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Ambiguous loci of the farthest distance mapping from compact convex sets

by

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Abstract. Let $\mathbb{E}$ be a strictly convex separable Banach space of dimension at least 2. Let $\mathcal{K}(\mathbb{E})$ be the space of all nonempty compact convex subsets of $\mathbb{E}$ endowed with the Hausdorff distance. Denote by $\mathcal{K}^0$ the set of all $X \in \mathcal{K}(\mathbb{E})$ such that the farthest distance mapping $a \mapsto M_X(a)$ is multivalued on a dense subset of $\mathbb{E}$. It is proved that $\mathcal{K}^0$ is a residual dense subset of $\mathcal{K}(\mathbb{E})$.

1. Introduction and preliminaries. Throughout the present paper $\mathbb{E}$ denotes a strictly convex separable Banach space of dimension at least 2, and $\mathcal{K}(\mathbb{E})$ (resp. $\mathcal{B}(\mathbb{E})$) the family of all nonempty compact convex (resp. closed bounded) subsets of $\mathbb{E}$. The spaces $\mathcal{K}(\mathbb{E})$ and $\mathcal{B}(\mathbb{E})$ are equipped with the Hausdorff distance h under which, as is well known, both are complete. For $X \in \mathcal{B}(\mathbb{E})$ and $a \in \mathbb{E}$ we set

$$e_X(a) = \sup\{\|x-a\| \mid x \in X\}.$$

Given $X \in \mathcal{B}(\mathbb{E})$ and $a \in \mathbb{E}$, let us consider the *maximization problem*, denoted $\max(a, X)$, which consists in finding some point $x \in X$ such that $\|x-a\| = e_X(a)$. Any such x is said a *solution* of $\max(a,X)$ and any sequence $\{x_n\} \subset X$ satisfying $\lim_{n\to\infty} \|x_n - a\| = e_X(a)$ is called a *maximizing sequence* of $\max(a, X)$.

In a metric space Z, $B_Z(z,r)$ (resp. $\widetilde{B}_Z(z,r)$) is an open (resp. closed) ball with center $z \in Z$ and radius $r > 0$ (resp. $r \ge 0$). For any $X \subset Z$, $\overline{X}$ and $\operatorname{diam} X$ ($X \ne \emptyset$) stand for the closure of X and the diameter of X, respectively.

A set $X \subset Z$ is called *everywhere uncountable* in Z if for every $z \in Z$ and $r > 0$ the set $X \cap B_Z(z,r)$ is nonempty and uncountable.

For $X \in \mathcal{K}(\mathbb{E})$ we denote by $M_X : \mathbb{E} \to \mathcal{K}(\mathbb{E})$ the *farthest distance mapping*, defined by

$$M_X(a) = \{x \in X \mid \|x-a\| = e_X(a)\}.$$

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We call $M_X(a)$ the *solution set* of the maximization problem $\max(a, X)$. Moreover, the set

$$A(M_X) = \{a \in \mathbb{E} \mid M_X(a) \text{ contains at least 2 points}\}$$

is called the *ambiguous locus* of M_X.

In this note we consider approximation problems for the mapping e_X from sets $X \in \mathcal{K}(\mathbb{E})$. It is known that, if $\mathbb{E}$ is also uniformly convex, then the ambiguous locus of any set $X \in \mathcal{K}(\mathbb{E})$ is σ-porous, thus of the first Baire category and of Lebesgue measure zero if $\mathbb{E} = \mathbb{R}^n$ (see [4] and, for similar results, Bartke and Berens [2] and Zajíček [13]). However, the set $A(M_X)$, though small from the category and the measure point of view, can be unexpectedly rich in points scattered all over $\mathbb{E}$. More precisely, we show that in every strictly convex separable Banach space $\mathbb{E}$ of dimension at least 2 there exists a nonempty compact convex set X for which the ambiguous locus $A(M_X)$ is everywhere uncountable in $\mathbb{E}$. Actually we prove more, namely that such a property of X is shared by *most* compact convex sets in $\mathcal{K}(\mathbb{E})$, in the Baire category sense.

For $a \in \mathbb{E}$ and $X \in \mathcal{B}(\mathbb{E})$ the set $M_X(a)$ can be empty (see Miyajima and Wada [11] for some examples). Under suitable assumptions on $\mathbb{E}$ and X, Asplund [1] and Lau [9] (see also Edelstein [7], Panda and Dwivedi [12], Deville and Zizler [5]) have proved that the set of all $a \in \mathbb{E}$ for which $M_X(a)$ is empty is of the Baire first category in $\mathbb{E}$. The question whether this set can be locally rich in points seems not yet settled.

Our approach is based on the Baire theorem. This has proven to be a useful tool in order to get existence results in several problems of geometry, starting with the classical work of Klee [10]. Developments of such ideas can be found in Gruber [8] and Zamfirescu [14], [15].

2. Lemmas

LEMMA 2.1. *Let $a, x_1, x_2 \in \mathbb{E}$, $x_1 \neq x_2$, be such that $\|x_1 - a\| = \|x_2 - a\|$. For $\theta \in \Delta = [d_1, d_2]$, $0 < d_1 \le d_2 \le 1$, set $a_i(\theta) = a + \theta(x_i - a)$, $i = 1, 2$. Then there exists an $\varepsilon_0 > 0$ such that, for every $\theta \in \Delta$,*

$$\|x_2 - a_1(\theta)\| > \|x_1 - a_1(\theta)\| + \varepsilon_0, \tag{2.1}$$

$$\|x_1 - a_2(\theta)\| > \|x_2 - a_2(\theta)\| + \varepsilon_0. \tag{2.2}$$

Proof. It suffices to prove (2.1) (the proof of (2.2) is analogous). If the statement is not true, there exists a $\theta \in \Delta$ such that $\|x_2 - a_1(\theta)\| \le \|x_1 - a_1(\theta)\|$. Furthermore,

$$\begin{aligned}\|x_1 - a\| &= \|a_1(\theta) - a\| + \|x_1 - a_1(\theta)\| \ge \|a_1(\theta) - a\| + \|x_2 - a_1(\theta)\| \\ &\ge \|a_1(\theta) - a\| + \|x_2 - a\| - \|a_1(\theta) - a\| = \|x_2 - a\|,\end{aligned}$$

which implies that

$$\|(x_2 - a_1(\theta)) + (a_1(\theta) - a)\| = \|x_2 - a_1(\theta)\| + \|a_1(\theta) - a\|.$$

Since $\mathbb{E}$ is strictly convex, for some $\beta > 0$ we have $x_2 - a_1(\theta) = \beta(a_1(\theta) - a)$. Hence $x_2 - a = (1+\beta)(a_1(\theta) - a) = (1+\beta)\theta(x_1 - a)$, which yields $x_2 = x_1$, a contradiction. This completes the proof.

LEMMA 2.2. *In addition to the assumptions of Lemma 2.1, set $b_\theta(t) = (1-t)a_1(\theta) + ta_2(\theta)$, $t \in [0,1]$. Then there exists an $\varepsilon > 0$ such that, for every $\theta \in \Delta$ and every $C_1 \subset B_{\mathbb{E}}(x_1, \varepsilon)$, $C_2 \subset B_{\mathbb{E}}(x_2, \varepsilon)$ with $C_1, C_2 \neq \emptyset$,*

$$e_{C_2}(a_1(\theta)) > e_{C_1}(a_1(\theta)), \tag{2.3}$$

$$e_{C_1}(a_2(\theta)) > e_{C_2}(a_2(\theta)). \tag{2.4}$$

Moreover, there exists a $t = t(\theta, C_1, C_2) \in]0,1[$ such that

$$e_{C_1}(b_\theta(t)) = e_{C_2}(b_\theta(t)). \tag{2.5}$$

Proof. By Lemma 2.1 there exists an $\varepsilon_0 > 0$ such that for every $\theta \in \Delta$, (2.1) and (2.2) are satisfied. Take $\varepsilon = \varepsilon_0/3$. Let $\theta \in \Delta$, and let $C_1 \subset B_{\mathbb{E}}(x_1, \varepsilon)$ and $C_2 \subset B_{\mathbb{E}}(x_2, \varepsilon)$ with $C_1, C_2 \neq \emptyset$. For $c_1 \in C_1$ and $c_2 \in C_2$ we have

$$\begin{aligned}\|c_2 - a_1(\theta)\| &\ge \|x_2 - a_1(\theta)\| - \|c_2 - x_2\| > \|x_1 - a_1(\theta)\| + \varepsilon_0 - \varepsilon \\ &\ge \|c_1 - a_1(\theta)\| - \|c_1 - x_1\| + \varepsilon_0 - \varepsilon \\ &\ge \|c_1 - a_1(\theta)\| - \varepsilon + \varepsilon_0 - \varepsilon = \|c_1 - a_1(\theta)\| + \varepsilon,\end{aligned}$$

which implies that $e_{C_2}(a_1(\theta)) > e_{C_1}(a_1(\theta))$. Hence (2.3) is proved. The proof of (2.4) is analogous. Furthermore, the function $t \to e_{C_1}(b_\theta(t)) - e_{C_2}(b_\theta(t))$ is continuous on $[0,1]$ and, by (2.3) and (2.4), assumes values of opposite sign at the end points of $[0,1]$. Thus there exists a $t = t(\theta, C_1, C_2) \in]0,1[$ for which (2.5) is satisfied. This completes the proof.

LEMMA 2.3. *Let $a \in \mathbb{E}$ and $0 < r < R$ and $x_1, x_2 \in \mathbb{E}$, $x_1 \neq x_2$, be such that $\|x_1 - a\| = \|x_2 - a\| = R$. Let $X \subset \widetilde{B}_{\mathbb{E}}(a, r)$ with $X \in \mathcal{K}(\mathbb{E})$. Set $\Delta = [d/8, d/4]$, where $d = (R-r)/R$. Define*

$$Z = \overline{\mathrm{co}}(X \cup \{x_1, x_2\})$$

and let $b_\theta(t)$ and $a_i(\theta)$ be defined as in the previous lemmas. Then:

(i) *For $\theta \in \Delta$ and $t \in [0,1]$, the maximization problem $\max(b_\theta(t), Z)$ has solution set $M_Z(b_\theta(t))$ satisfying*

$$M_Z(b_\theta(t)) \subset \{x_1, x_2\}. \tag{2.6}$$

Moreover, $M_Z(b_\theta(0)) = x_2$ and $M_Z(b_\theta(1)) = x_1$.

(ii) *For $\theta \in \Delta$ and $t \in [0,1]$, every maximizing sequence $\{z_n\}$ of $\max(b_\theta(t), Z)$ has a subsequence which converges to a point $z \in \{x_1, x_2\}$.*

Proof. For $\theta \in \Delta$ and $t \in [0,1]$, define $\varphi_{b_\theta(t)} : Z \to \mathbb{R}$ by $\varphi_{b_\theta(t)}(x) = \|x - b_\theta(t)\|$. As the function $\varphi_{b_\theta(t)}$ is continuous on the compact set Z, $\varphi_{b_\theta(t)}$ attains its supremum at some point, say $\overline{z} \in Z$. Set

$$E = \{z \in Z \mid \varphi_{b_\theta(t)}(z) = \varphi_{b_\theta(t)}(\overline{z})\},$$

and observe that $E \neq \emptyset$. We claim that $E \subset \{x_1, x_2\}$.

Indeed, as $\varphi_{b_\theta(t)}$ is strictly convex on Z, a convex set, we have $E \subset \operatorname{ext} Z$, where $\operatorname{ext} Z$ denotes the set of the extreme points of Z. Moreover, by Krein–Milman's theorem [6], $\operatorname{ext} Z \subset X \cup \{x_1, x_2\}$, and thus $E \subset X \cup \{x_1, x_2\}$. To prove the claim it suffices to show that $E \cap X = \emptyset$. Suppose otherwise, and let $u \in E \cap X$. Then

$$\varphi_{b_\theta(t)}(u) = \|u - b_\theta(t)\| \le \|u - a\| + \|a - b_\theta(t)\| \le r + \frac{R-r}{4},$$

since, by a simple calculation, $\|a - b_\theta(t)\| \le \theta R \le (d/4)R = (R-r)/4$. On the other hand, for $i = 1, 2$, we have

$$\varphi_{b_\theta(t)}(x_i) = \|x_i - b_\theta(t)\| \ge \|x_i - a\| - \|a - b_\theta(t)\| \ge R - \frac{R-r}{4}.$$

Hence $\varphi_{b_\theta(t)}(u) < \varphi_{b_\theta(t)}(x_i)$, $i = 1, 2$, which implies that $u \notin E$, a contradiction. Thus $E \subset \{x_1, x_2\}$. Since $E = M_Z(b_\theta(t))$, (2.6) is proved. Moreover, by Lemma 2.1, we have

$$\varphi_{b_\theta(0)}(x_2) = \|x_2 - a_1(\theta)\| > \|x_1 - a_1(\theta)\| = \varphi_{b_\theta(0)}(x_1),$$

which implies that $M_Z(b_\theta(0)) = x_2$. Similarly one can show that $M_Z(b_\theta(1)) = x_1$, and so (i) is proved.

To prove (ii), for given $\theta \in \Delta$ and $t \in [0,1]$, let $\{z_n\} \subset Z$ be a maximizing sequence of $\max(b_\theta(t), Z)$. As Z is compact, passing to a subsequence we can assume that $\lim_{n\to\infty} z_n = z$ for some $z \in Z$. This implies that $z \in M_Z(b_\theta(t))$ and so, by (i), $z \in \{x_1, x_2\}$. This completes the proof.

LEMMA 2.4. *Under the assumptions of Lemma* 2.3, *for every $\varepsilon > 0$ there exists a $\sigma > 0$ such that for every $Y \in B_{\mathcal{K}(\mathbb{E})}(Z, \sigma)$ and every $\theta \in \Delta$,*

(i) $$M_Y(b_\theta(0)) \subset B_{\mathbb{E}}(x_2, \varepsilon), \quad M_Y(b_\theta(1)) \subset B_{\mathbb{E}}(x_1, \varepsilon),$$

(ii) $$M_Y(b_\theta(t)) \subset B_{\mathbb{E}}(x_1, \varepsilon) \cup B_{\mathbb{E}}(x_2, \varepsilon) \quad \textit{for every } t \in [0,1].$$

Proof. For (i) it suffices to prove the first inclusion (the proof of the second being analogous). Suppose that, on the contrary, there exist an $\varepsilon > 0$, a sequence $\{Y_n\} \subset \mathcal{K}(\mathbb{E})$ converging to Z, and a sequence $\{\theta_n\} \subset \Delta$ such that

$$M_{Y_n}(a_1(\theta_n)) \not\subset B_{\mathbb{E}}(x_2, \varepsilon), \quad n \in \mathbb{N}.$$

Passing to a subsequence, we assume that $\{\theta_n\}$ converges to a $\theta \in \Delta$. Let $\{y_n\} \subset \mathbb{E}$ be a sequence such that

(2.7) $$y_n \in M_{Y_n}(a_1(\theta_n)) \backslash B_{\mathbb{E}}(x_2, \varepsilon), \quad n \in \mathbb{N}.$$

Thus $y_n \in Y_n$, and $\|y_n - a_1(\theta_n)\| = e_{Y_n}(a_1(\theta_n))$, $n \in \mathbb{N}$. Since $\{\theta_n\}$ converges to θ and $\{Y_n\}$ converges to Z, there exists a sequence $\{\sigma_n\}$, $\sigma_n > 0$, converging to zero such that

$$\|y_n - a_1(\theta)\| \ge e_Z(a_1(\theta)) - \sigma_n, \quad n \in \mathbb{N}.$$

As $y_n \in Y_n$ and $\{Y_n\}$ converges to Z, there exists a sequence $\{z_n\} \subset Z$ satisfying

(2.8) $$\lim_{n\to\infty} \|z_n - y_n\| = 0.$$

Clearly,

$$\|z_n - a_1(\theta)\| \ge e_Z(a_1(\theta)) - \sigma_n - \|z_n - y_n\|, \quad n \in \mathbb{N}.$$

Hence $\{z_n\}$ is a maximizing sequence of $\max(a_1(\theta), Z)$, and so, by Lemma 2.3(ii), there is a subsequence, say $\{z_n\}$, which converges to a point $z \in \{x_1, x_2\}$. Since $z \in M_Z(a_1(\theta))$ and, by Lemma 2.3(i), $M_Z(a_1(\theta)) = x_2$, we have $z = x_2$. Consequently, there exists an $n_0 \in \mathbb{N}$ such that $z_n \in B_{\mathbb{E}}(x_2, \varepsilon/2)$ for $n \ge n_0$. Thus, by (2.8), there exists an $n_1 \ge n_0$ such that $y_n \in B_{\mathbb{E}}(x_2, \varepsilon)$ for $n \ge n_1$, contrary to (2.7). We can conclude that, given $\varepsilon > 0$, there exists a $\sigma_0 > 0$ such that for every $Y \in B_{\mathcal{K}(\mathbb{E})}(Z, \sigma_0)$ and $\theta \in \Delta$, (i) is satisfied.

It remains to prove (ii). Suppose that it is not true. Then there exist an $\varepsilon > 0$, a sequence $\{Y_n\} \subset \mathcal{K}(\mathbb{E})$ converging to Z, and two sequences $\{\theta_n\} \subset \Delta$ and $\{t_n\} \subset [0,1]$ such that

$$M_{Y_n}(b_{\theta_n}(t_n)) \not\subset B_{\mathbb{E}}(x_1, \varepsilon) \cup B_{\mathbb{E}}(x_2, \varepsilon), \quad n \in \mathbb{N}.$$

Passing to subsequences, we can assume that $\{\theta_n\}$ converges to $\theta \in \Delta$, and that $\{t_n\}$ converges to $t \in [0,1]$. Now let $\{y_n\} \subset \mathbb{E}$ be a sequence such that

(2.9) $$y_n \in M_{Y_n}(b_{\theta_n}(t_n)) \backslash (B_{\mathbb{E}}(x_1, \varepsilon) \cup B_{\mathbb{E}}(x_2, \varepsilon)), \quad n \in \mathbb{N}.$$

As in the proof of (i), one can construct a sequence $\{z_n\} \subset Z$ which satisfies (2.8) and is maximizing for $\max(b_\theta(t), Z)$. Then, by Lemma 2.3(ii), a subsequence, say $\{z_n\}$, converges to a point $z \in \{x_1, x_2\}$. This and (2.8) imply that there exists an $n_0 \in \mathbb{N}$ such that $y_n \in B_{\mathbb{E}}(x_1, \varepsilon) \cup B_{\mathbb{E}}(x_2, \varepsilon)$ for $n \ge n_0$, contrary to (2.9). Hence, given $\varepsilon > 0$, there exists a $0 < \sigma \le \sigma_0$ such that for every $Y \in B_{\mathcal{K}(\mathbb{E})}(Z, \sigma)$ and $\theta \in \Delta$, (ii) as well as (i) are satisfied. This completes the proof.

3. Main result

THEOREM 3.1. *Let $\mathbb{E}$ be a strictly convex separable Banach space of dimension at least* 2. *Then*

$$\mathcal{K}^0 = \{X \in \mathcal{K}(\mathbb{E}) \mid A(M_X) \text{ is everywhere uncountable in } \mathbb{E}\}$$

is a residual dense subset of $\mathcal{K}(\mathbb{E})$.

Proof. We follow some ideas from Klee [10] and Zamfirescu [15]. For $a \in \mathbb{E}$ and $s > 0$, set

$$\mathcal{N}_{a,s} = \{X \in \mathcal{K}(\mathbb{E}) \mid A(M_X) \cap B_{\mathbb{E}}(a,s) \text{ is empty or at most countable}\}.$$

CLAIM. *$\mathcal{N}_{a,s}$ is nowhere dense in $\mathcal{K}(\mathbb{E})$.*

For this it suffices to show that, given $X \in \mathcal{K}(\mathbb{E})$ and $0 < \varrho < s$, both arbitrary, there exist $Z \in \mathcal{K}(\mathbb{E})$ and $\sigma > 0$ such that

$$B_{\mathcal{K}(\mathbb{E})}(Z,\sigma) \subset B_{\mathcal{K}(\mathbb{E})}(X,\varrho) \cap (\mathcal{K}(\mathbb{E}) \setminus \mathcal{N}_{a,s}). \tag{3.1}$$

Case 1. Suppose $X \neq \{a\}$. Take $x_0 \in X$ such that $\|x_0 - a\| = r$, where $r = e_X(a)$, and set

$$x_1 = a + \left(1 + \frac{\varrho}{4r}\right)(x_0 - a).$$

We have $\|x_1 - a\| = R$, where $R = r + \varrho/4$. Next take $x_2 \in \mathbb{E}$ such that

$$\|x_2 - a\| = \|x_1 - a\|, \qquad \|x_2 - x_1\| = \varrho/4.$$

Define $Z = \overline{\mathrm{co}}(X \cup \{x_1, x_2\})$. Clearly $Z \in \mathcal{K}(\mathbb{E})$. By construction, $\|x_1 - x_0\| = \varrho/4$ and $\|x_2 - x_0\| \le \|x_2 - x_1\| + \|x_1 - x_0\| = \varrho/2$, thus $h(Z,X) \le \varrho/2$. Set $\Delta = [d/8, d/4]$, where $d = (R-r)/R$. Now define $a_i(\theta) = a + \theta(x_i - a)$, $i = 1,2$, and $b_\theta(t) = (1-t)a_1(\theta) + ta_2(\theta)$, $t \in [0,1]$.

By Lemma 2.2, there exists an $\varepsilon > 0$ with

$$B_{\mathbb{E}}(x_1,\varepsilon) \cap B_{\mathbb{E}}(x_2,\varepsilon) = \emptyset \tag{3.2}$$

such that for every $\theta \in \Delta$, and every $C_1 \subset B_{\mathbb{E}}(x_1,\varepsilon)$, $C_2 \subset B_{\mathbb{E}}(x_2,\varepsilon)$ with $C_1, C_2 \neq \emptyset$, there exists a $t_\theta \in\,]0,1[$ (depending on C_1 and C_2) such that

$$e_{C_1}(b_\theta(t_\theta)) = e_{C_2}(b_\theta(t_\theta)). \tag{3.3}$$

By Lemma 2.4, given $\varepsilon/2$, there exists a σ with

$$0 < \sigma < \min\{\varepsilon/2, \varrho/2\}$$

such that for every $Y \in B_{\mathcal{K}(\mathbb{E})}(Z,\sigma)$ and every $\theta \in \Delta$ we have

$$M_Y(b_\theta(t)) \subset B_{\mathbb{E}}(x_1,\varepsilon/2) \cup B_{\mathbb{E}}(x_2,\varepsilon/2), \qquad t \in [0,1]. \tag{3.4}$$

Now, let $Y \in B_{\mathcal{K}(\mathbb{E})}(Z,\sigma)$ be arbitrary. Set $C_1 = Y \cap \widetilde{B}_{\mathbb{E}}(x_1,\varepsilon/2)$, $C_2 = Y \cap \widetilde{B}_{\mathbb{E}}(x_2,\varepsilon/2)$ and observe that C_1 and C_2 are compact, and also nonempty since $x_i \in Z$, $i = 1,2$, and $\sigma < \varepsilon/2$. Let $t_\theta \in\,]0,1[$ be such that (3.3) is satisfied, with C_1 and C_2 defined above.

We claim that

$$M_Y(b_\theta(t_\theta)) \cap \widetilde{B}_{\mathbb{E}}(x_i,\varepsilon/2) \neq \emptyset, \qquad i = 1,2. \tag{3.5}$$

Indeed, let $y_i \in C_i$, $i = 1,2$, be such that

$$\|y_i - b_\theta(t_\theta)\| = e_{C_i}(b_\theta(t_\theta)), \qquad i = 1,2.$$

Clearly, $e_{C_i}(b_\theta(t_\theta)) \le e_Y(b_\theta(t_\theta))$, $i = 1,2$. Suppose that for $i = 1$ or $i = 2$ the strict inequality holds. Then, by (3.3),

$$e_{C_i}(b_\theta(t_\theta)) < e_Y(b_\theta(t_\theta)), \qquad i = 1,2. \tag{3.6}$$

Now let $y \in Y$ be such that $\|y - b_\theta(t_\theta)\| = e_Y(b_\theta(t_\theta))$, thus $y \in M_Y(b_\theta(t_\theta))$. From (3.4) it follows that for $i \in \{1,2\}$, say $i = 1$, we have $y \in B_{\mathbb{E}}(x_1,\varepsilon/2)$. Hence $y \in C_1$ and so $e_{C_1}(b_\theta(t_\theta)) \ge \|y - b_\theta(t_\theta)\|$, which gives $e_{C_1}(b_\theta(t_\theta)) \ge e_Y(b_\theta(t_\theta))$, contrary to (3.6). Hence,

$$e_{C_i}(b_\theta(t_\theta)) = e_Y(b_\theta(t_\theta)), \qquad i = 1,2.$$

Since $C_i \subset Y$, $i = 1,2$, it follows that

$$M_{C_i}(b_\theta(t_\theta)) \subset M_Y(b_\theta(t_\theta)), \qquad i = 1,2. \tag{3.7}$$

Moreover,

$$M_{C_i}(b_\theta(t_\theta)) \subset \widetilde{B}_{\mathbb{E}}(x_i,\varepsilon/2), \qquad i = 1,2. \tag{3.8}$$

Combining (3.7) and (3.8) gives (3.5).

From (3.2) and (3.5) it follows that $b_\theta(t_\theta) \in A(M_Y)$. Furthermore, $b_\theta(t_\theta) \in B_{\mathbb{E}}(a,s)$, for

$$\|b_\theta(t_\theta) - a\| \le \theta R \le \frac{d}{4} R = \frac{\varrho}{16} < s.$$

Hence $b_\theta(t_\theta) \in A(M_Y) \cap B_{\mathbb{E}}(a,s)$. As the set of such points $b_\theta(t_\theta)$ with $\theta \in \Delta$ is uncountable, we see that $Y \in \mathcal{K}(\mathbb{E}) \setminus \mathcal{N}_{a,s}$. Since, in addition, $Y \in B_{\mathcal{K}(\mathbb{E})}(Z,\sigma)$ is arbitrary, we have

$$B_{\mathcal{K}(\mathbb{E})}(Z,\sigma) \subset \mathcal{K}(\mathbb{E}) \setminus \mathcal{N}_{a,s}. \tag{3.9}$$

On the other hand, each $Y \in B_{\mathcal{K}(\mathbb{E})}(Z,\sigma)$ satisfies $h(Y,X) \le h(Y,Z) + h(Z,X) < \sigma + \varrho/2 \le \varrho$ for, by construction, $\sigma \le \varrho/2$ and $h(Z,X) \le \varrho/2$. Hence,

$$B_{\mathcal{K}(\mathbb{E})}(Z,\sigma) \subset B_{\mathcal{K}(\mathbb{E})}(X,\varrho).$$

Combining this with (3.9) gives (3.1), and thus the claim that $\mathcal{N}_{a,s}$ is nowhere dense in $\mathcal{K}(\mathbb{E})$ is proved, in Case 1.

Case 2. Suppose $X = \{a\}$. Take an $x_0 \in \mathbb{E}$ with $\|x_0 - a\| = \varrho/4$, and fix $x_1, x_2 \in \mathbb{E}$ as in Case 1. Set $Z = \overline{\mathrm{co}}\{x_0, x_1, x_2\}$. Clearly $Z \in \mathcal{K}(\mathbb{E})$, and $h(Z,X) = \varrho/2$. From this point the proof is as in Case 1 and so it is omitted.

Now we are ready to prove that the set $\mathcal{K}^0$ is residual in $\mathcal{K}(\mathbb{E})$. To this end, let $D \subset \mathbb{E}$ be a countable set everywhere dense in $\mathbb{E}$, and let $\mathbb{Q}^+$ be the set of all strictly positive rationals. Define

$$\mathcal{K}^* = \bigcap_{\substack{a \in D \\ s \in \mathbb{Q}^+}} (\mathcal{K}(\mathbb{E}) \setminus \mathcal{N}_{a,s}).$$

Clearly, $\mathcal{K}^*$ is residual in $\mathcal{K}(\mathbb{E})$. Furthermore, $\mathcal{K}^* \subset \mathcal{K}^0$. Indeed, let $X \in \mathcal{K}^*$, $x \in \mathbb{E}$ and $r > 0$. Take $a \in A$ and $s \in \mathbb{Q}^+$ so that $B_{\mathbb{E}}(a,s) \subset B_{\mathbb{E}}(x,r)$. Since $X \notin \mathcal{N}_{a,s}$, the set $A(M_X) \cap B_{\mathbb{E}}(a,s)$ is nonempty and uncountable. This shows that $A(M_X)$ is everywhere uncountable in $\mathbb{E}$, and so $X \in \mathcal{K}^0$. Hence $\mathcal{K}^* \subset \mathcal{K}^0$, and $\mathcal{K}^0$ is residual in $\mathcal{K}(\mathbb{E})$, for $\mathcal{K}^*$ is so. As $\mathcal{K}(\mathbb{E})$ is complete, $\mathcal{K}^0$ is dense in $\mathcal{K}(\mathbb{E})$. This completes the proof.

Remark 3.1. Let $\mathbb{E} = \mathbb{R}^n$ be endowed with the Euclidean norm. From Theorem 3.1 and the Mazur property it follows that most $X \in \mathcal{K}(\mathbb{R}^n)$, in the Baire category sense, can be represented as the intersection of a family of closed balls containing X, having on their boundary at least two points of X.

Remark 3.2. If X is a nonempty closed convex bounded subset of $\mathbb{E}$, beside the *ambiguous locus of uniqueness* $A^{\mathrm{u}}(M_X)$ given by $A^{\mathrm{u}}(M_X) = A(M_X)$, one can consider the *ambiguous locus of existence* $A^{\mathrm{e}}(M_X) = \{a \in \mathbb{E} \mid M_X(a) = \emptyset\}$ and the *ambiguous locus of well posedness* $A^{\mathrm{w}}(M_X) = \{a \in \mathbb{E} \mid \max(a, X) \text{ is not well posed}\}$. We recall that a maximization problem $\max(a, X)$ is said to be *well posed* if it has one and only one solution, say x, and every maximizing sequence converges to x. Clearly, $A^{\mathrm{u}}(M_X) \cup A^{\mathrm{e}}(M_X) \subset A^{\mathrm{w}}(M_X)$. However, while the local cardinality of the set $A^{\mathrm{w}}(M_X)$ can be studied, under appropriate hypotheses, by adapting the preceding approach, the investigation of the sets $A^{\mathrm{u}}(M_X)$ and $A^{\mathrm{e}}(M_X)$ seems to require a different approach.

Whenever $X \in \mathcal{K}(\mathbb{E})$, we have $A^{\mathrm{e}}(M_X) = \emptyset$ and $A^{\mathrm{w}}(M_X) = A^{\mathrm{u}}(M_X) = A(M_X)$, where the latter set is the ambiguous locus considered in Theorem 3.1.

Finally, we observe that the main result of this paper, proved for the farthest distance mapping from sets $X \in \mathcal{K}(\mathbb{E})$, has no analog for the nearest distance mapping since, in this case, the corresponding ambiguous locus is empty for each $X \in \mathcal{K}(\mathbb{E})$. A comprehensive treatment of nearest distance problems from closed sets can be found in Borwein and Fitzpatrick [3].

References

[1] E. Asplund, *Farthest points in reflexive locally uniformly rotund Banach spaces*, Israel J. Math. 4 (1966), 213–216.

[2] K. Bartke and H. Berens, *Eine Beschreibung der Nichteindeutigkeitsmenge für die beste Approximation in der Euklidischen Ebene*, J. Approx. Theory 47 (1986), 54–74.

[3] J. M. Borwein and S. Fitzpatrick, *Existence of nearest points in Banach spaces*, Canad. J. Math. 41 (1989), 702–720.

[4] F. S. De Blasi, J. Myjak and P. L. Papini, *Porous sets in best approximation theory*, J. London Math. Soc. (2) 44 (1991), 135–142.

[5] R. Deville and V. Zizler, *Farthest points in w^*-compact sets*, Bull. Austral. Math. Soc. 38 (1988), 433–439.

[6] N. Dunford and J. T. Schwartz, *Linear Operators*, Vol. I, Interscience, New York, 1958.

[7] M. Edelstein, *Farthest points of sets in uniformly convex Banach spaces*, Israel J. Math. 4 (1966), 171–176.

[8] P. M. Gruber, *Die meisten konvexen Körper sind glatt, aber nicht zu glatt*, Math. Ann. 229 (1977), 259–266.

[9] K. S. Lau, *Farthest points in weakly compact sets*, Israel J. Math. 22 (1975), 168–174.

[10] V. Klee, *Some new results on smoothness and rotundity in normed linear spaces*, Math. Ann. 139 (1959), 51–63.

[11] S. Miyajima and F. Wada, *Uniqueness of a farthest point in a bounded closed set in Banach spaces*, preprint.

[12] B. B. Panda and K. Dwivedi, *On existence of farthest points*, Indian J. Pure Appl. Math. 16 (1985), 486–490.

[13] T. Zajíček, *On the Fréchet differentiability of distance functions*, Rend. Circ. Mat. Palermo (2) Suppl. 5 (1985), 161–165.

[14] T. Zamfirescu, *Using Baire category in geometry*, Rend. Sem. Mat. Univers. Politec. Torino 43 (1985), 67–88.

[15] —, *The nearest point mapping is single valued nearly everywhere*, Arch. Math. (Basel) 54 (1990), 563–566.

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