STUDIA MATHEMATICA 112 (2) (1995)

Commutativity of compact selfadjoint operators

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Abstract. The relationship between the *joint spectrum* $\gamma(A)$ of an n-tuple $A = (A_1, \ldots, A_n)$ of selfadjoint operators and the support of the corresponding *Weyl calculus* $T(A) : f \mapsto f(A)$ is discussed. It is shown that one always has $\gamma(A) \subset \operatorname{supp}(T(A))$. Moreover, when the operators are compact, equality occurs if and only if the operators A_j mutually commute. In the non-commuting case the equality fails badly: While $\gamma(A)$ is countable, $\operatorname{supp}(T(A))$ has to be an uncountable set. An example is given showing that, for non-compact operators, coincidence of $\gamma(A)$ and $\operatorname{supp}(T(A))$ no longer implies commutativity of the set $\{A_i\}$.

Introduction. A notion of joint spectrum $\gamma(A)$ for a *commuting* n-tuple of bounded linear operators $A = (A_1, \ldots, A_n)$ in a Banach space X was introduced by McIntosh and Pryde in [5]. Namely

$$\gamma(A) = \Big\{\lambda \in \mathbb{R}^n : 0 \in \sigma\Big(\sum_{j=1}^{n}(A_j - \lambda_j I)^2\Big)\Big\}, \tag{1}$$

where $\sigma(B)$ is the usual spectrum of a single operator B. For n-tuples A satisfying $\sigma(A_j) \subset \mathbb{R}$, $1 \le j \le n$, this particular joint spectrum $\gamma(A)$ coincides with most other known joint spectra [6], and has proved to be effective in the solution of certain linear systems of operator equations [5, 7].

For commuting n-tuples A satisfying an estimate of the form

$$\|e^{i\langle \xi, A\rangle}\| \le C(1 + |\xi|)^s, \qquad \xi \in \mathbb{R}^n,$$

for some positive constants C and s (where $\langle \xi, A\rangle = \sum_{j=1}^{n} \xi_j A_j$ and $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^n$) it turns out that $\gamma(A)$ is precisely the support, $\operatorname{supp}(T(A))$, of a certain functional calculus $T(A) : \mathcal{A}_s \to \mathcal{L}(X)$, that is,

$$\operatorname{supp}(T(A)) = \gamma(A) \tag{2}$$

1991 *Mathematics Subject Classification*: 47A13, 47A25, 47A60, 47B15.

The support of an Alexander von Humboldt Fellowship is gratefully acknowledged by the second author.

(see [5], for example). Here $\mathcal{A}_s$ is an algebra of functions containing the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of all rapidly decreasing, $\mathbb{C}$-valued functions on $\mathbb{R}^n$ and $\mathcal{L}(X)$ is the space of all bounded linear operators of X into itself. For the case $s = 0$, the algebra $\mathcal{A}_s$ reduces to $\mathcal{S}(\mathbb{R}^n)$ itself and the calculus $T(A) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{L}(X)$, given by the formula

$$(3) \qquad T(A)f = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle \xi, A\rangle} \widehat{f}(\xi)\, d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^n),$$

may be interpreted as an operator-valued distribution (where $\widehat{f}$ denotes the Fourier transform of f). In this case $T(A)$ is called the Weyl calculus of A [1, 2, 12], and $\operatorname{supp}(T(A))$ is precisely the support of $T(A)$ in the usual sense for distributions; it is always a non-empty compact subset of $\mathbb{R}^n$ [1, Lemma 2.3].

So, for commuting n-tuples A which generate bounded groups $\xi \mapsto e^{i\langle \xi, A\rangle}$, $\xi \in \mathbb{R}^n$, the joint spectral set $\gamma(A)$ is intimately related to the Weyl calculus $T(A)$. An examination of (1) shows that the definition of $\gamma(A)$, unlike many other joint spectra, also makes perfectly good sense for *non-commuting* n-tuples A. Moreover, the recent articles [8, 9] show that $\gamma(A)$ also has useful applications in the non-commutative setting. Of course, the Weyl calculus (3) is also well defined for certain non-commuting n-tuples A; indeed, it was introduced by H. Weyl precisely because of this point. So, the natural question is: *How closely related are the sets $\gamma(A)$ and* $\operatorname{supp}(T(A))$ *in general?*

The aim of this note is to give a detailed answer to this question for the case of n-tuples A of selfadjoint operators in Hilbert space. A suggestion as to what might be expected can be found in [3] where a detailed study is made of certain properties of the sets $\gamma(A)$ for (possibly) non-commutative A (call A commutative if the operators A_j, $1 \le j \le n$, mutually commute). For an n-tuple A of selfadjoint operators in a 2-dimensional Hilbert space it is known that A is commutative if and only if (2) holds [3, Proposition 8]. We show that the same is true in any finite-dimensional Hilbert space H; see Theorem 2. Moreover, commutativity of A turns out to be equivalent to $\operatorname{supp}(T(A))$ being a finite subset of $\mathbb{R}^n$ with at most k elements, where $k = \dim(H)$. For non-commutative n-tuples A (still with $\dim(H) < \infty$) the equality (2) "fails badly". Indeed, the set $\gamma(A)$ remains finite (always being a subset of $\sigma(A_1) \times \ldots \times \sigma(A_n)$) whereas $\operatorname{supp}(T(A))$ is necessarily an uncountable subset of $\mathbb{R}^n$; see Theorem 3. This dichotomy makes it somewhat unclear what to expect in arbitrary Hilbert spaces. Surprisingly, for n-tuples A consisting of *compact* selfadjoint operators the analogy with the finite-dimensional case is rather close. It turns out that equality in (2) is still equivalent to commutativity of A, which, in turn, is equivalent to $\operatorname{supp}(T(A))$ being a countable subset of $\mathbb{R}^n$ (cf. Theorem 4). So (curiously),

the commutativity of A is equivalent to the equality of a purely algebraic notion (namely, the set $\gamma(A)$) with a purely analytic notion (namely, the set $\operatorname{supp}(T(A))$).

The main ingredients in the proofs of the above results are the notion of the maximal abelian subspace of A (introduced in [3]), Theorem 1 below which states that particular kinds of isolated points of $\operatorname{supp}(T(A))$ (called hyperisolated) are joint eigenvalues of A, and the fact (cf. Proposition 4) that every isolated point of $\operatorname{supp}(T(A))$ is hyperisolated whenever $\operatorname{supp}(T(A))$ is a countable set.

Since any compact subset of $\mathbb{R}^n$ is the support of some (even commuting) n-tuple of bounded selfadjoint operators [1, p. 255], it cannot be expected that Theorem 4 has a larger range of applicability. Indeed, we exhibit a pair $A = (A_1, A_2)$ of bounded selfadjoint (but not compact) operators A_1 and A_2 in an infinite-dimensional Hilbert space for which equality in (2) does hold, but such that $A_1A_2 \ne A_2A_1$; see Example 1.

In the final section of the paper a study is made, for pairs $A = (A_1, A_2)$ of compact selfadjoint operators A_1 and A_2, of the connection between the sets $\gamma(A)$, $\operatorname{supp}(T(A))$ and $\sigma(A_1 + iA_2)$ with the aim of extending Proposition 10 of [3] from 2-dimensional spaces to finite-dimensional spaces.

1. Basic properties of $\gamma(A)$ and $\operatorname{supp}(T(A))$. In this section we collect together some basic facts about the sets $\gamma(A)$ and $\operatorname{supp}(T(A))$ which are needed in the sequel. We begin with a simple but useful result.

LEMMA 1. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of bounded selfadjoint operators in a Hilbert space H and M be a closed linear subspace of H which is invariant for A (i.e., invariant for each operator A_j, $j = 1, \ldots, n$).*

(i) *The orthogonal complement $M^\perp$ is invariant for each operator A_j, $1 \le j \le n$.*

(ii) *If A_M (respectively, $A_{M^\perp}$) denotes the selfadjoint n-tuple in the Hilbert space M (respectively, $M^\perp$) consisting of the restrictions of A_j, $1 \le j \le n$, to M (respectively, $M^\perp$), then*

(a) $\operatorname{supp}(T(A)) = \operatorname{supp}(T(A_M)) \cup \operatorname{supp}(T(A_{M^\perp}))$, *and*

(b) $\gamma(A) = \gamma(A_M) \cup \gamma(A_{M^\perp})$.

Proof. (i) follows from $A_j^*(M^\perp) \subset M^\perp$ and the selfadjointness of each A_j, $1 \le j \le n$.

(ii) We have $H = M \oplus M^\perp$ and $A_j = (A_j)_M \oplus (A_j)_{M^\perp}$ for each $j = 1, \ldots, n$.

(a) It follows that $(i\langle \xi, A\rangle)^r = (i\langle \xi, A_M\rangle)^r \oplus (i\langle \xi, A_{M^\perp}\rangle)^r$, $\xi \in \mathbb{R}^n$, $r \in \mathbb{N}$, and hence, via the power series expansion of the exponential function, that

$$e^{i\langle \xi, A\rangle} = e^{i\langle \xi, A_M\rangle} \oplus e^{i\langle \xi, A_{M^\perp}\rangle}, \qquad \xi \in \mathbb{R}^n.$$

It is then clear from the definition of $T(A)f$ as a Bochner integral with respect to the uniform operator topology of $\mathcal{L}(H)$ (see (3)) that

$$T(A)f = T(A_M)f \oplus T(A_{M^\perp})f, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

from which (a) follows.

(b) follows from the formulae

$$\sum_{j=1}^{n}(\lambda_j I - A_j)^2 = \sum_{j=1}^{n}(\lambda_j I - (A_j)_M)^2 \oplus \sum_{j=1}^{n}(\lambda_j I - (A_j)_{M^\perp})^2, \quad \lambda \in \mathbb{R}^n,$$

together with the fact that $U \oplus V$ is invertible in $H = M \oplus M^\perp$ if and only if U is invertible in M and V is invertible in $M^\perp$. ∎

We recall that $\lambda \in \mathbb{C}^n$ is called a *joint eigenvalue* of an n-tuple of bounded operators $A = (A_1, \ldots, A_n)$ if there exists a non-zero vector $x \in H$ such that $A_j x = \lambda_j x$ for each $j = 1, \ldots, n$. The vector x is then called a *joint eigenvector* of A corresponding to λ.

LEMMA 2. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of bounded selfadjoint operators in a Hilbert space H and $\lambda \in \mathbb{R}^n$ be a joint eigenvalue of A. Then $\lambda \in \gamma(A) \cap \operatorname{supp}(T(A))$.*

Proof. Let $x \neq 0$ be a joint eigenvector of A corresponding to λ. A simple calculation (using power series expansion) shows that $e^{i\langle \xi, A\rangle}x = e^{i\langle \xi, \lambda\rangle}x$, $\xi \in \mathbb{R}^n$. It then follows from (3) and the Fourier inversion theorem that

$$(4) \qquad [T(A)f]x = (2\pi)^{-n/2}\Big[\int_{\mathbb{R}^n} e^{i\langle \xi, \lambda\rangle}\widehat{f}(\xi)\,d\xi\Big]x = f(\lambda)x$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$. So, given any neighbourhood U of λ in $\mathbb{R}^n$ choose $f \in C_c^\infty(\mathbb{R}^n)$ satisfying $\operatorname{supp}(f) \subset U$ and $f(\lambda) = 1$. Then $[T(A)f]x = x \neq 0$, that is, $T(A)f \neq 0$. Accordingly, $\lambda \in \operatorname{supp}(T(A))$.

Since joint eigenvalues of A are also joint approximate eigenvalues, it follows from [3, Proposition 2] that $\lambda \in \gamma(A)$. ∎

LEMMA 3. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of bounded selfadjoint operators in a Hilbert space H. Then $\gamma(A) \subset \operatorname{supp}(T(A))$.*

Proof. Let $\lambda \in \gamma(A)$. Then λ is a joint approximate eigenvalue of A by [3, Proposition 2]. Choose vectors $x_n \in H$ satisfying $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and such that $\lim_{n\to\infty}\|A_j x_n - \lambda_j x_n\| = 0$ for $j = 1, \ldots, n$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and $H_\mathcal{U} = \ell^\infty(H)/c_\mathcal{U}(H)$ denote the $\mathcal{U}$-product of H (where $\ell^\infty(H)$ is the space of all bounded sequences in H and $c_\mathcal{U}(H)$ is the subspace of those sequences converging to 0 along $\mathcal{U}$; see [11, V.1]). Furthermore, let $(A_j)_\mathcal{U}$ be the canonical extension of A_j. Then $A_\mathcal{U} = ((A_1)_\mathcal{U}, \ldots, (A_n)_\mathcal{U})$ is an n-tuple of selfadjoint operators on the Hilbert space $H_\mathcal{U}$ and $x_\mathcal{U} = (x_n) + c_\mathcal{U}(H) \in H_\mathcal{U}$ is an eigenvector of $(A_j)_\mathcal{U}$ corresponding to λ_j, for each $1 \le j \le n$. That is, $x_\mathcal{U}$ is a joint eigenvector of $A_\mathcal{U}$ corresponding to λ. From Lemma 2 we conclude that $\lambda \in \operatorname{supp}(T(A_\mathcal{U}))$. So, it remains to show that $\operatorname{supp}(T(A_\mathcal{U})) = \operatorname{supp}(T(A))$. This will follow from the identity

$$(T(A)f)_\mathcal{U} = T(A_\mathcal{U})f, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

To establish this identity we note that the mapping $B \mapsto B_\mathcal{U}$ is an isometric homomorphism of the Banach algebra $\mathcal{L}(H)$ into $\mathcal{L}(H_\mathcal{U})$; see [11, V.1.2]. Thus we have $\langle \eta, A\rangle_\mathcal{U} = \langle \eta, A_\mathcal{U}\rangle$ for $\eta \in \mathbb{R}^n$. Then (by power series expansion) it follows that $(e^{i\langle \eta, A\rangle})_\mathcal{U} = e^{i\langle \eta, A_\mathcal{U}\rangle}$ and finally, for $f \in \mathcal{S}(\mathbb{R}^n)$, we have (by approximating the integral via Riemann sums)

$$(2\pi)^{n/2}(T(A)f)_\mathcal{U} = \Big(\int_{\mathbb{R}^n} e^{i\langle \eta, A\rangle}\widehat{f}(\eta)\,d\eta\Big)_\mathcal{U}$$
$$= \int_{\mathbb{R}^n} e^{i\langle \eta, A_\mathcal{U}\rangle}\widehat{f}(\eta)\,d\eta = (2\pi)^{n/2}T(A_\mathcal{U})f. \ ∎$$

DEFINITION 1. For $\lambda \in \mathbb{R}^n$ and $A = (A_1, \ldots, A_n)$ an n-tuple of selfadjoint operators in a Hilbert space H, define

$$H_\lambda(A) = \{0\} \cup \{x \in H : x \text{ is a joint eigenvector of } A \text{ for } \lambda\}.$$

Then $H_\lambda(A)$, called the *joint eigenspace* of λ, is a closed subspace of H. The orthogonal projection onto $H_\lambda(A)$ is denoted by $E_\lambda(A)$ and is called the *joint eigenprojection* of A corresponding to λ.

We recall that $M[A]$ denotes the *maximal abelian subspace* for A; see [3]. It is the largest closed subspace of H which is invariant for A and such that the restrictions $(A_j)_{M[A]}$ of A_j to $M[A]$, for $j = 1, \ldots, n$, mutually commute in the Hilbert space $M[A]$. The connection between $M[A]$ and the joint eigenprojections of A is given by the following

PROPOSITION 1. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of compact selfadjoint operators in a Hilbert space H. Then*

(i) *$E_\lambda(A)E_\mu(A) = 0 = E_\mu(A)E_\lambda(A)$ for all $\lambda, \mu \in \gamma(A)$ with $\lambda \neq \mu$, and*

(ii) *$M[A] = \bigoplus_{\lambda\in\gamma(A)} H_\lambda(A)$ is the closed subspace of H generated by the family of joint eigenspaces $\{H_\lambda(A) : \lambda \in \gamma(A)\}$.*

Proof. (i) Choose any index $j \in \{1, \ldots, n\}$ such that $\lambda_j \neq \mu_j$. Since A_j is selfadjoint and $\lambda_j, \mu_j \in \sigma(A_j)$ [3, Proposition 2], it follows that $\ker(A_j - \lambda_j I)$ is orthogonal to $\ker(A_j - \mu_j I)$. Since $H_\lambda(A) = \bigcap_{r=1}^n \ker(A_r - \lambda_r I)$ and $H_\mu(A) = \bigcap_{r=1}^n \ker(A_r - \mu_r I)$ it follows that $H_\lambda(A)$ is orthogonal to $H_\mu(A)$ and (i) follows.

(ii) It is clear that each closed subspace $H_\lambda(A)$, $\lambda \in \gamma(A)$, is invariant for each operator A_j, $1 \le j \le n$, and the restrictions of A_j to $H_\lambda(A)$ mutually commute. By definition of $M[A]$ it follows that the closed subspace of H generated by $\{H_\lambda(A) : \lambda \in \gamma(A)\}$ is contained in $M[A]$. On the other hand, the restrictions $(A_j)_{M[A]}$, $1 \le j \le n$, form a mutually commuting family of compact selfadjoint operators in the Hilbert space $M[A]$. Accordingly, there exists an orthonormal basis of $M[A]$ consisting of joint eigenvectors of $\{(A_j)_{M[A]} : 1 \le j \le n\}$. Each such joint eigenvector $x \in M[A]$ of $A_{M[A]}$ is also a joint eigenvector of A with the same joint eigenvalue μ as for $A_{M[A]}$. Lemma 2 implies that $\mu \in \gamma(A)$ and hence, $M[A]$ is contained in the closed subspace of H generated by $\{H_\lambda(A) : \lambda \in \gamma(A)\}$. ∎

The next result shows that for *compact* n-tuples A the Weyl calculus $T(A)$ almost takes its values in the compact operators on H.

PROPOSITION 2. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of compact selfadjoint operators in a Hilbert space H. Then $T(A)f - f(0)I$ is a compact operator for every $f \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. For $\xi \in \mathbb{R}^n$ fixed, a consideration of the power series expansion of $e^{i\langle\xi,A\rangle}$, together with the fact that each operator $(i\langle\xi, A\rangle)^r$, $r = 1, 2, \ldots$, is compact, shows that $e^{i\langle\xi,A\rangle} - I$ is compact. Let $B_N = \{x \in \mathbb{R}^n : |x| < N\}$ for each $N = 1, 2, \ldots$ and fix $f \in \mathcal{S}(\mathbb{R}^n)$. Since the map $\xi \mapsto e^{i\langle\xi,A\rangle}$, $\xi \in \mathbb{R}^n$, is continuous for the operator norm topology in $\mathcal{L}(H)$ the integral

$$K_N(f) = \int_{B_N} (e^{i\langle\xi,A\rangle} - I)\widehat{f}(\xi)\, d\xi$$

exists as the operator norm limit of Riemann sums and hence, is a compact operator. The conclusion follows from the identities

$$(2\pi)^{n/2}(T(A)f - f(0)I) = \int_{B_N} (e^{i\langle\xi,A\rangle} - I)\widehat{f}(\xi)\, d\xi + \int_{\mathbb{R}^n\setminus B_N} (e^{i\langle\xi,A\rangle} - I)\widehat{f}(\xi)\, d\xi,$$

together with the estimates

$$\Big\| \int_{\mathbb{R}^n\setminus B_N} (e^{i\langle\xi,A\rangle} - I)\widehat{f}(\xi)\, d\xi \Big\| \le 2 \int_{\mathbb{R}^n\setminus B_N} |\widehat{f}(\xi)|\, d\xi,$$

valid for $N = 1, 2, \ldots$, which show that $K_N(f) \to T(A)f - f(0)I$ as $N \to \infty$ in the operator norm topology. ∎

Given a function $f : \mathbb{R}^n \to \mathbb{C}$ and $\nu \in \mathbb{R}^n$ define the ν-translate $f_\nu : \mathbb{R}^n \to \mathbb{C}$ of f by $f_\nu(x) = f(x - \nu)$ for $x \in \mathbb{R}^n$. For a subset $K \subset \mathbb{R}^n$ let $K - \nu = \{x - \nu : x \in K\}$. Finally, if $A = (A_1, \ldots, A_n)$ is an n-tuple of elements from $\mathcal{L}(H)$ denote the n-tuple $(A_1 - \nu_1 I, \ldots, A_n - \nu_n I)$ by $A - \nu I$.

LEMMA 4. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of bounded selfadjoint operators in a Hilbert space H and $\lambda \in \mathbb{R}^n$. Then*

(i) *$T(A)f_\lambda = T(A - \lambda I)f$ for every $f \in \mathcal{S}(\mathbb{R}^n)$,*
(ii) *$\operatorname{supp}(T(A - \lambda I)) = \operatorname{supp}(T(A)) - \lambda$, and*
(iii) *$\gamma(A - \lambda I) = \gamma(A) - \lambda$.*

Proof. (i) follows from the definition of $T(A)f_\lambda$, the fact that $\widehat{f_\lambda} = e^{-i\langle\cdot,\lambda\rangle}\widehat{f}$ and the observation that

$$e^{-i\langle\xi,\lambda\rangle}e^{i\langle\xi,A\rangle} = e^{-i\langle\xi,\lambda I\rangle}e^{i\langle\xi,A\rangle} = e^{i\langle\xi,A-\lambda I\rangle}, \quad \xi \in \mathbb{R}^n,$$

since the operators $\langle\xi, \lambda I\rangle$ and $\langle\xi, A\rangle$ commute.

(ii) follows from (i), the definition of the support of a distribution, and the fact that $\operatorname{supp}(f_\lambda) = \lambda + \operatorname{supp}(f)$ for every $f \in C_c^\infty(\mathbb{R}^n)$.

(iii) follows from the definition of the sets involved. ∎

We conclude this section with a topological result needed later.

PROPOSITION 3. *Let K be a subset of $\mathbb{R}^n$ which is compact, infinite and countable. Let P denote the set of all isolated points of K. Then*

(i) *P is an infinite set, and*
(ii) *$K = \overline{P}$ (the bar denoting closure).*

Proof. (i) The set $K = \bigcup_{\lambda\in K}\{\lambda\}$ is a countable union. By Baire's Theorem at least one set $\{\lambda\}$ has non-empty interior, that is, $\lambda \in P$. Choose any $\lambda \in P$. Then $K\setminus\{\lambda\}$ is again compact, infinite and countable and hence, also has isolated points. Continuing this argument inductively it follows that P is infinite.

(ii) Suppose $\overline{P} \ne K$. Then $M = K \setminus \overline{P}$ is a non-empty, open subset of the compact space K. The set $M = \bigcup_{\lambda\in M}\{\lambda\}$ is a countable union, hence by Baire's Theorem at least one set $\{\lambda\}$, $\lambda \in M$, is open in M. Since M is open (in K), $\{\lambda\}$ is open in K, that is, $\lambda \in P$, a contradiction. ∎

2. Commutativity criteria.

The purpose of this section is to present some criteria which characterize commutativity of n-tuples $A=(A_1, \ldots, A_n)$ of compact selfadjoint operators. These results are consequences of the following important fact concerning the nature of particular kinds of isolated points of $\operatorname{supp}(T(A))$. First we require a new notion.

DEFINITION 2. Let M be a compact subset of $\mathbb{R}^n$. A point $\lambda \in M$ is called *hyperisolated* if it is isolated and there is a hyperplane (i.e. a maximal proper affine subspace of $\mathbb{R}^n$), say L, such that $L \cap M = \{\lambda\}$.

Analytically this means that there is a (necessarily non-zero) $\eta \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $|\langle\lambda - \mu, \eta\rangle| \ge \varepsilon$ for every $\mu \in M$ with $\mu \ne \lambda$.

Remark 1. λ is hyperisolated in M if and only if there exists a direction η and $\varepsilon > 0$ such that the n-dimensional strip

$$S(\lambda, \eta, \varepsilon) = \lambda + \{x \in \mathbb{R}^n : |\langle x, \eta\rangle| < \varepsilon\}$$

intersects M only at the point λ.

THEOREM 1. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of bounded selfadjoint operators in a Hilbert space H and $\lambda \in \operatorname{supp}(T(A))$ be hyperisolated. Then λ is a joint eigenvalue of A. Moreover, the decomposition $H = H_\lambda(A) \oplus H_\lambda(A)^\perp$ reduces the n-tuple A (that is, $A = A_{H_\lambda} \oplus A_{H_\lambda^\perp}$), one has $A_{H_\lambda} = \lambda I$ and $\operatorname{supp}(T(A_{H_\lambda^\perp})) = \operatorname{supp}(T(A)) \setminus \{\lambda\}$.*

Remark 2. (a) It will be shown in the course of the proof of Theorem 1 that the corresponding eigenprojection $E_\lambda(A)$ equals $T(A)\varphi$, where $\varphi \in C_c^\infty(\mathbb{R}^n)$ is supported in a neighbourhood U_λ of λ with $U_\lambda \cap \operatorname{supp}(T(A)) = \{\lambda\}$ and φ is constantly equal to 1 in a (smaller) neighbourhood of λ. It follows from Proposition 2 that $E_\lambda(A)$ is a finite rank projection whenever the operators A_j, $1 \le j \le n$, are compact and $\lambda \ne 0$.

(b) As a consequence of Theorem 1 we obtain $\operatorname{supp}(T(A)) = \{\lambda\}$ if and only if $A = \lambda I$. Furthermore, if $\operatorname{supp}(T(A))$ is a finite set, say $\operatorname{supp}(T(A)) = \{\lambda^{(1)}, \ldots, \lambda^{(m)}\}$, then we can successively split off the joint eigenspaces. After m steps we end up with the following representation of A: there exist non-zero orthogonal projections $P_1, \ldots, P_m$ satisfying $\sum_{j=1}^m P_j = I$ and $P_k P_j = P_j P_k = 0$ for $k \ne j$ such that $A = \sum_{j=1}^m \lambda^{(j)} P_j$. In particular, $A_r = \sum_{j=1}^m \lambda_r^{(j)} P_j$ for each $1 \le r \le n$, where $\lambda^{(j)} = (\lambda_1^{(j)}, \ldots, \lambda_n^{(j)})$.

For ease of reading we postpone the proof of Theorem 1 to the end of this section. We prefer first to establish some consequences. We begin with a finite-dimensional result.

THEOREM 2. *Let H be a Hilbert space of finite dimension $k \ge 1$ and $A = (A_1, \ldots, A_n)$ be an n-tuple of selfadjoint operators in H. The following statements are equivalent.*

(i) *The operators A_j, $1 \le j \le n$, mutually commute.*
(ii) $\operatorname{supp}(T(A))$ *is a finite subset of* $\mathbb{R}^n$.
(iii) $\operatorname{supp}(T(A))$ *has at most k elements.*
(iv) $\gamma(A) = \operatorname{supp}(T(A))$.

Proof. (i)$\Leftrightarrow$(ii) follows from the main Theorem in [10]; see also Remark 2(b). The implication (i)$\Rightarrow$(iv) is well known (cf. Introduction) and (iv)$\Rightarrow$(ii) since $\gamma(A) \subset \sigma(A_1) \times \ldots \times \sigma(A_n)$; see [3, Proposition 2]. Clearly (iii)$\Rightarrow$(ii). So, it remains to establish (ii)$\Rightarrow$(iii). Since each point of a finite set is hyperisolated it follows from Theorem 1 that each point of $\operatorname{supp}(T(A))$ is a joint eigenvalue of A. Since joint eigenvectors corresponding to distinct joint eigenvalues are eigenvectors of some A_j, $1 \le j \le n$, corresponding to distinct eigenvalues of A_j, these joint eigenvectors are necessarily orthogonal in H. So, there can be at most k points in $\operatorname{supp}(T(A))$. ∎

The next result illustrates "how different" the set $\operatorname{supp}(T(A))$ is when the n-tuple A does not commute.

THEOREM 3. *Let H be a Hilbert space of finite dimension $k \ge 1$ and $A = (A_1, \ldots, A_n)$ be an n-tuple of selfadjoint operators in H. Then $\operatorname{supp}(T(A))$ is either a set with at most k elements (in which case A commutes), or $\operatorname{supp}(T(A))$ is an uncountable set (in which case A is not commutative).*

Proof. Suppose that $\operatorname{supp}(T(A))$ has more than k elements, in which case it is an infinite set by Theorem 2. Suppose that it is a countable set. Then Proposition 3 implies that the set P of all isolated points of $\operatorname{supp}(T(A))$ is infinite as well. Since each point of P is hyperisolated (see the following Proposition 4) each such point is a joint eigenvalue of A by Theorem 1. This is impossible as H is finite-dimensional and joint eigenvectors of A corresponding to distinct joint eigenvalues are orthogonal. Accordingly, $\operatorname{supp}(T(A))$ is an uncountable subset of $\mathbb{R}^n$. ∎

PROPOSITION 4. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of bounded selfadjoint operators in a Hilbert space H. If $\operatorname{supp}(T(A))$ is a countable subset of $\mathbb{R}^n$, then*

(i) *each isolated point of* $\operatorname{supp}(T(A))$ *is hyperisolated, and*
(ii) $\operatorname{supp}(T(A)) = \gamma(A)$.

Proof. (i) By a suitable translation it suffices to consider the special case of 0 being an isolated point of $\operatorname{supp}(T(A))$; see Lemma 4. Since the countable union of hyperplanes $V = \bigcup\{\ker(\langle\cdot, \lambda\rangle) : \lambda \in \operatorname{supp}(T(A)) \setminus \{0\}\}$ cannot be all of $\mathbb{R}^n$ (hyperplanes have Lebesgue measure 0) there must exist a point $\eta \in \mathbb{R}^n \setminus V$. Then the hyperplane $\ker(\langle\cdot, \eta\rangle)$ intersects $\operatorname{supp}(T(A))$ only in 0. Here $\langle\cdot, \lambda\rangle$ denotes the linear functional $x \mapsto \langle x, \lambda\rangle$, $x \in \mathbb{R}^n$.

(ii) By Theorem 1 and part (i) all isolated points of $\operatorname{supp}(T(A))$ are joint eigenvalues. By Lemma 2 they belong to $\gamma(A)$. By Proposition 3, $\operatorname{supp}(T(A))$ is the closure of its isolated points. Since $\gamma(A)$ is a closed set [3, Proposition 1], it follows that $\operatorname{supp}(T(A)) \subset \gamma(A)$. The converse inclusion is just Lemma 3. ∎

The following result may be viewed as a natural extension of Theorem 2 to a class of operators in infinite-dimensional spaces.

THEOREM 4. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of compact selfadjoint operators in a Hilbert space H. The following statements are equivalent.*

(i) *The operators A_j, $1 \le j \le n$, mutually commute.*
(ii) $\operatorname{supp}(T(A))$ *is a countable subset of* $\mathbb{R}^n$.

(iii) $\operatorname{supp}(T(A))$ *is a countable subset of* $\mathbb{R}^n$ *with* 0 *as only possible limit point.*

(iv) $\gamma(A) = \operatorname{supp}(T(A))$.

Proof. (i)⇒(iv) is well known (cf. Introduction) and (iv)⇒(iii) by Corollary 3.1 of [3]. The implication (iii)⇒(ii) is obvious and (ii)⇒(iv) follows from Proposition 4.

So, it remains to establish (iv)⇒(i). Let $M = M[A]$ be the maximal abelian subspace of A, in which case $H = M \oplus M^\perp$. By Lemma 1 it follows that

$$\operatorname{supp}(T(A_{M^\perp})) \subset \operatorname{supp}(T(A)) = \gamma(A)$$

and hence, $\operatorname{supp}(T(A_{M^\perp}))$ is a countable set. Suppose that $M^\perp \neq \{0\}$. Then $\operatorname{supp}(T(A_{M^\perp}))$ is a nonempty, countable, compact set, hence it has an isolated point λ (by Proposition 3). By Proposition 4(i) it is hyperisolated and then by Theorem 1 it is a joint eigenvalue of $A_{M^\perp}$. Clearly a corresponding joint eigenvector $x \in M^\perp$ of $A_{M^\perp}$ is also a joint eigenvector of A. This is a contradiction since, by Proposition 1(ii), joint eigenvectors belong to M. ∎

Remark 3. Slightly more is true than proved in Theorem 4. Namely, let $A = (A_1, \ldots, A_n)$ be any n-tuple of bounded selfadjoint operators. It is *not* assumed that the operators A_j, $1 \le j \le n$, are compact. If $\operatorname{supp}(T(A))$ is a countable set, then the operators A_j, $1 \le j \le n$, mutually commute. This follows from the same argument as used to establish (iv)⇒(i) in the proof of Theorem 4, after noting that Proposition 4 implies $\gamma(A) = \operatorname{supp}(T(A))$.

Theorem 4 shows that the spectral set $\gamma(A)$, originally introduced for commuting n-tuples A actually *characterizes* commutativity of A for the case of n-tuples of compact selfadjoint operators. The following example shows that the hypothesis of the operators A_j, $1 \le j \le n$, being *compact* cannot be omitted.

EXAMPLE 1. Let $B = (B_1, B_2)$ where $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are considered as selfadjoint operators in $\mathbb{C}^2$. It will be shown in Example 2 of Section 3 that $\operatorname{supp}(T(B)) = \mathbb{D}$, where $\mathbb{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$. Since $B_1B_2 \neq B_2B_1$, it follows from [3, Proposition 7] that $\gamma(B) = \emptyset$. Let $\{\lambda^{(k)}\}_{k=1}^\infty$ be a countable dense subset of $\mathbb{D}$ and let T be the multiplication operator on ℓ^2 with the (bounded) sequence $(\lambda^{(k)})_{k=1}^\infty$. Then T is normal, hence $C_1 = \Re(T) = \frac{1}{2}(T + T^*)$ and $C_2 = \Im(T) = \frac{1}{2i}(T - T^*)$ commute. Thus, for the pair $C = (C_1, C_2)$, we have $\gamma(C) = \operatorname{supp}(T(C))$. Each $\lambda^{(k)}$ is a joint eigenvalue of C (the kth unit vector is a corresponding eigenvector) and so, by the closedness of $\gamma(C)$, it follows that $\mathbb{D} \subset \gamma(C)$. Since T is a contraction, for each $\nu = \nu_1 + i\nu_2$, $\nu \notin \mathbb{D}$, the operator

$$(\nu_1 I - C_1)^2 + (\nu_2 I - C_2)^2 = (\nu I - T)(\nu I - T)^*$$

is invertible, that is, $(\nu_1, \nu_2) \notin \gamma(C)$. Accordingly, $\gamma(C) = \operatorname{supp}(T(C)) = \mathbb{D}$.

Let $A_j = B_j \oplus C_j$, for $j \in \{1, 2\}$, act in the Hilbert space $H = \mathbb{C}^2 \oplus \ell^2$ $(\equiv \ell^2)$. Then Lemma 1 implies that

$$\operatorname{supp}(T(A)) = \operatorname{supp}(T(B)) \cup \operatorname{supp}(T(C)) = \mathbb{D} \cup \mathbb{D} = \mathbb{D}$$

and also that

$$\gamma(A) = \gamma(B) \cup \gamma(C) = \emptyset \cup \mathbb{D} = \mathbb{D}.$$

However, $A_1A_2 \neq A_2A_1$ since $B_1B_2 \neq B_2B_1$. ∎

Proof of Theorem 1. By translating to the origin (cf. Lemma 4) it suffices to prove the result for the case when 0 is a hyperisolated point of $\operatorname{supp}(T(A))$.

Choose a non-negative function $\varphi \in C_c^\infty(\mathbb{R}^n)$ which is supported inside a disc $B_\varepsilon = \{x \in \mathbb{R}^n : |x| < \varepsilon\}$, for some $\varepsilon > 0$, such that φ is constantly 1 near 0 (say, in $B_{\varepsilon/2}$) and $\overline{B}_\varepsilon \cap \operatorname{supp}(T(A)) = \{0\}$.

Define a distribution $\mathcal{U} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{L}(H)$ by

$$\mathcal{U}(f) = T(A)(\varphi f), \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Then $\operatorname{supp}(\mathcal{U}) = \{0\}$ and $\mathcal{U}$ is of finite order, say N (with N not exceeding the (finite) order of $T(A)$ [1, Lemma 3.8]). So, there exist bounded operators R_α for $|\alpha| \le N$ (multi-index notation) such that

$$\mathcal{U}(f) = \sum_{|\alpha| \le N} (D^\alpha f)(0) R_\alpha, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Being compactly supported, this distribution has a (unique) extension to $C^\infty(\mathbb{R}^n)$. For fixed $\xi \in \mathbb{R}^n$, let $e_\xi(x) = e^{i\langle x, \xi\rangle}$, $x \in \mathbb{R}^n$, in which case

$$T(A)(\varphi e_\xi) = \sum_{|\alpha| \le N} i^{|\alpha|} \xi^\alpha R_\alpha. \tag{5}$$

On the other hand, since $(\varphi e_\xi)^\wedge = (\widehat{\varphi})_\xi$ it follows that

$$T(A)(\varphi e_\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle y, A\rangle} \widehat{\varphi}(y - \xi)\, dy = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle u+\xi, A\rangle} \widehat{\varphi}(u)\, du$$

and hence, since $\|e^{i\langle u+\xi, A\rangle}\| = 1$, we have

$$\|T(A)(\varphi e_\xi)\| \le (2\pi)^{-n/2} \int_{\mathbb{R}^n} |\widehat{\varphi}(u)|\, du < \infty$$

for all $\xi \in \mathbb{R}^n$. It follows from (5) that $R_\alpha = 0$ whenever $|\alpha| > 0$. Denoting R_0 simply by R gives

$$T(A)(\varphi f) = f(0) R, \quad f \in C^\infty(\mathbb{R}^n). \tag{6}$$

FACT 1. *The operator* R *is non-zero, selfadjoint and coincides with* $T(A)\varphi$.

Proof. Substitute $f = \varphi$ into (6) and use the fact that $\varphi^2 = \varphi$ in a neighbourhood of $\operatorname{supp}(T(A))$ yields $R = T(A)\varphi$. That R is selfadjoint then follows from [1, Theorem 2.9]. To see that $R \neq 0$, let $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be any function supported in B_ε such that $T(A)f_\varepsilon \neq 0$; since $0 \in \operatorname{supp}(T(A))$ such a function f_ε exists. Then $f_\varepsilon\varphi$ coincides with f_ε in a neighbourhood of $\operatorname{supp}(T(A))$ and so $T(A)(f_\varepsilon\varphi) = T(A)f_\varepsilon \neq 0$. But $T(A)(f_\varepsilon\varphi) = f_\varepsilon(0)R$ by (6). Accordingly, $f_\varepsilon(0)R \neq 0$ and hence, also $R \neq 0$. ∎

Since 0 is hyperisolated there is $\eta \in \mathbb{R}^n$ with $|\eta| = 1$ and $\delta > 0$ such that the strip $S(\eta,\delta) = \{x \in \mathbb{R}^n : |\langle x,\eta\rangle| < 2\delta\}$ intersects $\operatorname{supp}(T(A))$ only in 0. Let $M_\eta = (m_{jk})_{1\le j,k\le n}$ be an orthogonal $(n\times n)$-matrix which maps the unit vector η onto the unit vector $e_1 = (1,0,\ldots,0)$, i.e. $M_\eta\eta = e_1$. Let $M_\eta A$ be the n-tuple of selfadjoint operators given by $(M_\eta A)_j = \sum_{k=1}^n m_{jk}A_k$. By [1, Theorem 2.9(a)] we have

$$T(M_\eta A)f = T(A)(f\circ M_\eta), \qquad f \in \mathcal{S}(\mathbb{R}^n). \tag{7}$$

Since both distributions $T(A)$ and $T(M_\eta A)$ have compact support, identity (7) also holds for $f \in C^\infty(\mathbb{R}^n)$. We choose a non-zero C^∞-function $\varphi_1 : \mathbb{R} \to [0,1]$ which is constantly 1 on a neighbourhood of 0 and vanishes on $\{t \in \mathbb{R} : |t| \geq \delta\}$. Define $\widetilde{\varphi} \in C^\infty(\mathbb{R}^n)$ by $\widetilde{\varphi}(x) = \varphi_1(\langle x,\eta\rangle)$. Then $\widetilde{\varphi}$ coincides with φ on a neighbourhood of $\operatorname{supp}(T(A))$ and hence,

$$R = T(A)\varphi = T(A)\widetilde{\varphi} = T(M_\eta A)(\widetilde{\varphi}\circ M_\eta^t),$$

where M_η^t denotes the transpose of M_η. We have

$$(\widetilde{\varphi}\circ M_\eta^t)(x) = \varphi_1(\langle M_\eta^t x,\eta\rangle) = \varphi_1(\langle x, M_\eta\eta\rangle) = \varphi_1(x_1).$$

That is, $\widetilde{\varphi}\circ M_\eta^t$ is a function depending on just one of the variables. It follows from Theorem 2.9(b) of [1] that

$$R = T(M_\eta A)(\widetilde{\varphi}\circ M_\eta^t) = T((M_\eta A)_1)\varphi_1.$$

Note that $M_\eta\eta = e_1$ implies $M_\eta^t e_1 = \eta$, that is, $m_{1j} = \eta_j$ for $j = 1,\ldots,n$. It follows that $(M_\eta A)_1 = \sum_{j=1}^n m_{1j}A_j = \sum_{j=1}^n \eta_j A_j = \langle\eta, A\rangle$. Thus we have

$$R = T(M_\eta A)(\widetilde{\varphi}\circ M_\eta^t) = T(\langle\eta,A\rangle)\varphi_1 = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{it\langle\eta,A\rangle}\widehat{\varphi}_1(t)\,dt.$$

By multiplicativity of the Weyl calculus for a single operator [1, Lemma 3.1] and the fact that $\widetilde{\varphi}^2 = \widetilde{\varphi}$ on a neighbourhood of $\operatorname{supp}(T(A))$ we have

$$R = T(A)(\widetilde{\varphi}^2) = T((M_\eta A)_1)(\varphi_1^2) = [T(\langle\eta,A\rangle)(\varphi_1)]^2 = R^2.$$

Thus we have established

FACT 2. *R is a projection.*

It is well known that the Weyl calculus for a single selfadjoint operator B coincides with the standard $C(\sigma(B))$-functional calculus $f \mapsto f(B)$ for selfadjoint operators; in particular, $T(B)f = f(B)$ for $f \in \mathcal{S}(\mathbb{R})$. Define $\varphi_{1,n}(t) = \varphi_1(nt)$, $t \in \mathbb{R}$, and $\widetilde{\varphi}_n(x) = \varphi_{1,n}(\langle x,\eta\rangle)$, $x \in \mathbb{R}^n$; the functions $\widetilde{\varphi}$ and $\widetilde{\varphi}_n$ coincide on a neighbourhood of $\operatorname{supp}(T(A))$, hence

$$R = T(A)\widetilde{\varphi}_n = T(\langle\eta,A\rangle)\varphi_{1,n} = \varphi_{1,n}(\langle\eta,A\rangle) \quad \text{for all } n \in \mathbb{N}.$$

For the $C(\sigma(B))$-functional calculus the operator norm of $\psi(B)$ can be estimated by the sup-norm of ψ, i.e. $\|\psi(B)\| \leq \sup_{t\in\mathbb{R}}|\psi(t)|$. Since $\sup_{t\in\mathbb{R}}|t\varphi_{1,n}(t)| \to 0$ as $n\to\infty$ it follows that we have $\|\langle\eta,A\rangle R\| = \|\langle\eta,A\rangle\varphi_{1,n}(\langle\eta,A\rangle)\| \leq \sup_{t\in\mathbb{R}}|t\varphi_{1,n}(t)| \to 0$ as $n\to\infty$. We conclude that

$$\langle\eta,A\rangle R = R\langle\eta,A\rangle = 0.$$

So far η was fixed. However, this holds for every η which generates a strip separating 0 from the rest of $\operatorname{supp}(T(A))$. By the compactness of $\operatorname{supp}(T(A))$ the set of all such η's forms an open subset of $\mathbb{R}^n$. In particular, there exist n *linearly independent* vectors $\eta^{(1)},\ldots,\eta^{(n)} \in \mathbb{R}^n$ such that

$$\langle\eta^{(j)},A\rangle R = R\langle\eta^{(j)},A\rangle = 0, \qquad 1\leq j\leq n.$$

In matrix notation $\boldsymbol{\eta}(AR) = \boldsymbol{\eta}(RA) = 0$, where $\boldsymbol{\eta}$ is the invertible $(n\times n)$-matrix having rows $\eta_1,\ldots,\eta_n$. Multiplying by $\boldsymbol{\eta}^{-1}$ from the right we obtain $AR = RA = 0$, i.e.

$$A_jR = RA_j = 0 \quad \text{for all } 1\leq j\leq n.$$

From this it follows that R maps H into the joint eigenspace $H_0(A)$. Actually, R is also onto as the following argument shows. Suppose $x \in H_0(A)$. Then by (4) and (6) we have $Rx = [T(A)\varphi]x = \varphi(0)x = x$, that is, $x \in R(H)$. Summarizing these results and taking into account that $R \neq 0$ (by Fact 1) we obtain

FACT 3. *The joint eigenspace $H_0(A)$ is non-zero and R is the joint eigenprojection $E_0(A)$ of A corresponding to 0.*

The joint eigenspace $H_0(A)$, being invariant for every A_j, reduces the n-tuple A, that is, A can be considered as a direct sum $A = A_{H_0}\oplus A_{H_0^\perp}$ corresponding to the decomposition $H = H_0(A)\oplus H_0(A)^\perp$. We have $A_{H_0} = 0$, hence $\operatorname{supp}(T(A_{H_0})) = \{0\}$. Since $\operatorname{supp}(T(A)) = \operatorname{supp}(T(A_{H_0}))\cup\operatorname{supp}(T(A_{H_0^\perp}))$ (by Lemma 1), it follows that either $\operatorname{supp}(T(A_{H_0^\perp})) = \operatorname{supp}(T(A))$ or $\operatorname{supp}(T(A_{H_0^\perp})) = \operatorname{supp}(T(A))\setminus\{0\}$. Actually 0 cannot be an element of $\operatorname{supp}(T(A_{H_0^\perp}))$, since otherwise the result proved above, applied to the n-tuple $A_{H_0^\perp}$ on the Hilbert space $H_0(A)^\perp$, implies that there is a joint eigenvector $x \in H_0(A)^\perp$ of A corresponding to 0, a contradiction. So, we have finally shown that $\operatorname{supp}(T(A_{H_0^\perp})) = \operatorname{supp}(T(A))\setminus\{0\}$ and Theorem 1 is proved. ∎

3. Pairs of selfadjoint operators. The aim of this section is to extend Proposition 10 of [3], formulated for 2-dimensional Hilbert spaces, to arbitrary finite-dimensional Hilbert spaces. First a preliminary result is needed.

PROPOSITION 5. *Let $A = (A_1, A_2)$ be a pair of bounded selfadjoint operators in a Hilbert space H. Then*

(i) $\gamma(A) \subset \sigma(A_1 + iA_2)$.

Suppose, in addition, that A_1 and A_2 are compact.

(ii) *If $A_1A_2 = A_2A_1$, then $\gamma(A) = \sigma(A_1 + iA_2)$.*

Proof. (i) Choose $\lambda \in \gamma(A)$. Then $0 \in \sigma(S)$, where $S = (A_1 - \lambda_1 I)^2 + (A_2 - \lambda_2 I)^2$. Since S is selfadjoint, there are unit vectors x_n such that $Sx_n \to 0$ in H as $n \to \infty$. Then also $(Sx_n, x_n) \to 0$ and hence, $(A_j - \lambda_j I)x_n \to 0$ as $n \to \infty$, for each $j \in \{1, 2\}$, from which the result follows.

(ii) Since $A_1 + iA_2$ is a compact (normal) operator its spectrum is a countable set with 0 as only possible limit point. Suppose that $\lambda \in \sigma(A_1 + iA_2) \setminus \{0\}$, in which case λ is an eigenvalue of $A_1 + iA_2$. So, there is $x \neq 0$ such that $(A_1 + iA_2)x = (\lambda_1 + i\lambda_2)x$, where $\lambda = \lambda_1 + i\lambda_2$. That is, $[(A_1 - \lambda_1 I) + i(A_2 - \lambda_2 I)]x = 0$ and hence, also

$$[(A_1 - \lambda_1 I) - i(A_2 - \lambda_2 I)][(A_1 - \lambda_1 I) + i(A_2 - \lambda_2 I)]x = 0.$$

Expanding this identity and using $A_1A_2 = A_2A_1$ gives

$$[(A_1 - \lambda_1 I)^2 + (A_2 - \lambda_2 I)^2]x = 0.$$

Since $x \neq 0$ it follows that $\lambda = (\lambda_1, \lambda_2)$ belongs to $\gamma(A)$. This shows that $\sigma(A_1 + iA_2) \setminus \{0\} \subset \gamma(A)$. If 0 is also an eigenvalue of $A_1 + iA_2$, then the same argument shows that $0 \in \gamma(A)$. Otherwise, 0 is a limit point of $\sigma(A_1 + iA_2)$, in which case the closedness of $\gamma(A)$ ensures that $0 \in \gamma(A)$. Hence, $\sigma(A_1 + iA_2) \subset \gamma(A)$ and so, by part (i), $\sigma(A_1 + iA_2) = \gamma(A)$. ∎

PROPOSITION 6. *Let H be a Hilbert space of dimension $k < \infty$ and $A = (A_1, A_2)$ be a pair of selfadjoint operators in H. The following statements are equivalent.*

(i) $A_1A_2 = A_2A_1$.

(ii) $A_1 + iA_2$ *is a normal operator.*

(iii) *The standard (polynomial) functional calculus $S(A_1 + iA_2) : C^{(k-1)}(\mathbb{R}^2) \to \mathcal{L}(H)$ of the single operator $A_1 + iA_2$, when restricted to $C^\infty(\mathbb{R}^2)$, agrees with the extension of the Weyl calculus $T(A) : \mathcal{S}(\mathbb{R}^2) \to \mathcal{L}(H)$ to $C^\infty(\mathbb{R}^2)$.*

(iv) $S(A_1 + iA_2)\overline{\lambda} = T(A)\overline{\lambda}$, *where $\overline{\lambda}(x, y) = x - iy$.*

(v) *The Weyl calculus $T(A)$ is multiplicative in $C^\infty(\mathbb{R}^2)$.*

(vi) $\operatorname{supp}(T(A))$ *is a finite subset of $\mathbb{R}^2$.*

(vii) $T(A)$ *has order zero as a distribution.*

(viii) $\operatorname{supp}(T(A)) = \gamma(A)$.

(ix) $\operatorname{supp}(T(A)) = \sigma(A_1 + iA_2)$, *where $\lambda_1 + i\lambda_2 \in \mathbb{C}$ is identified with $(\lambda_1, \lambda_2) \in \mathbb{R}^2$.*

Proof. The mutual equivalence of the first five statements follows from [3, Proposition 9]. The equivalence of (i) with both (vi) and (vii) is the main Theorem in [10]; see also [4] for the case $k = 2$. Theorem 2 implies that (i)⇔(viii). Clearly (ix)⇒(vi). Finally, (i)⇒(ix) by Proposition 5 and the implication (i)⇒(viii). ∎

Remark 4. In Proposition 10 of [3] it is shown (for 2-dimensional Hilbert spaces) that each of the statements in Proposition 6 is equivalent to

(x) $\gamma(A) \neq \emptyset$.

(xi) $\sigma(A_1 + iA_2) = \gamma(A)$, *where $\lambda_1 + i\lambda_2 \in \mathbb{C}$ is identified with $(\lambda_1, \lambda_2) \in \mathbb{R}^2$.*

It is shown in Remark 2 of [3] that the statements of Proposition 6 are *not* equivalent to statement (x) for $\dim(H) > 2$. We conclude with an example which shows that the statements of Proposition 6 are also *not* equivalent to statement (xi) for $2 < \dim(H) < \infty$. Indeed, it is shown that $\gamma(A) = \sigma(A_1 + iA_2)$ is a proper subset of $\operatorname{supp}(T(A))$.

EXAMPLE 2. Let B_1 and B_2 be the (2×2)-selfadjoint matrices given in Example 1 and $u \in \mathbb{R}^2$. Let $A_j = \begin{pmatrix} B_j & 0 \\ 0 & u_j \end{pmatrix}$ for $j \in \{1, 2\}$, considered as operators in the Hilbert space $H = \mathbb{C}^3$. Since $B_1 + iB_2$ is nilpotent (of order 2) it follows that $\sigma(B_1 + iB_2) = \{0\}$. Since $A_1 + iA_2$ equals the direct sum $(B_1 + iB_2) \oplus (u_1 + iu_2)I$ in $\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}$ it follows that

$$\sigma(A_1 + iA_2) = \sigma(B_1 + iB_2) \cup \sigma((u_1 + iu_2)I) = \{0\} \cup \{u_1 + iu_2\}.$$

It was shown in Remark 2 of [3] that $\gamma(A) = \{(u_1, u_2)\}$, where $A = (A_1, A_2)$. Moreover, Lemma 1 implies that

$$\operatorname{supp}(T(A)) = \operatorname{supp}(T(B)) \cup \{(u_1, u_2)\}.$$

Putting $u_1 = u_2 = 0$ shows that $\gamma(A) = \sigma(A_1 + iA_2)$, even though $A_1A_2 \neq A_2A_1$. However, as must be the case, $\sigma(A_1 + iA_2) = \gamma(A)$ is a proper subset of $\operatorname{supp}(T(A))$.

It remains to show that $\operatorname{supp}(T(B)) = \mathbb{D}$. Let $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ be the 2-dimensional sphere and m denote normalized surface measure on $\mathbb{S}^2$. For $f \in \mathcal{S}(\mathbb{R}^2)$, the functions

$$x = (x_1, x_2, x_3) \mapsto f(x_1, x_2) \pm \partial_1 f(x_1, x_2) + (x_1 + x_2)\partial_2 f(x_1, x_2), \quad x \in \mathbb{S}^2,$$

are denoted by $f \pm \partial_1 f + (x_1 + x_2)\partial_2 f$. It follows from Theorem 4.1 of [1] that

(8) $T(B)f$

$$= \begin{pmatrix} \int_{\mathbb{S}^2}(f+\partial_1 f+(x_1+x_2)\partial_2 f)\,dm(x) & \int_{\mathbb{S}^2}\partial_2 f\,dm(x) \\ \int_{\mathbb{S}^2}\partial_2 f\,dm(x) & \int_{\mathbb{S}^2}(f-\partial_1 f+(x_1+x_2)\partial_2 f)\,dm(x) \end{pmatrix}$$

for every $f \in \mathcal{S}(\mathbb{R}^2)$; see [4] for the details. It is clear from (8) that $\mathrm{supp}(T(B)) \subset \mathbb{D}$. Since $\mathrm{supp}(T(B))$ is equal to the union of the supports of the four distributions forming the entries of the right-hand side of (8) it suffices to show that the support of the $\mathbb{C}$-valued distribution

$$V: f \mapsto \int_{\mathbb{S}^2} \partial_2 f\,dm(x), \quad f \in \mathcal{S}(\mathbb{R}^2),$$

contains $\mathbb{D}$. But $Vf = 2\int_{\mathbb{S}^2_+} \partial_2 f\,dm(x)$ where $\mathbb{S}^2_+ = \{x \in \mathbb{S}^2 : x_3 \ge 0\}$ and so the problem reduces to showing that the support of the distribution

$$\mathcal{U}: f \mapsto \int_{\mathbb{S}^2_+} \partial_2 f\,dm(x), \quad f \in \mathcal{S}(\mathbb{R}^2),$$

contains $\mathbb{D}$. Let $\varphi(u,v) = (1-u^2-v^2)^{-1/2}$ for $u^2+v^2<1$. Then a transformation of measure shows that $\mathcal{U}f = \int_{\mathbb{D}} \varphi(u,v)\partial_2 f(u,v)\,du\,dv$. By considering functions of the form $f(u,v) = g(u)h(v)$, for suitable g and h, it can be shown that all interior points of $\mathbb{D}$ belong to $\mathrm{supp}(\mathcal{U})$ and hence, $\mathbb{D} \subset \mathrm{supp}(\mathcal{U})$. ∎

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Received August 13, 1992 (2985)
Revised version June 21, 1994