$, i.e. it is a group extension of $(T \times S, \lambda)$ via G_1 . Since the latter is ergodic, $\pi_1(H) = G_1$.

The proof of the equality $\pi_2(H) = G_2$ is similar. ■

The next lemma immediately follows from Lemma 6.1.

LEMMA 6.2. The subgroups H_1 and H_2 are normal in G_1 and G_2 respectively. ■

LEMMA 6.3. (a) If $(g_1, g_2) \in H$ and $(g_1, \tilde{g}_2) \in H$ then $\tilde{g}_2^{-1}g_2 \in H_2$.

(b) If $(g_1, g_2) \in H$ and $(\tilde{g}_1, g_2) \in H$ then $\tilde{g}_1^{-1}g_1 \in H_1$.

(c) $(g_1, g_2) \in H$ iff $g_1 H_1 \times g_2 H_2 \subset H$.

Proof. (a) If $(g_1, g_2) \in H$ and $(g_1, \tilde{g}_2) \in H$ then $(g_1^{-1}, g_2^{-1}) \in H$ and $H \ni (g_1^{-1}, \tilde{g}_2^{-1})(g_1, g_2) = (e_1, \tilde{g}_2^{-1}g_2)$. Therefore $\tilde{g}_2^{-1}g_2 \in H_2$. The proof of (b) is similar.

(c) Assume that $(g_1, g_2) \in H$. Take $h_1 \in H_1$ and $h_2 \in H_2$. Then $(h_1, e_2) \in H$ and $(e_1, h_2) \in H$. Therefore $(h_1, h_2) \in H$ and $H \ni (g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2)$. Because h_1 and h_2 were arbitrary, $g_1 H_1 \times g_2 H_2 \subset H$. ■

We define a map $v : G_1/H_1 \rightarrow G_2/H_2$ by

$$(5) \quad v(g_1 H_1) = \pi_2((g_1 H_1 \times G_2) \cap H).$$

LEMMA 6.4. The map v is a continuous group isomorphism.

Proof. By Lemma 6.3, v is well defined. The continuity of v is evident. Obviously v is bijective. Because $H_1 \times H_2 \subset H$, $v(H_1) = H_2$. We now prove that v is a group homomorphism.

Take $gH_1, \bar{g}H_1 \in G_1/H_1$. Set $v(gH_1 \bar{g}H_1) = \tilde{g}H_2$, $v(gH_1) = g_1 H_2$ and $v(\bar{g}H_1) = \bar{g}_1 H_2$. Then $g\bar{g}H_1 \times \tilde{g}H_2 \subset H$, $gH_1 \times g_1 H_2 \subset H$ and $\bar{g}H_1 \times \bar{g}_1 H_2 \subset H$. This implies $g\bar{g}H_1 \times g_1 \bar{g}_1 H_2 \subset H$. By Lemma 6.3, $\tilde{g}H_2 = g_1 \bar{g}_1 H_2$, i.e. $v(gH_1 \bar{g}H_1) = v(gH_1)v(\bar{g}H_1)$.

Obviously $v(g^{-1}H_1) = v(gH_1)^{-1}$. ■

As an immediate consequence of Lemmas 6.3 and 6.4 we have

LEMMA 6.5. $H = \bigcup_{g \in G_1} gH_1 \times v(gH_1)$. ■

Let

$$\begin{aligned} (T \times S)_{\varphi_i, H_i} &: X \times Y \times G_i/H_i \rightarrow X \times Y \times G_i/H_i, \\ (T \times S)_{\varphi_i, H_i}(x_1, x_2, gH_i) &= (Tx_1, Sx_2, \varphi_i(x_i)gH_i), \quad i = 1, 2. \end{aligned}$$

Then $(X \times Y \times G_i/H_i, \lambda \times \nu_i, (T \times S)_{\varphi_i, H_i})$, $i = 1, 2$, is an ergodic dynamical system.

Our next aim is to define an isomorphism \bar{I} of $(T \times S)_{\varphi_1, H_1}$ and $(T \times S)_{\varphi_2, H_2}$. It will have the form

$$\begin{aligned} \bar{I} &= I_{f,v} : X \times Y \times G_1/H_1 \rightarrow X \times Y \times G_2/H_2, \\ I_{f,v}(x, y, gH_1) &= (x, y, f(x, y)v(gH_1)), \end{aligned}$$

for some measurable map $f : X \times Y \rightarrow G_2/H_2$.

Let $\alpha : (G_1 \times G_2)/H \rightarrow G_2/H_2$ be the (open) map given by

$$(6) \quad \alpha((g_1, g_2)H) = g_2 v(g_1^{-1} H_1).$$

We have to prove that α is well defined. Assume that $(g_1, g_2)H = (\tilde{g}_1, \tilde{g}_2)H$. Then $(g_1^{-1} \tilde{g}_1, g_2^{-1} \tilde{g}_2) \in H$ and therefore

$$(*) \quad v(g_1^{-1} \tilde{g}_1 H_1) = g_2^{-1} \tilde{g}_2 H_2.$$

We will show that $(g_2 v(g_1^{-1} H_1))^{-1} \tilde{g}_2 v(\tilde{g}_1 H_1) = H_2$.

Indeed, by (*),

$$\begin{aligned} (g_2 v(g_1^{-1} H_1))^{-1} \tilde{g}_2 v(\tilde{g}_1 H_1) &= v(g_1 H_1) g_2^{-1} \tilde{g}_2 H_2 v(\tilde{g}_1^{-1} H_1) \\ &= v(g_1 H_1) v(g_1^{-1} \tilde{g}_1 H_1) v(\tilde{g}_1^{-1} H_1) = H_2. \end{aligned}$$

Thus α is well defined.

Having α we can define the desired function $f : X \times Y \rightarrow G_2/H_2$ by setting

$$(7) \quad f(x, y) = \alpha(\tau(x, y)),$$

where τ is defined by (3) and it satisfies (4) for $\varphi = \varphi_1 \times \varphi_2$.

Now, one easily checks that

$$(T \times S)_{\varphi_2, H_2} \circ \bar{I} = \bar{I} \circ (T \times S)_{\varphi_1, H_1}.$$

We will also use the following

LEMMA 6.6.

$$(a) \quad \tau(x, y) = \bigcup_{g \in G_1} g H_1 \times f(x, y) v(g H_1) \quad \lambda\text{-a.s.},$$

$$(b) \quad \tilde{\lambda} \left(\bigcup_{\substack{(x, y) \in X \times Y \\ g \in G_1}} \{(x, y)\} \times g H_1 \times f(x, y) v(g H_1) \right) = 1.$$

Proof. (a) Fix $(x, y) \in X \times Y$. Set $\tau(x, y) = (a, b)H$. Then by (6), (7) and Lemma 6.5,

$$\begin{aligned} &\bigcup_{g \in G_1} g H_1 \times f(x, y) v(g H_1) \\ &= \bigcup_{g \in G_1} g H_1 \times b v(a^{-1} H_1) v(g H_1) \\ &= \bigcup_{g \in G_1} g H_1 \times b v(a^{-1} g H_1) = \bigcup_{g \in G_1} a g H_1 \times b v(g H_1) \\ &= (a, b) \bigcup_{g \in G_1} g H_1 \times v(g H_1) = (a, b)H = \tau(x, y). \end{aligned}$$

(b) Using (a) we have

$$\begin{aligned} 1 &= \tilde{\lambda} \left(\bigcup_{(x, y) \in X \times Y} \{(x, y)\} \times \tau(x, y) \right) \\ &= \tilde{\lambda} \left(\bigcup_{(x, y) \in X \times Y} \left(\{(x, y)\} \times \bigcup_{g \in G_1} g H_1 \times f(x, y) v(g H_1) \right) \right) \\ &= \tilde{\lambda} \left(\bigcup_{\substack{(x, y) \in X \times Y \\ g \in G_1}} \{(x, y)\} \times g H_1 \times f(x, y) v(g H_1) \right). \quad \blacksquare \end{aligned}$$

Proof of Theorem 2. By Lemma 6.6 we can define an isomorphism

$$U : (X \times Y \times G_1/H_1 \times G_2/H_2, \tilde{\lambda}, (T \times S)_{\varphi_1 \times \varphi_2, H_1 \times H_2}) \rightarrow ((X \times Y \times G_1/H_1) \times (X \times Y \times G_2/H_2), (\lambda \times \nu_1)_{\bar{I}}, (T \times S)_{\varphi_1, H_1} \times (T \times S)_{\varphi_2, H_2})$$

by

$$U(x, y, g H_1, f(x, y) v(g H_1)) = (x, y, g H_1, x, y, f(x, y) v(g H_1)).$$

Then U sends the measure $\tilde{\lambda}$ to $(\lambda \times \nu_1)_{\bar{I}}$ and we have the formula

$$\begin{aligned} \tilde{\lambda}(A \times B \times C) &= \int_{X \times Y \times G_1/H_1} \chi_{A \times B}(x, y, g H_1) \\ &\quad \times \chi_{A \times C}(x, y, f(x, y) v(g H_1)) d(\lambda \times \nu_1)(x, y, g H_1) \end{aligned}$$

for $A \subset X \times Y$, $B \subset G_1/H_1$ and $C \subset G_2/H_2$.

Therefore for $A \times C_1 \subset X \times G_1$ and $B \times C_2 \subset Y \times G_2$,

$$\begin{aligned} \tilde{\lambda}(A \times C_1 \times B \times C_2) &= \int_{X \times Y \times G_1/H_1} E(\chi_{A \times Y \times C_1} | H_1)(x, y, g H_1) \\ &\quad \times E(\chi_{X \times B \times C_2} | H_2)(x, y, f(x, y) v(g H_1)) d\lambda d\nu_1, \end{aligned}$$

which finishes the proof of Theorem 2. \blacksquare

COROLLARY 6.1. Assume $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is an ergodic semi-simple automorphism. Let G be a compact metric group equipped with the normalized Haar measure ν , let $\varphi : X \rightarrow G$ be such that T_φ is ergodic, and suppose $\tilde{\lambda} \in J^e(T_\varphi, T_\varphi)$ is an extension of some $\lambda \in J^e(T, T)$. Then there are normal closed subgroups $H_1, H_2 \subset G$, a continuous group isomorphism $v : G/H_1 \rightarrow G/H_2$ and a Borel map $f : X \times X \rightarrow G/H_2$ such that for any Borel sets $A, B \subset X$ and $C_1, C_2 \subset G$ we have

$$\begin{aligned} &\tilde{\lambda}(A \times C_1 \times B \times C_2) \\ &= \int_{X \times X \times G/H_1} E(\chi_{A \times X \times C_1} | H_1)(x, y, g H_1) \\ &\quad \times E(\chi_{X \times B \times C_2} | H_2)(x, y, f(x, y) v(g H_1)) d(\lambda \times \nu)(x, y, g H_1). \end{aligned}$$

Assume that $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is an ergodic automorphism and $\varphi : X \rightarrow G$ a cocycle such that T_φ is ergodic. Suppose that $S \in C(T)$ has an extension to $\tilde{S} \in C(T_\varphi)$. If we assume that additionally S is invertible then it is rather well known that

$$(8) \quad \tilde{S}(x, g) = S_{f,v}(x, g) = (Sx, f(x)v(g)),$$

where $f : X \rightarrow G$ is measurable and $V : G \rightarrow G$ is a continuous group epimorphism (this result can be directly deduced from Theorem 2). In general we obtain the following

PROPOSITION 6.1. *\tilde{S} is of the form (8), where $v : G \rightarrow G$ is a continuous group homomorphism (not necessarily onto).*

Proof. Write $\tilde{S}(x, g) = (Sx, \psi(x, g))$. Since $\tilde{S}T_\varphi = T_\varphi\tilde{S}$ we get

$$\psi(T_\varphi(x, g)) = \varphi(Sx)\psi(x, g).$$

Writing $\sigma_g(x, h) = (x, hg)$ we have $\sigma_g \in C(T_\varphi)$. Set

$$F_g(x, h) = \psi(x, h)^{-1}\psi\sigma_g(x, h) = \psi(x, h)^{-1}\psi(x, hg).$$

We have

$$\begin{aligned} F_gT_\varphi(x, h) &= (\psi T_\varphi(x, h))^{-1}\psi\sigma_gT_\varphi(x, h) = (\psi T_\varphi(x, h))^{-1}\psi T_\varphi(x, hg) \\ &= (\varphi(Sx)\psi(x, h))^{-1}\varphi(Sx)\psi(x, hg) = F_g(x, h). \end{aligned}$$

Thus F_g is a constant function. Set

$$v(g) = F_g(\cdot, \cdot).$$

Clearly $v : G \rightarrow G$ is measurable. We now show that v is a group homomorphism. We have $v(e) = e$ and

$$\begin{aligned} v(g_1g_2) &= \psi(x, h)^{-1}\psi(x, hg_1g_2) \\ &= \psi(x, h)^{-1}\psi(x, hg_1)\psi(x, hg_1)^{-1}\psi(x, (hg_1)g_2) = v(g_1)v(g_2). \end{aligned}$$

In particular, v is continuous.

Put

$$f(x, h) = \psi(x, h)v(h)^{-1} \quad \text{a.s.}$$

Take any $g \in G$. Then for a.e. (x, h) we have

$$\begin{aligned} f\sigma_g(x, h) &= f(x, hg) = \psi(x, hg)v(hg)^{-1} \\ &= \psi(x, h)\psi(x, h)^{-1}\psi(x, hg)v(g)^{-1}v(h)^{-1} = \psi(x, h)v(h) = f(x, h). \end{aligned}$$

Therefore f depends only on x . ■

Assume that $\mathcal{A} \subset \mathcal{B}$ is a factor of T . We call it a *canonical* (resp. *weakly canonical*) factor of T if for each isomorphic copy \mathcal{A}' of \mathcal{A} we have $\mathcal{A}' = \mathcal{A}$ (resp. $\mathcal{A}' \subset \mathcal{A}$).

Suppose now that $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is semisimple.

PROPOSITION 6.2. *Let $\hat{T} : (\hat{X}, \hat{\mathcal{B}}, \hat{\mu}) \rightarrow (\hat{X}, \hat{\mathcal{B}}, \hat{\mu})$ be an arbitrary ergodic distal extension of T . Then T is a weakly canonical factor of \hat{T} .*

Suppose that \mathcal{B}' is a factor of $\hat{\mathcal{B}}$ isomorphic to \mathcal{B} . Let \mathcal{A} be the smallest factor containing \mathcal{B} and \mathcal{B}' . Since T is semisimple, $\mathcal{A} \rightarrow \mathcal{B}$ rel.w.m. However, $\hat{\mathcal{B}} \rightarrow \mathcal{A} \rightarrow \mathcal{B}$, and $\hat{\mathcal{B}} \rightarrow \mathcal{B}$ is a distal extension. Hence \mathcal{A} and $\hat{\mathcal{B}}$ are relatively (over \mathcal{B}) disjoint, and consequently $\mathcal{B} = \mathcal{A}$. ■

Remark. Notice that the centralizer of a semisimple automorphism need not be a group; for instance, take $T = T_1 \times T_1 \times \dots$, where T_1 has MSJ.

Remark. D. Newton [Ne] asked about canonicity of automorphisms, i.e. whether there are automorphisms which are canonical factors in an arbitrary ergodic extension. As shown in [Le], the only ones with this property are those with discrete spectrum. Let us ask what is the class of automorphisms which are canonical factors in an arbitrary ergodic distal extension. The above proposition says that semisimple coalescent automorphisms enjoy this property. The question arises whether they are the only ones.

It follows from Proposition 6.2 that a semisimple automorphism sits weakly canonically in any of its ergodic group extensions. In particular, if $\tilde{S} \in C(T_\varphi)$, then $\tilde{S}^{-1}(\mathcal{B}) \subset \mathcal{B}$ and we can apply Proposition 6.1. Hence we obtain the following generalization of the results from [An], [Ne], [Me]:

COROLLARY 6.2. *If $T_\varphi : (X \times G, \tilde{\mu}) \rightarrow (X \times G, \tilde{\mu})$ is an ergodic group extension of a semisimple automorphism and $\tilde{S} \in C(T_\varphi)$ then there are $S \in C(T)$, a Borel map $f : X \rightarrow G$ and a continuous group homomorphism $v : G \rightarrow G$ such that*

$$\tilde{S}(x, g) = (Sx, f(x)v(g)).$$

If, additionally, T is coalescent, then v is onto. ■

7. A natural family of factors for group extensions of simple maps. Throughout this section we assume that $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is 2-fold simple weakly mixing. Let $\varphi : X \rightarrow G$ be a cocycle such that T_φ is weakly mixing.

LEMMA 7.1. *Let $\tilde{\lambda} \in J^0(T_\varphi, T_\varphi)$ with $\tilde{\lambda}|_{\mathcal{B} \otimes \mathcal{B}}$ an isomorphism. Then $\mathcal{B}_1(\tilde{\lambda}) = \tilde{\mathcal{B}}_{H_1}$ and $\mathcal{B}_2(\tilde{\lambda}) = \tilde{\mathcal{B}}_{H_2}$ for some H_1 and H_2 which are normal.*

Proof. Let $H \subset G \times G$ be the stabilizer of $\tilde{\lambda}$. By Lemma 6.1, $\pi_i(H) = G$, $i = 1, 2$. Since $\mathcal{B}_1(\tilde{\lambda})$ and $\mathcal{B}_2(\tilde{\lambda})$ are two factors between \mathcal{B} and $\tilde{\mathcal{B}}$ ($\tilde{\lambda}$ is an isomorphism on the base), it follows that $\mathcal{B}_1(\tilde{\lambda}) = \tilde{\mathcal{B}}_{H_1}$ and $\mathcal{B}_2(\tilde{\lambda}) = \tilde{\mathcal{B}}_{H_2}$, where H_1 and H_2 are closed subgroups of G (this easily follows from the relativized version of Veech's Theorem). We now prove that for each $g \in G$,

$$(9) \quad \tilde{\mathcal{B}}_{g^{-1}H_1g} \subset \mathcal{B}_1(\tilde{\lambda}).$$

Fix $g \in G$. Since $\pi_i(H) = G$, $i = 1, 2$, there exists $g_2 \in G$ such that $(g, g_2) \in H$. We have

$$\sigma_g(\tilde{\mathcal{B}}_{H_1}) = \tilde{\mathcal{B}}_{g^{-1}H_1g}, \quad \sigma_{g_2}(\tilde{\mathcal{B}}_{H_2}) = \tilde{\mathcal{B}}_{g_2^{-1}H_2g_2},$$

so (by the definition of $\mathcal{B}_1(\tilde{\lambda})$) it is enough to show that

$$\sigma_g \tilde{\mathcal{B}}_{H_1} = \sigma_{g_2} \tilde{\mathcal{B}}_{H_2} \text{ mod } \tilde{\lambda}.$$

This, however, is obvious, because if $A \in \tilde{\mathcal{B}}_{H_1}$, $B \in \tilde{\mathcal{B}}_{H_2}$ and

$$\tilde{\lambda}(A \times (X \times G) \Delta (X \times G) \times B) = 0$$

then

$$\begin{aligned} & \tilde{\lambda}(\sigma_g A \times (X \times G) \Delta (X \times G) \times \sigma_{g_2} B) \\ &= \tilde{\lambda}(A \times (X \times G) \Delta (X \times G) \times B) = 0. \end{aligned}$$

Therefore (9) follows. The proof is complete by symmetry. ■

PROPOSITION 7.1. *If $T_\varphi : (X \times G, \tilde{\mu}) \rightarrow (X \times G, \tilde{\mu})$ is a weakly mixing group extension of a weakly mixing 2-fold simple map T then the family*

$$\eta = \{\tilde{\mathcal{B}}_H : H \text{ is a normal closed subgroup of } G\} \cup \{\mathcal{N}\}$$

is a natural family of factors.

PROOF. Since obviously η is closed under taking intersections (the smallest closed subgroup generated by a family of closed normal subgroups is normal) and Lemma 7.1 holds true, it remains to show that if $\tilde{S} : \tilde{\mathcal{B}}_{H_1} \rightarrow \tilde{\mathcal{B}}_{H_2}$ is an isomorphism of two natural factors then \tilde{S} sends natural factors contained in $\tilde{\mathcal{B}}_{H_1}$ to natural factors contained in $\tilde{\mathcal{B}}_{H_2}$. By Proposition 6.1, $\tilde{S}(x, gH_1) = (Sx, f(x)v(gH_1))$, where $v : G/H_1 \rightarrow G/H_2$ is a continuous group isomorphism, $S \in C(T)$ and $f : X \rightarrow G/H$ is measurable. If H' is a closed normal subgroup containing H_1 then by the form of \tilde{S} we have $\tilde{S}\tilde{\mathcal{B}}_{H'} = \tilde{\mathcal{B}}_{v(H'/H_1)}$ and it is clear that $v(H'/H_1)$ is a normal subgroup of G/H_2 . ■

Remark. From Proposition 7.1 and the Structure Theorem we immediately get the result on the structure of factors for group extensions of rotations proved in [Me].

Remark. If we assume that a 2-fold simple map is not weakly mixing, then in fact it has discrete spectrum (see [Ju-Ru]) and then both Lemma 7.1 and Proposition 7.1 are valid for each ergodic cocycle $\varphi : X \rightarrow G$.

8. Coalescence of factors of group extensions of automorphisms with discrete spectrum. Since a semisimple map need not be coales-

cent and it is not known whether or not factors of simple maps are coalescent, we will concentrate on group extensions of discrete spectrum automorphisms. The question whether or not each factor of a coalescent automorphism is again coalescent was stated by D. Newton in 1970 ([Ne1]) and the negative answer is contained in [Le] (see also a recent paper by A. Fieldsteel and D. Rudolph [Fi-Ru]). An ergodic group extension of a rotation need not be coalescent, but we will assume that this is the case and ask about the coalescence of all factors. Our goal is to prove the following theorem (which is a generalization of a result from [Le] for the abelian case).

THEOREM 3. *If $T_\varphi : (X \times G, \tilde{\mu}) \rightarrow (X \times G, \tilde{\mu})$ is an ergodic group extension of an automorphism T with discrete spectrum and η denotes the natural family of factors (from Section 7), then all factors of T_φ are coalescent whenever all natural factors are.*

PROOF. Let \mathcal{E} be a factor of T_φ which is isomorphic to a proper factor $\mathcal{E}' \subsetneq \mathcal{E}$. To simplify notation we assume that $\hat{\mathcal{E}} = \tilde{\mathcal{B}}$. Now, $\mathcal{E}' \subset \mathcal{E}$ and they are isomorphic, so by the coalescence of natural factors we have $\hat{\mathcal{E}}' = \hat{\mathcal{E}} = \mathcal{B}$.

Let $\mathcal{H}(\mathcal{E})$ be the compact subgroup contained in $C(T_\varphi)$ that determines \mathcal{E} . Let \tilde{S} be the (noninvertible) element of the centralizer of \mathcal{E} which gives rise to an isomorphism of \mathcal{E} and \mathcal{E}' . Denote by \hat{S} an extension of \tilde{S} to $C(T_\varphi)$. Now \hat{S} is invertible. Moreover, the factor $\mathcal{E}' = \tilde{S}^{-1}\mathcal{E}$ is determined by $\hat{S}^{-1}\mathcal{H}(\mathcal{E})\hat{S}$. Consequently,

$$(*) \quad \hat{S}^{-1}\mathcal{H}(\mathcal{E})\hat{S} \subset \mathcal{H}(\mathcal{E})$$

and the inclusion is strict.

Set

$$H = \{g \in G : \sigma_g \in \mathcal{H}(\mathcal{E})\},$$

where $\sigma_g(x, h) = (x, hg)$. Note that $\sigma_g \in C(T_\varphi)$ and can be written as Id_{g, τ_g} , where $\tau_g(h) = g^{-1}hg$. Now, each $\hat{U} \in \mathcal{H}(\mathcal{E})$ is of the form $\hat{U} = U_{f, v}$ (Proposition 6.1) and if two elements $\hat{U}, \hat{V} \in \mathcal{H}(\mathcal{E})$ have the same projections on the first coordinate (i.e. they are liftings of the same $U \in C(T)$) then $\hat{U} = \hat{V} \circ \sigma_g$ for some $g \in H$. Suppose that $\hat{S} = S_{f, w}$, where $w : G \rightarrow G$ is an automorphism. Then

$$\hat{S}^{-1} = (S^{-1})_{w^{-1}[(fS^{-1})^{-1}], w^{-1}}$$

(w^{-1} denotes the inverse in the sense of composition of maps) and

$$(S_{f, w})^{-1} \circ \sigma_g \circ S_{f, w} = \sigma_{w^{-1}(g)}.$$

Take the factor $\tilde{\mathcal{B}}_H$ which is determined by the group $\mathcal{H}(\mathcal{E}) \cap \{\sigma_g : g \in G\} \subset C(T_\varphi)$ and consider $\hat{S}^{-1}\tilde{\mathcal{B}}_H$. The latter factor is determined by

$$\hat{S}^{-1}\mathcal{H}(\tilde{\mathcal{B}}_H)\hat{S} = \{\sigma_{w^{-1}(g)} : g \in H\}.$$

Set $H' = \{g \in G : \sigma_{w^{-1}(g)} \in \hat{S}^{-1}\mathcal{H}(\mathcal{E})\hat{S}\}$. Then H is a proper subgroup of H' because $\hat{S}^{-1}\mathcal{H}(\mathcal{E})\hat{S}$ determines $\bar{S}^{-1}\mathcal{E} = \mathcal{E}'$ and \mathcal{E}' is a proper factor of \mathcal{E} . Thus $\mathcal{B}_{H'}$ is a proper factor of \mathcal{B}_H . Moreover, $\hat{S}^{-1}(\mathcal{B}_H) = \mathcal{B}_{H'}$ and therefore

$$\{\sigma_{w^{-1}(g)} : g \in H\} = \hat{S}^{-1}\mathcal{H}(\mathcal{B}_H)\hat{S} = \mathcal{H}(\hat{S}^{-1}(\mathcal{B}_H)) = \mathcal{H}(\mathcal{B}_{H'}).$$

This implies that $\tilde{\mathcal{B}}_H$ has a proper factor $\tilde{\mathcal{B}}_{w^{-1}(H)}$ isomorphic to it. The result follows from Lemma 8.1 below.

LEMMA 8.1. *Let T be semisimple and coalescent, $\varphi : X \rightarrow G$ ergodic, $H \subset G$ a closed subgroup and $\tilde{S} \in C(T_\varphi)$. Assume that $\tilde{\mathcal{B}}_H$ is \tilde{S} -invariant. If \hat{S} is invertible on $\tilde{\mathcal{B}}$ then it is so on $\tilde{\mathcal{B}}_H$.*

PROOF. We have $\hat{S}(x, g) = S_{f,v}(x, g) = (Sx, f(x)v(g))$, where $v : G \rightarrow G$ is a group automorphism. We have assumed that $\hat{S}^{-1}\tilde{\mathcal{B}}_H \subset \tilde{\mathcal{B}}_H$, which means that

$$\forall (x, g) \in X \times G \quad S_{f,v}(x, gH) \in X \times G/H.$$

But $S_{f,v}(x, gH) = (Sx, f(x)v(gH))$ so

$$\forall (x, g) \in X \times G \quad f(x)v(g)v(H) \in G/H,$$

hence $v(H) = (f(x)v(g))^{-1}g_{(x,g)}H$ and $v(H) = g_0H$. But $v(H)$ is a subgroup, so $g_0 = e$ and hence $v(H) = H$. Thus, on $X \times G/H$,

$$\hat{S}(x, gH) = S_{f,v,H}(x, gH) = (Sx, f(x)v(g)H),$$

and one directly checks that $S_{f,v,H}$ is invertible. ■

COROLLARY 8.1. *If \bar{T} is an isometric ergodic extension of a semisimple automorphism T and the group cover of T is coalescent then so is \bar{T} .*

9. Questions

QUESTION 1. What can we say about ergodic joinings of a semisimple automorphism with an arbitrary one?

QUESTION 2. Is semisimplicity a generic property in the group of automorphisms of a fixed Lebesgue space?

QUESTION 3. Is it true in general that if T has a natural family of factors and each natural factor is coalescent then all factors are coalescent?

Warning. It is perfectly possible to have a topological group G and its compact subgroup H with

$$g_0Hg_0^{-1} \not\subseteq H.$$

An example where $G = C(\bar{T})$, for a special \bar{T} , is contained in [Le].

QUESTION 4. Are Interval Exchange Transformations semisimple?

QUESTION 5. How to define semisimplicity of higher orders?

References

- [An] H. Anzai, *Ergodic skew product transformations on the torus*, Osaka J. Math. 3 (1951), 83–99.
- [Fi-Ru] A. Fieldsteel and D. Rudolph, *An ergodic transformation with trivial Kakutani centralizer*, Ergodic Theory Dynamical Systems 12 (1992), 459–478.
- [Fu] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton, N.J., 1981.
- [Gl-Ho-Ru] E. Glasner, B. Host and D. Rudolph, *Simple systems and their higher order self-joinings*, Israel J. Math. 78 (1992), 131–142.
- [Ju-Ru] A. del Junco and D. Rudolph, *On ergodic actions whose self-joinings are graphs*, Ergodic Theory Dynamical Systems 7 (1987), 531–557.
- [Ju-Th] A. del Junco and J.-P. Thouvenot, *The theory for Gaussian–Kronecker automorphisms*, preprint.
- [Ke-Ne1] H. B. Keynes and D. Newton, *Choquet Theory and Ergodic Measures for Compact Group Extensions*, Lecture Notes in Math. 318, Springer, 1973.
- [Ke-Ne2] —, —, *The structure of ergodic measures for compact group extensions*, Israel J. Math. 18 (1974), 363–389.
- [Ke-Ne3] —, —, *Ergodic measures for nonabelian compact group extensions*, Compositio Math. 32 (1976), 53–70.
- [Le] M. Lemańczyk, *Ergodic abelian group extensions of rotations*, preprint, Toruń 1990.
- [Le-Me] M. Lemańczyk and M. K. Mentzen, *Compact subgroups in the centralizer of natural factors of an ergodic group extension of a rotation determine all factors*, Ergodic Theory Dynamical Systems 10 (1990), 763–776.
- [Me] M. K. Mentzen, *Ergodic properties of group extensions of dynamical systems with discrete spectra*, Studia Math. 101 (1991), 19–31.
- [Ne] D. Newton, *On canonical factors of ergodic dynamical systems*, J. London Math. Soc. 19 (1979), 129–136.
- [Ne1] —, *Coalescence and spectrum of automorphisms of a Lebesgue space*, Z. Wahrsch. Verw. Gebiete 19 (1971), 117–122.
- [Ru] D. Rudolph, *An example of measure preserving map with minimal self-joinings, and applications*, J. Analyse Math. 35 (1979), 97–122.
- [Ru1] —, *Classifying the isometric extensions of a Bernoulli shift*, ibid. 34 (1978), 36–60.
- [Th] J.-P. Thouvenot, *The metrical structure of some Gaussian processes*, in: Proc. Ergodic Theory and Related Topics II, Georghenthal 1986, 195–198.

- [Ve] W. A. Veech, *A criterion for a process to be prime*, Monatsh. Math. 94 (1982), 335–341.
 [Zi] R. Zimmer, *Extensions of ergodic group actions*, Illinois J. Math. 20 (1976), 373–409.
 [Zi1] —, *Ergodic actions with generalized discrete spectrum*, *ibid.*, 555–588.

DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF TORONTO
 TORONTO, ONTARIO
 CANADA M5S 1A1
 E-mail: DELJUNCO@MATH.TORONTO.EDU

DEPARTMENT OF MATHEMATICS AND INFORMATICS
 NICHOLAS COPERNICUS UNIVERSITY
 TORONTO, ONTARIO
 CHOPINA 12/18
 87-100 TORUŃ, POLAND
 E-mail: MLEM@MAT.UNI.TORUN.PL
 MENTZEN@MAT.UNI.TORUN.PL

Received March 26, 1993

(3088)

Revised version December 6, 1993 and April 4, 1994

Derivability, variation and range of a vector measure

by

L. RODRÍGUEZ-PIAZZA (Sevilla)

Abstract. We prove that the range of a vector measure determines the σ -finiteness of its variation and the derivability of the measure. Let F and G be two countably additive measures with values in a Banach space such that the closed convex hull of the range of F is a translate of the closed convex hull of the range of G ; then F has a σ -finite variation if and only if G does, and F has a Bochner derivative with respect to its variation if and only if G does. This complements a result of [Ro] where we proved that the range of a measure determines its total variation. We also give a new proof of this fact.

Answering a question of Anantharaman and Diestel [AD], we proved in [Ro] that if the ranges of two measures with values in a Banach space have the same closed convex hull, then they have the same total variation. So we can say that the range of a vector measure determines its total variation. The purpose of this paper is to show two other properties of a vector measure which are determined by its range: the σ -finiteness of its variation, and the Bochner derivability.

In Section 1 we introduce the notation and collect some known results we will use throughout the paper. We first establish some properties of the Bartle integral and vector measures with scalar density with respect to another vector measure; and we finish with a result about the determination of real-valued symmetric measures defined on the euclidean unit sphere (Theorem 1.3).

The fact that the range determines the total variation does not imply directly that the range determines the σ -finiteness of the variation. If we know that Z , the closed convex hull of the range of a vector measure F , is also the closed convex hull of the range of another vector measure of σ -finite variation, what we know is that Z can be decomposed as $Z = \sum_{n \in \mathbb{N}} Z_n$, where each Z_n is the closed convex hull of the range of a measure of finite

1991 *Mathematics Subject Classification*: 46G10, 28B05.

Key words and phrases: vector measures, range, variation, Bochner derivability, zonoid.

Research partially supported by DGICYT grant PB90-893 and by *La Junta de Andalucía*.