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ω-Calderón–Zygmund operators

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Abstract. We prove a $T1$ theorem and develop a version of Calderón–Zygmund theory for ω-CZO when $\omega \in A_\infty$.

1. Introduction. Let T be a linear operator mapping test functions on $\mathbb{R}^n$ continuously into distributions and with an associated kernel $K(x,y)$, $x \neq y$ (in the sense that $\langle Tf, g\rangle = \iint g(x)K(x,y)f(y)\,dx\,dy$ whenever f and g are test functions with compact support). Let ω be an A_∞ weight with critical exponent p_0, and set $\omega_t(x) = t^{-n}\int_{|x-y|<t}\omega(y)\,dy$. We say that $K(x,y)$ satisfies *ω-standard estimates* if for some $\varepsilon > 0$,

$$|K(x,y)| \le C\omega_{|x-y|}(x)\frac{1}{|x-y|^n} \tag{1.1}$$

and

$$\begin{gathered}|K(x,y) - T_{x'}K(x,y)| \le C\omega_{|x-x'|}(x)\frac{|x-x'|^\varepsilon}{|x-y|^{n+\varepsilon}} \quad \text{if } |x-x'| < \tfrac12|x-y|\,,\\ |K(x,y) - T_{y'}K(x,y)| \le C\omega_{|y-y'|}(y)\frac{|y-y'|^\varepsilon}{|x-y|^{n+\varepsilon}} \quad \text{if } |y-y'| < \tfrac12|x-y|\,,\end{gathered} \tag{1.2}$$

where

$$\begin{aligned}T_{x'}K(x,y) = K(x',y) &+ ((x-x')\cdot D_x)K(x',y)\\ &+ \frac{1}{2!}((x-x')\cdot D_x)^2K(x',y)\\ &+ \ldots + \frac{1}{N!}((x-x')\cdot D_x)^N K(x',y)\end{aligned}$$

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and

$$T_{y'}K(x,y) = K(x,y') + ((y-y')\cdot D_y)K(x,y') + \frac{1}{2!}((y-y')\cdot D_y)^2K(x,y') + \ldots + \frac{1}{N!}((y-y')\cdot D_y)^N K(x,y')$$

(with $D_x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and $D_y = (\partial/\partial y_1, \ldots, \partial/\partial y_n)$) are the Taylor polynomials of degree N of $K(x,y)$ with respect to x (at x') and y (at y'), respectively, and $N = [n(p_0-1)]$, where $[a]$ stands for the biggest integer $\le a$.

An example of such an operator is the fractional integral

$$I_\alpha f(x) = \begin{cases} \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\,dy & \text{if } \alpha > 0 \text{ and } \alpha \ne n+2l, \\ \int_{\mathbb{R}^n} y|y|^{\alpha-n-1} f(x-y)\,dy & \text{if } \alpha > 0 \text{ and } \alpha = n+2l, \end{cases}$$

where l is an integer. In these cases, we can take $\omega = |x|^\alpha$.

For the case $\omega \in A_1$, a version of the $T1$ Theorem has been proved and a Calderón–Zygmund theory for ω-CZO has been developed in [3]. In this paper, we will give a version of the $T1$ Theorem for $\omega \in A_\infty$ and prove a Calderón–Zygmund theory for ω-CZO which contains the theory in [3] as a special case.

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2. The statement of the $T1$ Theorem. Let ω be an A_∞ weight with critical exponent p_0. Suppose $N = [n(p_0-1)]$. Let T be a continuous linear operator from test functions to distributions. We say that T satisfies the ω-*WBP* (weak boundedness property) if for each ball B and any test functions ϕ_1, ϕ_2 supported in B that satisfy the estimate $\|\phi_i'\|_\infty \le |B|^{-1/n}$ for $i = 1, 2$ and the cancellation law

$$\int x^k \phi(x)\,dx = 0 \quad \text{for } k = 0, 1, \ldots, N,$$

where $\phi = \phi_1$ or ϕ_2, we have

$$|\langle T\phi_1, \phi_2\rangle| \le C\omega(B).$$

This will hold if T is associated with an antisymmetric kernel satisfying the first standard estimate (1.1).

The space $BMO_\omega = (H^1(\omega))^*$ has been studied in [5]. The definition is, for $\omega \in A_p$, $1 \le p < \infty$,

$$BMO_\omega = \Big\{ f : f/\omega \in L^{p'}_{\mathrm{loc}}(\omega) \text{ and for every interval } I, \Big(\omega(I)^{-1} \int_I |f(x) - P_I f(x)|^{p'} \omega(x)^{1-p'}\,dx\Big)^{1/p'} \le C \Big\},$$

where $1/p + 1/p' = 1$ and $P_I f$ is a polynomial of degree $\le N$ such that

$$\int_I (f(x) - P_I f(x)) x^k\,dx = 0 \quad \text{for } k = 0, 1, \ldots, N,$$

and the BMO_ω norm of $f \in BMO_\omega$ is the smallest constant C in the inequality defining BMO_ω. For more details, see [5]. We also refer to [5] or [6] for the definition of weighted Hardy spaces $H^p(\omega)$.

THEOREM 2.1. *Suppose $\omega \in A_\infty$. Let T be a continuous linear operator from test functions to distributions that is associated with an ω-standard kernel. Then $T : H^1(\omega) \to L^1$ iff T satisfies the ω-WBP and $T1$ and T^t1 lie in BMO_ω.*

A corollary of this theorem is

COROLLARY 2.2. *Same assumptions as in Theorem 2.1. Then $T : H^1(\omega) \to L^1$ and $T : L^\infty \to BMO_\omega$ iff T satisfies the ω-WBP and $T1$ and T^t1 lie in BMO_ω.*

3. The proof of the theorem. In the following, all C's are constants, and they need not be the same at each appearance. Moreover, $1/p + 1/p' = 1$.

The proof of Theorem 2.1 follows the same steps as in [3] and [4]. We prove the sufficiency first. Because $\omega \in A_\infty$, we have $\omega \in A_p$, for some $2 < p < \infty$.

LEMMA 3.1. *Suppose $\phi, \psi \in C^\infty$, $\operatorname{supp}\phi, \psi \subseteq \{|x| \le 1\}$, $\int x^k \psi(x)\,dx = 0$ for $k = 0, 1, \ldots, N$, and $b \in BMO_\omega$. Define $P_t f = \phi_t * f$, $Q_t f = \psi_t * f$, where $\phi_t(x) = t^{-n}\phi(x/t)$. Then the kernel of the paraproduct operator*

$$T_b f = \int_0^\infty Q_t((Q_t b)(P_t f))\,\frac{dt}{t}$$

satisfies the second ω-standard estimate (1.2) *and $T_b : H^{p'}(\omega) \to L^{p'}(\omega^{1-p'})$.*

Proof. The kernel of the operator is

$$K(x,y) = \int_0^\infty l_t(x,y)\,\frac{dt}{t} = \int_0^\infty \int_{\mathbb{R}^n} \psi_t(x-z) Q_t b(z) \phi_t(z-y)\,dz\,\frac{dt}{t}.$$

We have

$$\begin{aligned}
&|l_t(x,y)-T_{x'}l_t(x,y)|\\
&= \left|\frac{1}{(N+1)!}((x-x')\cdot D_x)^{N+1}l_t(\xi,y)\right|\\
&= \frac{1}{(N+1)!}\left|\int ((x-x')\cdot D)^{N+1}\psi_t(\xi-z)Q_tb(z)\phi_t(z-y)\,dz\right|
\end{aligned}$$

for some $\xi = x'+\theta(x-x')$, $0\le\theta\le 1$, and $|x-x'|<\frac{1}{2}|x-y|$. Because $b\in BMO_\omega$ and $\psi_t\in H^1(\omega)$ (in fact, $(\omega_t(z))^{-1}\psi_t(z-\cdot)$ is a $(1,\infty)$-atom), we have

$$|Q_tb(z)| = \left|\int \psi_t(z-u)b(u)\,du\right| \le \|\psi_t\|_{H^1(\omega)}\|b\|_* = C_\psi\|b\|_*\omega_t(z)\,.$$

Then for any x with $|z-x|<t$, we have

$$|Q_tb(z)|\le C\|b\|_*\omega_t(z)\le C\|b\|_*\omega_t(x)\,,$$

and

$$|l_t(x,y)-T_{x'}l_t(x,y)|\le \frac{C_{\phi,\psi}}{(N+1)!}|x-x'|^{N+1}\frac{\omega_t(x)}{t^{N+n+1}}\chi_{\{|x-y|<4t\}}(t)$$

whenever $|x-x'|<\frac{1}{2}|x-y|$. Therefore,

$$\begin{aligned}
|K(x,y)-T_{x'}K(x,y)| &\le \int_0^\infty |l_t(x,y)-T_{x'}l_t(x,y)|\,\frac{dt}{t}\\
&\le C|x-x'|^{N+1}\int_{|x-y|/4}^\infty \frac{\omega_t(x)}{t^{N+n+1}}\,\frac{dt}{t}\\
&= C|x-x'|^{N+1}\int_{|x-y|/4}^\infty \frac{\omega(B(x,t))}{t^{N+2n+1}}\,\frac{dt}{t}\,.
\end{aligned}$$

Take $\overline{p}_0>p_0$ such that $N+1-(n\overline{p}_0-n)>0$ and $\omega\in A_{\overline{p}_0}$. Then

$$\omega(B(x,t))\le\left(\frac{t}{|x-x'|}\right)^{n\overline{p}_0}\omega(B(x,|x-x'|)) = \frac{t^{n\overline{p}_0}}{|x-x'|^{n\overline{p}_0-n}}\omega_{|x-x'|}(x)\,,$$

and we obtain

$$\begin{aligned}
&|K(x,y)-T_{x'}K(x,y)|\\
&\le C|x-x'|^{N-(n\overline{p}_0-n)+1}\omega_{|x-x'|}(x)\int_{|x-y|/4}^\infty \frac{t^{n\overline{p}_0}}{t^{N+2n+1}}\,\frac{dt}{t}\\
&= C\omega_{|x-x'|}(x)|x-x'|^{N+1-(n\overline{p}_0-n)}\frac{1}{|x-y|^{n+N+1-(n\overline{p}_0-n)}}
\end{aligned}$$

whenever $|x-x'|<\frac{1}{2}|x-y|$, where $\varepsilon=N+1-(n\overline{p}_0-n)>0$.

The same argument as above can be applied to the estimate of $|K(x,y)-T_{y'}K(x,y)|$. This proves that the kernel of T_b satisfies the second ω-standard estimate (1.2) for $\varepsilon=N+1-(n\overline{p}_0-n)$.

We check the boundedness of $T_b: H^{p'}(\omega)\to L^{p'}(\omega^{1-p'})$ as follows. For $g\in L^p(\omega)$, we have

$$\begin{aligned}
&\left|\int T_bf(x)\cdot g(x)\,dx\right|\\
&= \left|\int_0^\infty\int_{\mathbb{R}^n} Q_t(g)Q_t(b)P_t(f)\,\frac{dx\,dt}{t}\right|\\
&\le \left(\int\left(\int_0^\infty |Q_tg|^2\,\frac{dt}{t}\right)^{p/2}\omega(x)\,dx\right)^{1/p}\\
&\quad\times\left(\int\left(\int_0^\infty |Q_tb|^2|P_tf|^2\,\frac{dt}{t}\right)^{p'/2}\omega(x)^{1-p'}\,dx\right)^{1/p'}\\
&\le C\|g\|_{L^p(\omega)}\left(\int\left(\int_0^\infty |Q_tb|^2|P_tf|^2\,\frac{dt}{t}\right)^{p'/2}\omega(x)^{1-p'}\,dx\right)^{1/p'}.
\end{aligned}$$

Now

$$\begin{aligned}
&\int\left(\int_0^\infty |Q_tb|^2|P_tf|^2\,\frac{dt}{t}\right)^{p'/2}\omega(x)^{1-p'}\,dx\\
&\qquad= \int\left(\int_0^\infty |Q_tb|^{p'}|P_tf|^{p'}g_t(x)\,\frac{dt}{t}\right)\omega(x)^{1-p'}\,dx\,,
\end{aligned}$$

for some $g_t\ge 0$ with $\int_0^\infty |g_t(x)|^r\,\frac{dt}{t}=1$ for a.e. $x\in\mathbb{R}^n$, where $1/r+p'/2=1$.

CLAIM. *$|Q_tb|^{p'}g_t(x)\omega(x)^{1-p'}\,\frac{dx\,dt}{t}$ is an ω-Carleson measure.*

Proof. Because $\omega^{1-p'}\in A_{p'}$, for any interval I, we have

$$\begin{aligned}
&\int_0^{|I|^{1/n}}\int_I |Q_tb|^{p'}g_t(x)\omega(x)^{1-p'}\,\frac{dt\,dx}{t}\\
&\qquad= \int_0^{|I|^{1/n}}\int_I |Q_t((b-P_Ib)\chi_{I^*})|^{p'}g_t(x)\,\frac{dt}{t}\,\omega(x)^{1-p'}\,dx\\
&\qquad\le \int\left(\int_0^\infty |Q_t((b-P_Ib)\chi_{I^*})|^2\,\frac{dt}{t}\right)^{p'/2}\omega(x)^{1-p'}\,dx
\end{aligned}$$

$$\le \int_{I^*} |b - P_I b|^{p'} \omega(x)^{1-p'}\,dx \le C\omega(I)\|b\|_*^{p'}.$$

Therefore,

$$\int \Big(\int_0^\infty |Q_t b|^2 |P_t f|^2 \frac{dt}{t} \Big)^{p'/2} \omega(x)^{1-p'}\,dx$$

$$= \int\int |P_t f|^{p'} |Q_t b|^{p'} g_t(x) \omega(x)^{1-p'} \frac{dx\,dt}{t}$$

$$\le C\|b\|_*^{p'} \int f^{*p'} \omega(x)\,dx = C\|b\|_*^{p'} \|f\|_{H^{p'}(\omega)}^{p'}.$$

where $f^*(x) = \sup_{|y-x|<t} |P_t f(y)|$. Consequently,

$$\Big| \int T_b f(x) \cdot g(x)\,dx \Big| \le C\|g\|_{L^p(\omega)} \|f\|_{H^{p'}(\omega)} \|b\|_*.$$

This proves that $T_b : H^{p'}(\omega) \to L^{p'}(\omega^{1-p'})$.

Unlike the case in [3] and [4], we cannot obtain the same estimate for T_b^t by duality. Therefore we take another approach.

Lemma 3.2. *Let T be a continuous linear operator from test functions to distributions that is associated with a kernel $K(x,y)$ which satisfies the second ω-standard estimate* (1.2). *If $T : H^q(\omega) \to L^q(\omega^{1-q})$ for some $1 \le q < \infty$, then $T^t : L^\infty \to BMO_\omega$.*

Proof. For any $f \in \mathcal{S}$ (Schwartz class), and any interval I, we can write

$$f = f\chi_{I^*} + f\chi_{I^{*c}} = f_1 + f_2, \quad \text{where } I^* = 4I.$$

We consider $T^t f_2$ first. Because T is associated with the kernel $K(x,y)$,

$$T^t f_2(y) = \int K(x,y) f_2(x)\,dx.$$

Take $y_0 \in 2I \setminus \frac{3}{2}I$, let $Pf_2(y) = \int T_{y_0} K(x,y) f_2(x)\,dx$, so $Pf_2(y)$ is a polynomial of degree $\le N$. Take $p > \max(2,q)$ such that $\omega \in A_p$. Then

$$\int_I |T^t f_2(y) - Pf_2(y)|^{p'} \omega^{1-p'}(y)\,dy$$

$$\le \int_I \Big(\int |K(x,y) - T_{y_0} K(x,y)| |f_2(x)|\,dx \Big)^{p'} \omega(y)^{1-p'}\,dy$$

$$\le \int_I \Big(\int_{I^{*c}} \frac{C\omega_{|y-y_0|}(y)|y-y_0|^\varepsilon}{|x-y|^{n+\varepsilon}} |f_2(x)|\,dx \Big)^{p'} \omega(y)^{1-p'}\,dy$$

$$\le \int_I (C\omega_{|y-y_0|}(y)\|f\|_\infty)^{p'} \omega(y)^{1-p'}\,dy$$

$$\le C\|f\|_\infty^{p'} \int_I \Big(\frac{\omega(I)}{|I|}\Big)^{p'} \omega(y)^{1-p'}\,dy$$

$$= C\|f\|_\infty^{p'} \frac{1}{|I|^{p'}} \Big(\int_I \omega(y)^{1-p'}\,dy \Big) \Big(\int_I \omega(y)\,dy \Big)^{p'}.$$

Because $\omega \in A_p$,

$$\Big(\frac{1}{\omega(I)} \int_I |T^t f_2(y) - Pf_2(y)|^{p'} \omega(y)^{1-p'}\,dy \Big)^{1/p'}$$

$$\le C\|f\|_\infty \Big(\frac{1}{|I|} \int_I \omega\,dy \Big)^{1-1/p'} \Big(\frac{1}{|I|} \int_I \omega^{1-p'}\,dy \Big)^{1/p'} \le C\|f\|_\infty.$$

For $T^t f_1$, we have

$$\Big(\frac{1}{\omega(I)} \int_I |T^t f_1 - P_I T^t f_1|^{p'} \omega^{1-p'}\,dx \Big)^{1/p'}$$

$$= \sup_{\substack{\operatorname{supp} a \subset I \\ \int x^k a\,dx = 0,\ k=0,1,\ldots,N \\ (\omega(I)^{-1} \int_I |a|^p \omega\,dx)^{1/p} \le \omega(I)^{-1}}} \Big| \int (T^t f_1) a\,dx \Big|$$

$$= \sup_{a \text{ a } (1,p)\text{-atom}} \Big| \int f_1 Ta\,dx \Big| \le \sup_{a \text{ a } (1,p)\text{-atom}} \int_{I^*} |Ta|\,dx \cdot \|f\|_\infty.$$

Now for a a $(1,p)$-atom, we have

$$\int_{I^*} |Ta|\,dx \le \Big(\int_{I^*} |Ta|^q \omega^{1-q}\,dx \Big)^{1/q} \Big(\int_{I^*} \omega\,dx \Big)^{1/q'}$$

$$\le C\omega(I)^{1/q'} \Big(\int |Ta|^q \omega^{1-q}\,dx \Big)^{1/q} \le C\omega(I)^{1/q'} \|a\|_{H^q(\omega)}.$$

Following the same method as in [5, Chapter 2, §3], we get

$$\|a\|_{H^q(\omega)} \le C\omega(I)^{-1/q'}$$

(C does not depend on a). Thus

$$\Big(\frac{1}{\omega(I)} \int_I |T^t f_1 - P_I T^t f_1|^{p'} \omega^{1-p'}\,dx \Big)^{1/p'} \le C\|f\|_\infty.$$

Combining the two parts, we have proved

$$\|T^t f\|_{BMO_\omega} \le C\|f\|_\infty.$$

This means $T^t : L^\infty \to BMO_\omega$.

As an immediate consequence, $T_b^t : L^\infty \to BMO_\omega$. By duality, we have $T_b : H^1(\omega) \to L^1$.

LEMMA 3.3. *Let T be a continuous linear operator from test functions to distributions that is associated with a kernel $K(x,y)$ which satisfies the second ω-standard estimate* (1.2). *If $T : L^\infty \to BMO_\omega$, then $T : H^1(\omega) \to L^1$.*

The proof of this lemma is almost the same as in [7, pp. 49–51]. For completeness, we present it here.

Proof. We need only estimate $\int |Ta|\,dx$ for a a $(1,\infty)$-atom. Suppose a is a $(1,\infty)$-atom associated with an interval Q, and $\overline{Q} = 4Q$. Then

$$\int_{\overline{Q}^c} |Ta|\,dx = \int_{\overline{Q}^c} \Big| \int_Q K(x,y)a(y)\,dy \Big|\,dx\,.$$

Because $\int x^k a\,dx = 0$ for $k = 0,1,\ldots,N$, it follows that, for any y_0 in $2Q \setminus \frac{3}{2}Q$,

$$\int_{\overline{Q}^c} |Ta|\,dx = \int_{\overline{Q}^c} \Big| \int_Q (K(x,y) - T_{y_0}K(x,y))a(y)\,dy \Big|\,dx$$

$$\le C \int_{\overline{Q}^c} \int_Q \frac{\omega_{|y-y_0|}(y)|y-y_0|^\varepsilon}{|x-y|^{n+\varepsilon}} |a(y)|\,dy\,dx \le C\,,$$

where C does not depend on a.

Let Q' be a cube of the same size as $\overline{Q}$, and such that Q' and $\overline{Q}$ have a face in common. Because $T : L^\infty \to BMO_\omega$, $\|Ta\|_{BMO_\omega} \le C/\omega(Q)$. By the above calculation,

$$\int_{Q'} |Ta|\,dx \le C\,.$$

Now,

$$\frac{1}{\omega(\overline{Q})} \int_{\overline{Q}} |Ta|\,dx$$

$$\le \frac{1}{\omega(\overline{Q})} \int_{\overline{Q}} |Ta - P_{\overline{Q}}Ta|\,dx + \frac{1}{\omega(\overline{Q})} \int_{\overline{Q}} |P_{\overline{Q}}Ta|\,dx$$

$$\le \Big(\frac{1}{\omega(\overline{Q})} \int_{\overline{Q}} |Ta - P_{\overline{Q}}Ta|^{p'} \omega^{1-p'}\,dx \Big)^{1/p'} + \frac{1}{\omega(\overline{Q})} \int_{\overline{Q}} |P_{\overline{Q}}Ta|\,dx$$

$$\le C\|a\|_\infty + \frac{1}{\omega(\overline{Q})} \int_{\overline{Q}} |P_{\overline{Q}}Ta|\,dx$$

$$\le C\omega(Q)^{-1} + \frac{1}{\omega(\overline{Q})} \int_{\overline{Q}} |P_{\overline{Q}}Ta|\,dx\,,$$

and

$$\int_{\overline{Q}} |P_{\overline{Q}}Ta|\,dx$$

$$\le \int_{\overline{Q}} |P_{\overline{Q}}Ta - P_{Q'}Ta|\,dx + \int_{\overline{Q}} |P_{Q'}Ta|\,dx$$

$$\le \int_{\overline{\overline{Q}}} |P_{Q'}(Ta - P_{\overline{\overline{Q}}}Ta)|\,dx + \int_{\overline{\overline{Q}}} |P_{\overline{Q}}(Ta - P_{\overline{\overline{Q}}}Ta)|\,dx + \int_{\overline{\overline{Q}}} |P_{Q'}Ta|\,dx,$$

where $\overline{\overline{Q}}$ is a cube containing $\overline{Q} \cup Q'$ and with size comparable to Q. (For example, $|\overline{\overline{Q}}| \le 10^n|Q|$.) Because

$$\int_{\overline{Q}} |P_Q f|\,dx \le C \int_Q |f|\,dx$$

for any $\overline{Q} \supset Q$ and $|\overline{Q}| \le 2^n|Q|$,

$$\int_{\overline{Q}} |P_{\overline{Q}}Ta|\,dx$$

$$\le C\Big(\int_{\overline{\overline{Q}}} |Ta - P_{\overline{\overline{Q}}}Ta|\,dx + \int_{\overline{\overline{Q}}} |Ta - P_{\overline{\overline{Q}}}Ta|\,dx + \int_{Q'} |Ta|\,dx \Big)$$

$$\le C\|a\|_\infty \omega(\overline{\overline{Q}}) + C \le C\,.$$

Therefore,

$$\frac{1}{\omega(\overline{Q})} \int_{\overline{Q}} |Ta|\,dx \le C\omega(Q)^{-1} + C\frac{1}{\omega(\overline{Q})}\,,$$

and then $\int_{\overline{Q}} |Ta|\,dx \le C$, which implies

$$\int |Ta|\,dx \le \int_{\overline{Q}} |Ta|\,dx + \int_{\overline{Q}^c} |Ta|\,dx \le C\,.$$

This proves

$$T : H^1(\omega) \to L^1\,.$$

Up to now, we have proved that $T_b : H^1(\omega) \to L^1$ and $T_b^t : H^1(\omega) \to L^1$. Take ϕ, ψ as in Lemma 3.1 such that $\int_0^\infty Q_t^2 \frac{dt}{t} = I$, $\int \phi\,dx = 1$, and $b = T1$, $b_1 = T^t1$. Then $M = T - T_b - T_{b_1}^t$ satisfies $M1 = M^t1 = 0$ and all the hypotheses of Theorem 2.1 except the first ω-standard estimate (1.1). Therefore, it suffices to prove the sufficiency of Theorem 2.1 in the case where $T1=0$, and $T^t1=0$. As in [3] and [4], we can write

$$T=\Big(\int_0^\infty Q_s^2\frac{ds}{s}\Big)T\Big(\int_0^\infty Q_t^2\frac{dt}{t}\Big)=\int_0^\infty\int_0^\infty Q_s(Q_sTQ_t)Q_t\,\frac{dt}{t}\,\frac{ds}{s}.$$

The fact that $T:H^1(\omega)\to L^1$ now follows from the following two lemmas.

LEMMA 3.4. *Same notations and assumptions as in Theorem 2.1 and Lemma 3.1. Assume also that ϕ,ψ satisfy the above hypotheses, and $T1=0$, $T^t1=0$. Then the kernel $K_{s,t}(x,y)$ of Q_sTQ_t satisfies:*

(1) *for* $0\le s\le t$,

$$|K_{s,t}(x,y)|\le\Big(\frac{s}{t}\Big)^{\varepsilon}\frac{t^{\varepsilon}}{(|x-y|+t)^{n+\varepsilon}}\omega_s(x),$$

(2) *for* $0\le t\le s$,

$$|K_{s,t}(x,y)|\le\Big(\frac{t}{s}\Big)^{\varepsilon}\frac{s^{\varepsilon}}{(|x-y|)^{n+\varepsilon}}\omega_t(x).$$

The proof of this lemma is the same as that of [3, Lemma 2.3], so we omit it.

LEMMA 3.5. *If $K_{s,t}(x,y)$ satisfies the estimates in Lemma 3.4 and $f\in H^{p'}(\omega)$, $g\in L^p(\omega)$, then*

$$\int_0^\infty\int_0^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|Q_sg(x)||K_{s,t}(x,y)||Q_tf(y)|\,dx\,dy\,\frac{ds}{s}\,\frac{dt}{t}\le C\|f\|_{H^{p'}(\omega)}\|g\|_{L^p(\omega)}.$$

Proof. By assumption, the integral is controlled by

$$C\iint_{s<t}\iint_{\mathbb{R}^{2n}}|Q_sg(x)|\omega_s(x)\Big(\frac{s}{t}\Big)^{\varepsilon}\times\frac{t^{\varepsilon}}{(|x-y|+t)^{n+\varepsilon}}|Q_tf(y)|\,dx\,dy\,\frac{ds}{s}\,\frac{dt}{t}\tag{3.6}$$

plus an analogous term for $s>t$. Setting

$$P_t^{\varepsilon}h(x)=\int_{\mathbb{R}^n}\frac{t^{\varepsilon}}{(|x-y|+t)^{n+\varepsilon}}h(y)\,dy,$$

we rewrite (3.6) and dominate it by

$$C\int_0^\infty\int_{\mathbb{R}^n}\int_0^t|Q_sg(x)|\omega_s(x)\Big(\frac{s}{t}\Big)^{\varepsilon}P_t^{\varepsilon}(|Q_tf|)(x)\frac{ds}{s}\frac{dx\,dt}{t}$$

$$=C\int_0^\infty\int_{\mathbb{R}^n}\Big(\int_0^t|Q_sg(x)|\frac{1}{s^n}\int_{|z-x|<s}\omega(z)\,dz\Big(\frac{s}{t}\Big)^{\varepsilon}\frac{ds}{s}\Big)P_t^{\varepsilon}(|Q_tf|)(x)\frac{dx\,dt}{t}$$

$$\le C\int_0^\infty\int_{\mathbb{R}^n}\omega(z)\int_0^t\Big(\frac{s}{t}\Big)^{\varepsilon}\int_{|x-z|<s}|Q_sg(x)|\times\frac{s}{(s+|z-x|)^{n+1}}P_t^{\varepsilon}(|Q_tf|)(x)\,dx\,\frac{ds}{s}\,\frac{dz\,dt}{t}$$

$$\le C\int_0^\infty\int_{\mathbb{R}^n}\omega(z)\Big(\int_0^t\Big(\frac{s}{t}\Big)^{\varepsilon}\frac{ds}{s}\Big)\times\Big(\int_{|x-z|<s}\frac{s}{(s+|z-x|)^{n+1}}|Q_sg(x)|^2\,dx\Big)^{1/2}\times\Big(\int_{|x-z|<t}\frac{s}{(s+|z-x|)^{n+1}}(P_t^{\varepsilon}(|Q_tf|)(x))^2\,dx\Big)^{1/2}\frac{dz\,dt}{t}$$

$$\le C\int_0^\infty\int_{\mathbb{R}^n}\omega(z)\int_0^t\Big(\frac{s}{t}\Big)^{\varepsilon}Q_{1,s}(|g|)(z)\frac{ds}{s}\cdot Q_{2,t}(|f|)(z)\frac{dz\,dt}{t}$$

$$\le C\int_{\mathbb{R}^n}\omega(z)\Big(\int_0^\infty\Big(\int_0^t\Big(\frac{s}{t}\Big)^{\varepsilon}Q_{1,s}(|g|)(z)\frac{ds}{s}\Big)^2\frac{dt}{t}\Big)^{1/2}\times\Big(\int_0^\infty Q_{2,t}^2(|f|)(z)\frac{dt}{t}\Big)^{1/2}dz$$

$$\le C\Big(\int_{\mathbb{R}^n}\omega(z)\Big(\int_0^\infty\Big(\int_0^t\Big(\frac{s}{t}\Big)^{\varepsilon}Q_{1,s}(|g|)(z)\frac{ds}{s}\Big)^2\frac{dt}{t}\Big)^{p/2}dz\Big)^{1/p}\times\Big(\int_{\mathbb{R}^n}\omega(z)\Big(\int_0^\infty Q_{2,t}^2(|f|)(z)\frac{dt}{t}\Big)^{p'/2}dz\Big)^{1/p'}$$

$$\le C\Big(\int_{\mathbb{R}^n}\omega(z)\Big(\int_0^\infty\int_0^t\Big(\frac{s}{t}\Big)^{\varepsilon}Q_{1,s}^2(|g|)(z)\frac{ds}{s}\frac{dt}{t}\Big)^{p/2}dz\Big)^{1/p}\|f\|_{H^{p'}(\omega)}$$

$$\le C\Big(\int_{\mathbb{R}^n}\omega(z)\Big(\int_0^\infty Q_{1,s}^2(|g|)(z)\frac{ds}{s}\Big)^{p/2}dz\Big)^{1/p}\|f\|_{H^{p'}(\omega)}$$

$$\le C\Big(\int_{\mathbb{R}^n}|g^*|^p\omega(z)\,dz\Big)^{1/p}\|f\|_{H^{p'}(\omega)}\le C\|g\|_{L^p(\omega)}\|f\|_{H^{p'}(\omega)},$$

because $\omega \in A_p$. The seventh and ninth inequalities are true according to the results in [2]. Now we have proved the sufficiency of Theorem 2.1. By Lemma 3.3 and duality, we obtain the sufficiency of Corollary 2.2.

The necessity of Theorem 2.1 and Corollary 2.2 is proved by using Lemma 3.3 and duality. Because $T : H^1(\omega) \to L^1$, by duality, we have $T^t : L^\infty \to BMO_\omega$, and then by Lemma 3.3, we have $T^t : H^1(\omega) \to L^1$, and so $T : L^\infty \to BMO_\omega$. Therefore, $T1 \in BMO_\omega$ and $T^t 1 \in BMO_\omega$.

The fact that T satisfies the ω-WBP is an easy consequence of $T : H^1(\omega) \to L^1$.

4. The Calderón–Zygmund theory for ω-CZO. The interpolation result in [9, Chapter 2, §3] enables us to establish the Calderón–Zygmund theory for ω-CZO.

THEOREM 4.1. *Suppose that $\omega \in A_\infty$. Let T be a continuous linear operator from test functions to distributions that is associated with an ω-standard kernel. Suppose further that $T : H^1(\omega) \to L^1$. Then $T : H^p(\omega) \to (H^{p'}(\omega))^*$ for $1 < p < \infty$, where $1/p + 1/p' = 1$ and $(H^{p'}(\omega))^*$ is the dual space of $H^{p'}(\omega)$. Precisely, we have*

$$\int \Big(\sum_{2I \ni x} \Big((Tf)_I \frac{|I|}{\omega(I)}\Big)^2 |I|^{-1}\Big)^{p/2} \omega(x)\, dx \le C\|f\|^p_{H^p(\omega)} \tag{4.2}$$

where $(Tf)_I = \int Tf\, \psi_I\, dx$ with the notations as in [9, *Chapter* 2].

Proof. By Corollary 2.2, we have $T : L^\infty \to BMO_\omega$. Applying Tom Wolff's theorem [8] and the interpolation result in [9], we obtain $[H^1(\omega), L^\infty]_\theta = H^p(\omega)$, where $1/p = 1 - \theta$. By the duality theorem and the reiteration theorem [1] and Tom Wolff's theorem [8],

$$\begin{aligned}[L^1, BMO_\omega]^\theta &\subset [(L^\infty)^*, BMO_\omega]^\theta = [(L^\infty)^*, (H^1(\omega))^*]^\theta \\ &= (H^r(\omega))^* = [L^1, BMO_\omega]_\theta\,,\end{aligned}$$

where $1/r = \theta$. Therefore we have $T : H^p(\omega) \to (H^{p'}(\omega))^*$, where $1 < p < \infty$, $1/p + 1/p' = 1$. (4.2) is a direct consequence of the duality result in [9].

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