[5] W. Żelazko, *On certain open problems in topological algebras*, Rend. Sem. Mat. Fis. Milano 59 (1989) (1992), 49–58.

[6] —, *Example of an algebra which is non-topologizable as a locally convex algebra*, Proc. Amer. Math. Soc. 110 (1990), 947–949.

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Precompactness in the uniform ergodic theory

by

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Abstract. We characterize the Banach space operators T whose arithmetic means $\{n^{-1}(I+T+\ldots+T^{n-1})\}_{n\ge1}$ form a precompact set in the operator norm topology. This occurs if and only if the sequence $\{n^{-1}T^n\}_{n\ge1}$ is precompact and the point 1 is at most a simple pole of the resolvent of T. Equivalent geometric conditions are also obtained.

Let T be a bounded linear operator on a complex Banach space X. The uniform ergodic theory deals with the asymptotic behaviour of the arithmetic means

$$M_n(T)=\frac{I+T+\ldots+T^{n-1}}{n}$$

in the operator norm (uniform) topology, as $n\in\mathbb{N}=\{1,2,3,\ldots\}$ tends to infinity. The basic result is due to Dunford [2, Theorem 3.16]:

THEOREM 1. *The sequence $\{M_n(T)\}$ uniformly converges if and only if*

1° $\lim n^{-1}\|T^n\|=0$, *and*

2° *the point* 1 *is at most a simple pole of the resolvent* $R_\lambda(T)=(T-\lambda I)^{-1}$.

Condition 2° means that either 1 does not belong to the *spectrum* $\sigma(T)$, or 1 is really a simple pole of $R_\lambda(T)$. In the latter case 1 is an isolated point of $\sigma(T)$, and the corresponding *Riesz projection*

$$P=-\frac{1}{2\pi i}\int R_\lambda(T)\,d\lambda \qquad (1)$$

has the image $\operatorname{Im} P=\{x\in X:\ Tx=x\}$. Moreover, $X=\operatorname{Im}P\oplus\operatorname{Ker}P$, and $\operatorname{Ker}P$ is a T-invariant closed subspace such that $1\notin\sigma(T|\operatorname{Ker}P)$.

A stronger asymptotic property is the convergence of the powers T^n. For this a spectral criterion was established by Koliha [8], [9] and Li [10]:

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THEOREM 2. *The sequence $\{T^n\}$ uniformly converges if and only if*

1° $\sup \|T^n\| < \infty$, *and*

2° *the point* 1 *is at most a simple pole of the resolvent* $R_\lambda(T)$, *and there are no other points* $\lambda \in \sigma(T)$ *with* $|\lambda| = 1$.

Note that condition 1° always implies $|\sigma(T)| \le 1$; in fact, it can be replaced by $|\sigma(T)| \le 1$ in Theorem 2.

Condition 2° implies that the *peripheral spectrum* $\{\lambda \in \sigma(T) : |\lambda| = 1\}$ is either $\{1\}$ or empty.

A wider problem concerns the precompactness of the sequence $\{T^n\}$, instead of the convergence. This situation was characterized by Kaashoek and West [6, Theorem 3], [7, Theorem I.2.3], and independently by Święch [14, Theorem 2]:

THEOREM 3. *The sequence $\{T^n\}$ is uniformly precompact if and only if*

1° $\sup \|T^n\| < \infty$, *and*

2° *every point* $\lambda \in \sigma(T)$ *with* $|\lambda| = 1$ *is a simple pole of the resolvent* $R_\lambda(T)$.

In this case the peripheral spectrum is finite (possibly empty), and all its points are eigenvalues. Also here condition 1° can be replaced by $|\sigma(T)| \le 1$.

A natural question arises: what is a similar criterion for $\{M_n(T)\}$ to be uniformly precompact? In this paper we answer this question:

THEOREM 4. *The sequence $\{M_n(T)\}$ is uniformly precompact if and only if*

1° *the sequence* $\{n^{-1}T^n\}$ *is uniformly precompact, and*

2° *the point* 1 *is at most a simple pole of the resolvent* $R_\lambda(T)$.

The case where T is a Riesz operator was considered in [16]. The question also appeared in the Banach algebra setting [13]; we consider this situation at the end of the paper.

Note that the *sufficiency* of conditions 1° and 2° is very easy in each of these four theorems. For instance, in Theorem 4 one can use the Riesz projection P (see (1)). Then $M_n(T)|\operatorname{Im} P$ is the identity, so we can assume that $1 \notin \sigma(T)$. But then

$$M_n(T) = (T - I)^{-1}\frac{T^n - I}{n}$$

is precompact by condition 1°.

The necessity of condition 1° in Theorem 4 follows immediately from the formula

$$\frac{T^n}{n} = \frac{n+1}{n}M_{n+1}(T) - M_n(T), \tag{2}$$

which also yields the necessity of 1° in Theorem 1.

Theorem 4 is the most general of all the above theorems as regards the *necessity* of the resolvent conditions 2°. This is obvious for Theorem 1. As for Theorem 3, we note that if $\{T^n\}$ is precompact, then $\{M_n(T)\}$, a subset of the convex hull of $\{T^n\}_{n\ge 0}$, is also precompact by Mazur's theorem. Replacing T by $\lambda^{-1}T$ for $\lambda \in \sigma(T)$, $|\lambda| = 1$, and applying Theorem 4 we get the necessity of 2° in Theorem 3. Now we can pass to Theorem 2 by noting that if $Tx = \lambda x$ with $|\lambda| = 1$, $\lambda \ne 1$, $x \ne 0$, then $T^n x = \lambda^n x$ is not convergent.

We have the chain of implications

$$\{T^n\} \text{ convergent} \Rightarrow \{T^n\} \text{ precompact} \Rightarrow$$
$$\Rightarrow \{M_n(T)\} \text{ convergent} \Rightarrow \{M_n(T)\} \text{ precompact},$$

where the second implication is a consequence of Theorems 3 and 1.

Let us give some examples to show that all these properties are distinct (that is, none of the above implications can be reversed).

EXAMPLE 1. Let $T = -I$. Then $\{T^n\}$ is precompact, but not convergent.

EXAMPLE 2. Let $T = -(I + V)^{-1}$, where V is the Volterra operator on the Hilbert space $X = L_2[0,1]$, defined by

$$(Vf)(t) = \int_0^t f(s)\,ds.$$

Then $\sigma(T) = \{-1\}$, and $\|T^n\| = 1$ for $n \in \mathbb{N}$ (see [5, Problems 146 and 150]). Thus, $\{M_n(T)\}$ converges by Theorem 1, but $\{T^n\}$ is not precompact by Theorem 3, because the point -1 is not a simple pole of the resolvent of T (in fact, it is an essential singularity). Note that such an example cannot be found within the Riesz operators.

EXAMPLE 3. Let

$$T = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

on $X = \mathbb{C}^2$, with $\lambda \ne 1$, $|\lambda| = 1$. Then

$$T^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}.$$

Thus, $\{M_n(T)\}$ does not converge, because $n^{-1}T^n$ does not tend to zero. But $\{M_n(T)\}$ is precompact.

Passing to the proof of necessity of condition 2° in Theorem 4 we start with some lemmas, which seem to be of independent interest.

LEMMA 1. *Let $\{M_n(T)\}$ be uniformly precompact. Then every limit point L of this sequence satisfies the equation*

$$(T - I)^2 L^2 = 0. \tag{3}$$

Proof. Let N be a subsequence of $\mathbb{N}$ such that the limits

$$L = \lim_N M_n(T) \quad \text{and} \quad S = \lim_N n^{-1}T^n$$

exist as $n \to \infty$, $n \in N$. Since

$$M_{n+1}(T) = \frac{n}{n+1} T M_n(T) + \frac{1}{n+1} I,$$

we have

$$\lim_N M_{n+1}(T) = TL.$$

Now, it follows from (2) that

$$(T - I)L = S. \tag{4}$$

On the other hand,

$$S^2 = \lim_N n^{-2}T^{2n} = 0, \tag{5}$$

because $\{n^{-1}T^{2n}\}$ is bounded a fortiori. Thus, (3) follows from (4) and (5), since L commutes with T. ∎

COROLLARY 1. *Suppose that* 1 *is not an eigenvalue of* T. *If* $\{M_n(T)\}$ *is uniformly precompact, then every limit point* L *of this sequence has* $L^2 = 0$.

Remark 1. Under the same conditions one can show that $L_1 L_2 = 0$ for any pair of limit points L_1, L_2.

Using Corollary 1 we prove

LEMMA 2. *Let* $\{M_n(T)\}$ *be uniformly precompact. If* 1 *is not an eigenvalue of* T, *then* $1 \notin \sigma(T)$.

Proof. Since $\|T^n\| = O(n)$ as $n \to \infty$, we have $|\sigma(T)| \le 1$. Consequently, if $1 \in \sigma(T)$, then 1 belongs to the approximate spectrum of T. This allows us to find a sequence $\{x_n\}$ of vectors such that $\|x_n\| = 1$ and $\|Tx_n - x_n\| < 1/n^3$. Then for $k \in \mathbb{N}$ we have

$$\|T^k x_n - x_n\| < \frac{1}{n^3} \sum_{j=0}^{k-1} \|T^j\| \le \frac{Ck(k+1)}{2n^3},$$

since $\|T^j\| \le Cj$ with some constant $C \ge 1$. Therefore,

$$\|M_n(T)x_n - x_n\| \le \frac{C}{2n},$$

and we see that $\lim \|M_n(T)x_n - x_n\| = 0$.

Let N be a subsequence of $\mathbb{N}$ such that $\lim_N M_n(T) = L$. Since $\|x_n\| = 1$, we have

$$\|Lx_n - x_n\| \le \|L - M_n(T)\| + \|M_n(T)x_n - x_n\|,$$

and hence

$$\lim_N \|Lx_n - x_n\| = 0. \tag{6}$$

Then also

$$\lim_N \|L^2 x_n - Lx_n\| = 0. \tag{7}$$

Since $L^2 = 0$ by Corollary 1, we conclude from (6) and (7) that $\lim_N \|x_n\| = 0$, contrary to $\|x_n\| = 1$. Thus, $1 \notin \sigma(T)$. ∎

Proof of Theorem 4. As we have already seen, it remains to show that the uniform precompactness of $\{M_n(T)\}$ implies condition 2°. To this end, consider the ergodic subspace

$$E = \{x \in X : \lim M_n(T)x \text{ exists}\},$$

which is obviously closed and T-invariant. The uniform precompactness implies that $\{M_n(T|E)\}$ converges uniformly to the operator P on E defined by

$$Px = \lim M_n(T)x, \quad x \in E.$$

By Theorem 1, the point 1 is at most a simple pole of $R_\lambda(T|E)$. (This conclusion can also be derived in the present context: Notice that $P^2 = P$, and $(T - I)\operatorname{Im} P = 0$. Also $\operatorname{Ker} P$ is T-invariant, and 1 is not an eigenvalue of $T|\operatorname{Ker} P$, hence $1 \notin \sigma(T|\operatorname{Ker} P)$ by Lemma 2; consequently, $(T - I)\operatorname{Ker} P = \operatorname{Ker} P$. Thus, 1 is at most a simple pole of $R_\lambda(T|E)$ by [1, Lemma 3.4.2] or [15, p. 330].)

Next, we pass to the factor space $\widetilde{X} = X/E$ and to the corresponding factor operator $\widetilde{T}$. Obviously, $\{M_n(\widetilde{T})\}$ is uniformly precompact. We shall show that 1 is not an eigenvalue of $\widetilde{T}$. Then $1 \notin \sigma(\widetilde{T})$ by Lemma 2. Now it is easy to verify that 1 is at most a simple pole of $R_\lambda(T)$.

So suppose that $\widetilde{T}\widetilde{x} = \widetilde{x}$ for the class $\widetilde{x} \in \widetilde{X}$ of a vector $x \in X$. Then $Tx - x \in E$, which means that $\lim M_n(T)(T - I)x$ exists. This limit is in fact the vector $v = \lim n^{-1}T^n x$, whence $T^k x = kv + o(k)$ as $k \to \infty$, which yields

$$M_n(T)x = \frac{n-1}{2} v + o(n) \quad \text{as } n \to \infty.$$

Since the left-hand side of the preceding formula is bounded, it follows that $v = 0$, which in turn implies that $x \in E$ by a known description of E (see [3, Theorem VIII.5.1]). Thus, $\widetilde{x} = 0$, and 1 is not an eigenvalue of $\widetilde{T}$. ∎

COROLLARY 2. *If* $\{M_n(T)\}$ *is uniformly precompact and* $\lim n^{-1}\|T^n\| = 0$, *then* $\{M_n(T)\}$ *is uniformly convergent.*

COROLLARY 3. *Let* $\{M_n(T)\}$ *be uniformly precompact, and let* L *be a limit point of this sequence. Then* $L = P + Q$, *where* P *is the Riesz projection* (1), *and* $Q^2 = 0$, $PQ = QP = 0$. *In particular*, $L^2 = P$, *and* $\sigma(L) \subset \{0, 1\}$.

Proof. We have $L = LP + L(I-P)$. Recall that $M_n(T)|\operatorname{Im} P$ is the identity, hence so is $L|\operatorname{Im} P$. Therefore, $LP = P$. The operator $Q = L(I-P)$ has the properties required: it satisfies $Q^2 = 0$ by Corollary 1, and $PQ = 0$ since P commutes with L; $QP = 0$ trivially. ■

Remark 2. It follows from Corollary 3 and Remark 1 that $L_1L_2 = P$ for any pair of limit points L_1, L_2 of the uniformly precompact sequence $\{M_n(T)\}$.

The results of Dunford [2] were complemented in [11] and [12] by clarifying the geometrical meaning of condition 2° in Theorem 1: this spectral condition can be replaced by the closedness of $\operatorname{Im}((T-I)^m)$ for some (in fact, any) $m \geq 1$ (the case $m = 2$ being already obtained by Dunford). Now we can give the corresponding counterpart of Theorem 4.

Note that

$$\sup \|M_n(T)\| < \infty \tag{8}$$

implies that

$$\operatorname{Im}(T-I) \cap \operatorname{Ker}(T-I) = 0. \tag{9}$$

Indeed, let u be in this intersection. Then $u = Tv - v$ for some $v \in X$, and $Tu = u$. Consequently, $T^n v = v + nu$, hence

$$M_n(T)v = v + \frac{n-1}{2}u,$$

so that $u = 0$ by (8).

Notice also that (9) does not follow from the precompactness of $\{n^{-1}T^n\}$: see the matrix T in Example 3, this time with $\lambda = 1$.

THEOREM 5. *Let $\{n^{-1}T^n\}$ be uniformly precompact and suppose that* (9) *holds. Then the following conditions are equivalent*:

1° $\{M_n(T)\}$ *is uniformly precompact*;

2° $\operatorname{Im}(T-I) + \operatorname{Ker}(T-I) = X$;

3° $\operatorname{Im}(T-I) + \operatorname{Ker}(T-I)$ *is closed*;

4° $\operatorname{Im}(T-I)$ *is closed*;

5° $\operatorname{Im}((T-I)^m)$ *is closed for some* $m \geq 1$.

Proof. The implication 1°⇒2° is a consequence of Theorem 4 and the Riesz decomposition corresponding to the point 1 (see (1)).

The implication 2°⇒3° is trivial.

Condition 3°, with the aid of (9), implies 4° in view of the following general fact:

If B is a bounded linear operator on X and $X = \operatorname{Im} B + Y$, where Y is a closed subspace such that $\operatorname{Im} B \cap Y = 0$, then $\operatorname{Im} B$ is closed; see [1, Lemma 3.2.4] or [15, Theorem IV.5.10].

The implication 4°⇒5° is trivial.

To complete the proof, we shall show that 5°⇒4°⇒1°. The first of these implications is a backward induction as in [12]: assuming that $\operatorname{Im}((T-I)^m)$ is closed for some $m > 1$, one can show, by a standard argument, that $\operatorname{Im}((T-I)^{m-1}) + \operatorname{Ker}(T-I)$ is closed, and then apply (9) together with the general fact cited above to conclude that $\operatorname{Im}((T-I)^{m-1})$ is closed. Thus, 5°⇒4°.

Notice that the a priori assumption of the uniform precompactness of $\{n^{-1}T^n\}$ has not yet been used. It will be essential in the final step 4°⇒1°, which is similar to the corresponding step in [11]. Having $\operatorname{Im}(T-I)$ closed, there is a constant $K > 0$ such that for every $y \in \operatorname{Im}(T-I)$ the equation $y = (T-I)x$ has a solution $x(y)$ with $\|x(y)\| \leq K\|y\|$. Then

$$\left\| M_n(T)y - \frac{T^n}{n}x(y) \right\| = \frac{\|x(y)\|}{n} \leq \frac{K\|y\|}{n}.$$

Take any subsequence N of $\mathbb{N}$. One can assume that $\{n^{-1}T^n\}_{n\in N}$ uniformly converges. Given $\varepsilon > 0$, we have $\|m^{-1}T^m - n^{-1}T^n\| < \varepsilon$ for all $m, n \in N$ sufficiently large. Then

$$\|(M_m(T) - M_n(T))y\| \leq K\|y\|(m^{-1} + n^{-1} + \varepsilon)$$

for these m, n, and all $y \in \operatorname{Im}(T-I)$. Hence $\{M_n(T)\}_{n\in N}$ is convergent on $\operatorname{Im}(T-I)$. We see that $\{M_n(T)\}$ is uniformly precompact on $\operatorname{Im}(T-I)$. It follows, by Theorem 4, that 1 is at most a simple pole for $T|\operatorname{Im}(T-I)$. Then the corresponding Riesz projection together with (9) yield that $\operatorname{Im}(T-I) = \operatorname{Im}((T-I)^2)$. Since (9) also gives $\operatorname{Ker}(T-I) = \operatorname{Ker}((T-I)^2)$, it follows that 1 is at most a simple pole of the resolvent of T on X by [1, Lemma 3.4.2] or [15, p. 330]. Thus, 1° holds by Theorem 4. ■

Remark 3. We have seen that $\operatorname{Im}((T-I)^m)$ does not depend on m provided that $\{M_n(T)\}$ is uniformly precompact. Also [12, Théorème 1] is a consequence of Theorem 5 and Corollary 2.

Remark 4. Concerning the final argument in the proof of Theorem 5 let us note that the formula

$$\begin{aligned}(\lambda - 1)I &= (T - \lambda I)\big(R_\lambda(T|\operatorname{Im}(T-I))(T-I) - I\big)\\ &= \big(R_\lambda(T|\operatorname{Im}(T-I))(T-I) - I\big)(T - \lambda I)\end{aligned}$$

implies that 1 is automatically a pole of order at most 2 for $R_\lambda(T)$, if $R_\lambda(T|\operatorname{Im}(T-I))$ had a simple pole at 1. Example 3 with $\lambda = 1$ shows that order 2 may occur in general. This cannot happen, however, if condition (9) is satisfied.

As another application of Theorem 4 we get immediately the following improvement of conditions 2° and 3° in [16, Theorem 6], a result related to the classical theorem of Gelfand [4].

COROLLARY 4. *If $\{M_n(T)\}$ is uniformly precompact and $\sigma(T) = \{1\}$, then $T = I$.*

EXAMPLE 4. The operator $T = (I+V)^{-1}$, where V is the Volterra operator from Example 2, shows that, in Corollary 4, the uniform precompactness cannot be replaced by just boundedness; it can, however, be replaced by the boundedness of both $\{M_n(T)\}$ and $\{M_n(T^{-1})\}$ (see [12, Théorème 2] or [16, Theorem 6]).

In conclusion let us remark that the above results can be extended to

$$M_n(a) = \frac{1 + a + \ldots + a^{n-1}}{n},$$

where a is an element of a unital Banach algebra A. It is enough to embed A isometrically into $L(A)$, the Banach algebra of bounded linear operators on A, by the left regular representation $T_a x = ax$, $x \in A$. Moreover, some additional information can be obtained knowing the algebraic surrounding of the element in question.

THEOREM 6. *Let A be a unital Banach algebra without non-zero nilpotent elements. If for some $a \in A$ the sequence $\{M_n(a)\}$ is precompact, then it is convergent.*

Proof. As we know from (5), all limit points of the sequence $\{n^{-1}a^n\}$ are nilpotent. Thus, $\lim n^{-1}a^n = 0$. It remains to apply Corollary 2. ■

Remark 5. Theorem 6 says, in other words, that the unital Banach algebra generated by an element a such that $\{M_n(a)\}$ is precompact, but not convergent, must contain a non-zero nilpotent. See Example 3 and Corollary 3.

Remark 6. If a unital Banach algebra A does contain a non-zero nilpotent element x, then there exists an $a \in A$ such that the sequence $\{M_n(a)\}$ is precompact, but not convergent. Indeed, one can assume that $x^2 = 0$ and take $a = \lambda + x$ with $|\lambda| = 1$, $\lambda \neq 1$.

References

[1] S. R. Caradus, W. E. Pfaffenberger and B. Yood, *Calkin Algebras and Algebras of Operators on Banach Spaces*, Marcel Dekker, New York, 1974.

[2] N. Dunford, *Spectral theory. I. Convergence to projections*, Trans. Amer. Math. Soc. 54 (1943), 185–217.

[3] N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*, Interscience Publishers, New York, 1958.

[4] I. Gelfand, *Zur Theorie der Charaktere der Abelschen topologischen Gruppen*, Mat. Sb. 9 (1941), 49–50.

[5] P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, 1967.

[6] M. A. Kaashoek and T. T. West, *Locally compact monothetic semi-algebras*, Proc. London Math. Soc. 18 (1968), 428–438.

[7] —, —, *Locally Compact Semi-Algebras with Applications to Spectral Theory of Positive Operators*, North-Holland, Amsterdam, 1974.

[8] J. J. Koliha, *Power convergence and pseudoinverses of operators in Banach spaces*, J. Math. Anal. Appl. 48 (1974), 446–469.

[9] —, *Some convergence theorems in Banach algebras*, Pacific J. Math. 52 (1974), 467–473.

[10] H. Li, *Equivalent conditions for the convergence of a sequence $\{B^n\}_{n=1}^{\infty}$*, Acta Math. Sinica 29 (1986), 285–288 (in Chinese).

[11] M. Lin, *On the uniform ergodic theorem*, Proc. Amer. Math. Soc. 43 (1974), 337–340.

[12] M. Mbekhta et J. Zemánek, *Sur le théorème ergodique uniforme et le spectre*, C. R. Acad. Sci. Paris Sér. I Math. 317 (1993), 1155–1158.

[13] H. C. Rönnefarth, *Charakterisierung des Verhaltens der Potenzen eines Elementes einer Banach-Algebra durch Spektraleigenschaften*, Diplomarbeit, Technische Universität Berlin, Berlin, 1993.

[14] A. Święch, *Spectral characterization of operators with precompact orbit*, Studia Math. 96 (1990), 277–282; 97 (1991), 266.

[15] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, Wiley, New York, 1980.

[16] J. Zemánek, *On the Gelfand–Hille theorems*, in: Functional Analysis and Operator Theory, J. Zemánek (ed.), Banach Center Publ. 30, Warszawa, 1994, 369–385.

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