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On topologization of countably generated algebras

by

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Abstract. We prove that any real or complex countably generated algebra has a complete locally convex topology making it a topological algebra. Assuming the continuum hypothesis, it is the best possible result expressed in terms of the cardinality of a set of generators. This result is a corollary to a theorem stating that a free algebra provided with the maximal locally convex topology is a topological algebra if and only if the number of variables is at most countable. As a byproduct we obtain an example of a semitopological (non-topological) algebra with every commutative subalgebra topological.

A *topological algebra* is a real or complex algebra whose underlying vector space is a topological (Hausdorff) vector space and whose multiplication is jointly continuous. In the case when the multiplication is merely separately continuous, we say that the algebra in question is *semitopological*. A *locally convex algebra* is a topological algebra whose underlying vector space is a locally convex space. Its topology can be given by means of a family $(\|x\|_\alpha)_{\alpha\in\mathfrak{A}}$ of seminorms such that for each index α there is a $\beta\in\mathfrak{A}$ so that

$$\|xy\|_\alpha \le \|x\|_\beta\|y\|_\beta \tag{1}$$

for all x and y in the algebra in question. For more information on locally convex algebras the reader is referred to [1] and [4].

The *maximal locally convex topology* on a vector space X is the topology given by means of all seminorms. It is clearly a Hausdorff topology. This topology will be denoted by $\tau_{\max}^{\mathrm{LC}}$. It is known (see [3], Example on p. 56) that $(X,\tau_{\max}^{\mathrm{LC}})$ is a complete locally convex space. In [5] we observed that any real or complex algebra topologized with $\tau_{\max}^{\mathrm{LC}}$ is a semitopological algebra. On the other hand, we proved in [6] that for any real or complex vector space X the algebra $L(X)$ of all its endomorphisms has a locally convex topology making it a locally convex algebra if and only if the dimension of X is finite. A natural question arises: under what conditions a given algebra is topologizable as a locally convex algebra? In this paper we give an answer

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to this question in terms of the cardinality of a set of generators of the algebra in question (Theorem 2).

Let $\mathbf{t} = (t_i)_{i\in I}$ be a family of variables. The (real or complex) *free algebra* in the variables $\mathbf{t}$, denoted by $F(\mathbf{t})$, is the algebra of all non-commuting polynomials in the variables $\mathbf{t}$ with scalar coefficients; we assume that $F(\mathbf{t})$ has a unit element denoted by e. Put $I^{(\infty)} = \bigcup_{n=0}^{\infty} I^n$, where $I^0 = (0)$ and 0 is not an element of I. For any $\mathbf{i}$ in $I^{(\infty)}$, $\mathbf{i} = (i_1, \ldots, i_k)$, or $\mathbf{i} = 0$, we put $\mathbf{t}^{\mathbf{i}} = t_{i_1} \ldots t_{i_k}$, or, respectively, $\mathbf{t}^0 = e$. With this notation every element x of $F(\mathbf{t})$ can be written in the form

$$x = \sum_{\mathbf{i}\in I^{(\infty)}} \xi_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}, \tag{2}$$

where the $\xi_{\mathbf{i}}$ are real or complex coefficients, and only a finite number of the coefficients are different from zero. Writing $\mathbf{ij} = (i_1, \ldots, i_k, j_1, \ldots, j_l)$ if $\mathbf{i} = (i_1, \ldots, i_k)$, $\mathbf{j} = (j_1, \ldots, j_l)$, and $\mathbf{i}0 = 0\mathbf{i} = \mathbf{i}$, for $\mathbf{i}, \mathbf{j} \in I^{(\infty)}$, we have

$$xy = \sum_{\mathbf{k}\in I^{(\infty)}} \Big(\sum_{\mathbf{ij}=\mathbf{k}} \xi_{\mathbf{i}}\eta_{\mathbf{j}}\Big)\mathbf{t}^{\mathbf{k}}, \tag{3}$$

where y is of the form (2) with coefficients $\eta_{\mathbf{i}}$ instead of $\xi_{\mathbf{i}}$.

We shall need the following

LEMMA. *Assume that the index set I is non-void and at most countable. Let $\mathbf{i} \to a_{\mathbf{i}}$ be a positive function defined on $I^{(\infty)}$ such that $a_0 = 1$. Then there is a positive function b on $I^{(\infty)}$ with $b_0 = 1$ such that for all $\mathbf{i}, \mathbf{j}$ in $I^{(\infty)}$ we have*

$$a_{\mathbf{ij}} \le b_{\mathbf{i}} b_{\mathbf{j}}. \tag{4}$$

Proof. Clearly $I^{(\infty)}$ is countable, so that we can arrange all its elements in a sequence $\mathbf{i}_1, \mathbf{i}_2, \ldots$ with $\mathbf{i}_1 = 0$. We can now rewrite (4) as

$$a_{\mathbf{i}_i \mathbf{i}_j} \le b_{\mathbf{i}_i} b_{\mathbf{i}_j} \tag{5}$$

for $i, j = 1, 2, \ldots$ We prove (5) by induction. Put first $b_0 = 1$ and suppose that we have defined $b_{\mathbf{i}_1}, \ldots, b_{\mathbf{i}_k}$ so that (5) is satisfied for $i, j \le k$. We now put

$$b_{\mathbf{i}_{k+1}} = \max\left\{[a_{\mathbf{i}_{k+1}\mathbf{i}_{k+1}}]^{1/2}, \max_{1\le j\le k}\left[\frac{a_{\mathbf{i}_{k+1}\mathbf{i}_j}}{b_{\mathbf{i}_j}}, \frac{a_{\mathbf{i}_j\mathbf{i}_{k+1}}}{b_{\mathbf{i}_j}}\right]\right\}.$$

It is clear that with $b_{\mathbf{i}_{k+1}}$ so defined, (5) holds for all $i, j \le k+1$. The conclusion follows.

We shall prove the following

THEOREM 1. *Let I be a non-void set of indices and consider the real or complex free algebra $F(\mathbf{t})$, $\mathbf{t} = \{(t_i) : i \in I\}$. Then $(F(\mathbf{t}), \tau_{\max}^{\mathrm{LC}})$ is a (complete) locally convex topological algebra if and only if the set I is at most countable.*

Proof. First we show that for I at most countable the multiplication in $(F(\mathbf{t}), \tau_{\max}^{\mathrm{LC}})$ is jointly continuous. We have to show that for a given seminorm $|x|$ on $F(\mathbf{t})$ there is a seminorm $\|x\|$ such that for all x and y in $F(\mathbf{t})$,

$$|xy| \le \|x\|\|y\|. \tag{6}$$

Without loss of generality we can assume $|e| \le 1$. Writing x in the form (2) we have

$$|x| \le \sum_{\mathbf{i}} |\xi_{\mathbf{i}}||\mathbf{t}^{\mathbf{i}}|.$$

Setting $a_{\mathbf{i}} = \max(1, |\mathbf{t}^{\mathbf{i}}|)$ we obtain a positive function on $I^{(\infty)}$ satisfying $a_0 = 1$. For x of the form (2), set

$$|x|_a = \sum_{\mathbf{i}} a_{\mathbf{i}}|\xi_{\mathbf{i}}|. \tag{7}$$

Then

$$|x| \le |x|_a \tag{8}$$

for all x in $F(\mathbf{t})$.

Let b be the function of the Lemma satisfying (4), and let $|x|_b$ be the seminorm of the form (7) with b instead of a. By (3), (4) and (8) we obtain

$$|xy| \le \sum_{\mathbf{k}\in I^{(\infty)}} a_{\mathbf{k}} \Big|\sum_{\mathbf{ij}=\mathbf{k}} \xi_{\mathbf{i}}\eta_{\mathbf{j}}\Big| \le \sum_{\mathbf{k}} \sum_{\mathbf{ij}=\mathbf{k}} b_{\mathbf{i}} b_{\mathbf{j}} |\xi_{\mathbf{i}}||\eta_{\mathbf{j}}|$$
$$= \sum_{\mathbf{i}\in I^{(\infty)}} b_{\mathbf{i}}|\xi_{\mathbf{i}}| \sum_{\mathbf{j}\in I^{(\infty)}} b_{\mathbf{j}}|\eta_{\mathbf{j}}| = |x|_b|y|_b.$$

Thus (6) holds with $\|x\| = |x|_b$, and $(F(\mathbf{t}), \tau_{\max}^{\mathrm{LC}})$ is a topological algebra.

We now show that $(F(\mathbf{t}), \tau_{\max}^{\mathrm{LC}})$ fails to be a topological algebra for I uncountable. If $\mathrm{card}(I) \ge \mathfrak{c}$ (continuum), we can assume that the unit interval $[0,1]$ is a subset of I and put $I_0 = [0,1]$. If $\mathrm{card}(I) \le \mathfrak{c}$ (which can happen if we reject the continuum hypothesis) we can assume $I \subset [0,1]$ and put $I_0 = I$. In both cases I_0 is an uncountable subset of the unit interval. Assume that our algebra is topological. Then for every seminorm $|x|$ there is a seminorm $\|x\|$ such that (6) holds. Since the products $t_s t_r$, $s, r \in I_0$, are linearly independent, there exists a seminorm $|x|$ on $F(\mathbf{t})$ satisfying

$$|t_r t_s| = \begin{cases} 1 & \text{if } r = s, \\ |r-s|^{-1} & \text{otherwise.} \end{cases}$$

Indeed, the elements $t_r t_s$, $r, s \in I_0$, can be included in a Hamel basis (h_α), and for such a basis and any non-negative function $\alpha \to a_\alpha$, the formula

$$\Big|\sum_\alpha \xi_\alpha h_\alpha\Big| = \sum_\alpha a_\alpha |\xi_\alpha|$$

defines on $F(\mathbf{t})$ a seminorm assuming on h_α the given non-negative values a_α. The formula (6) now implies

$$|r-s|^{-1} \le \varphi(r)\varphi(s), \quad r \ne s,\ r, s \in I_0, \tag{9}$$

where $\varphi(q) = \|t_q\|$, $q \in I_0$. But such a (finite) function φ cannot exist because for a given natural n, (9) implies that $\varphi(t) > n$ except for finitely many points in I_0, and so $\varphi(t) = \infty$ except for countably many points in I_0. The conclusion follows.

As a corollary we obtain the following result:

THEOREM 2. *Let A be a countably or finitely generated real or complex algebra. Then A can be topologized as a complete locally convex topological algebra. More precisely, the algebra $(A, \tau_{\max}^{\mathrm{LC}})$ is a topological algebra.*

Proof. It is well known that any algebra is a quotient algebra of a free algebra with the set of variables of the same cardinality as a set of generators of the algebra in question. Thus A is the quotient of $F(\mathbf{t})$ by a two-sided ideal J, and $\mathbf{t}$ is at most countable. Since in the topological vector space $(X, \tau_{\max}^{\mathrm{LC}})$, X any real or complex vector space, any linear subspace is closed, the ideal J is closed and the algebra A with the quotient topology is a locally convex topological algebra. It remains to be shown that the quotient topology on A coincides with $\tau_{\max}^{\mathrm{LC}}$. Denote elements of A (cosets) by $[x]$, with $x \in F(\mathbf{t})$. Take a Hamel basis $([h_\alpha])$ in A. The elements (h_α) are linearly independent in $F(\mathbf{t})$ so they can be included in a Hamel basis (l_β) for $F(\mathbf{t})$ obtained by adding to (h_α) any Hamel basis for the ideal J. Let $|[x]|$ be any seminorm on A. For any element $[x]$ of A we have $[x] = \sum_\alpha \xi_\alpha [h]_\alpha$ and

$$|[x]| \le \sum_\alpha |\xi_\alpha||[h_\alpha]| = \sum_\alpha a_\alpha |\xi_\alpha| =: |[x]|_a.$$

Thus $|[x]|$ is dominated by the seminorm $|[x]|_a$ defined above. It is easy to verify that the seminorm $|[x]|_a$ is the quotient of the seminorm $|x|_b$ on $F(\mathbf{t})$ given by means of the Hamel basis (l_β) and the function $\beta \to b_\beta$, where

$$b_\beta = \begin{cases} a_\alpha & \text{if } l_\beta = h_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that the seminorm $|[x]|$ is continuous on A provided with the quotient topology. Since it was an arbitrary seminorm, the conclusion follows.

Remark. Assuming the continuum hypothesis, Theorem 2, or rather its first part offers the best possible result expressed in terms of the cardinality of a set of generators, even if we relax the completeness requirement. In fact, if X is a countable-dimensional vector space, then the algebra $L(X)$ of all its endomorphisms is non-topologizable as a locally convex topological algebra (see [6]). But then $L(X)$ has cardinality continuum and so it has continuum many generators (not less, by Theorem 2). Even if we relax the requirement of local convexity the result does not improve: in [2], a commutative algebra is constructed with the set of generators of cardinality continuum, which is not topologizable as a topological algebra.

As another corollary of Theorem 1 we obtain the following result. It answers the question whether a semitopological algebra A with every commutative subalgebra being a topological algebra in the topology inherited from A, is topological itself.

THEOREM 3. *There exists a locally convex semitopological* (*non-topological*) *algebra with every commutative subalgebra topological.*

Proof. Let A be the algebra $F(\mathbf{t})$ with the set $\mathbf{t}$ of variables of cardinality continuum, provided with the topology $\tau_{\max}^{\mathrm{LC}}$. By Theorem 1 it is a complete locally convex semitopological algebra which is not topological. Let $\mathcal{A}$ be a commutative subalgebra of A. If $x \in \mathcal{A}$ and x is not of the form λe, then $\mathcal{A}$ is contained in the commutant $(x)' = \{y \in A : xy = yx\}$. But x is of the form (2), involving a non-void finite set of variables, say $t_1, \ldots, t_k$. It is easy to see that any y in $(x)'$ depends only upon $t_1, \ldots, t_k$ and so it is contained in the subalgebra $F(t_1, \ldots, t_k)$ of $F(\mathbf{t})$. As in the proof of Theorem 2 we show that the topology of $F(\mathbf{t})$ restricted to $F(t_1, \ldots, t_k)$ is $\tau_{\max}^{\mathrm{LC}}$. By Theorem 1 the algebra $F(t_1, \ldots, t_k)$, and so $\mathcal{A}$, is a topological algebra. The conclusion follows.

Acknowledgments. In the first draft of the paper we proved Theorem 1 under the assumption of the continuum hypothesis. The author is greatly indebted to the referee for calling his attention to the fact that the same proof works as well without this assumption.

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Precompactness in the uniform ergodic theory

by

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Abstract. We characterize the Banach space operators T whose arithmetic means $\{n^{-1}(I+T+\ldots+T^{n-1})\}_{n\ge1}$ form a precompact set in the operator norm topology. This occurs if and only if the sequence $\{n^{-1}T^n\}_{n\ge1}$ is precompact and the point 1 is at most a simple pole of the resolvent of T. Equivalent geometric conditions are also obtained.

Let T be a bounded linear operator on a complex Banach space X. The uniform ergodic theory deals with the asymptotic behaviour of the arithmetic means

$$M_n(T)=\frac{I+T+\ldots+T^{n-1}}{n}$$

in the operator norm (uniform) topology, as $n\in\mathbb{N}=\{1,2,3,\ldots\}$ tends to infinity. The basic result is due to Dunford [2, Theorem 3.16]:

THEOREM 1. *The sequence* $\{M_n(T)\}$ *uniformly converges if and only if*

1° $\lim n^{-1}\|T^n\|=0$, *and*

2° *the point* 1 *is at most a simple pole of the resolvent* $R_\lambda(T)=(T-\lambda I)^{-1}$.

Condition 2° means that either 1 does not belong to the *spectrum* $\sigma(T)$, or 1 is really a simple pole of $R_\lambda(T)$. In the latter case 1 is an isolated point of $\sigma(T)$, and the corresponding *Riesz projection*

$$P=-\frac{1}{2\pi i}\int R_\lambda(T)\,d\lambda \qquad (1)$$

has the image $\operatorname{Im}P=\{x\in X: Tx=x\}$. Moreover, $X=\operatorname{Im}P\oplus\operatorname{Ker}P$, and $\operatorname{Ker}P$ is a T-invariant closed subspace such that $1\notin\sigma(T|\operatorname{Ker}P)$.

A stronger asymptotic property is the convergence of the powers T^n. For this a spectral criterion was established by Koliha [8], [9] and Li [10]:

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