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On unbounded hyponormal operators III

by

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Abstract. The paper deals mostly with spectral properties of unbounded hyponormal operators. Some nontrivial examples of such operators are given.

I. Introduction. In this work we continue our previous study of unbounded hyponormal operators, [1], [2]. We concentrate on some of their basic spectral properties, and on their polar factors. We also find when the square of a hyponormal operator is the generator of a holomorphic semigroup. The paper ends up with two examples of new classes of unbounded hyponormal operators.

Let H be a complex Hilbert space and let T be a densely defined linear operator in H with domain $D(T)$.

We say that T is *hyponormal* if $D(T) \subset D(T^*)$ and $\|T^*f\| \le \|Tf\|$, $f \in D(T)$. We refer to [1] for basic facts concerning unbounded hyponormal operators. Throughout the paper $\sigma(T)$, $W(T)$ and $R(\lambda, T)$ denote the spectrum, the numerical range and the resolvent of T, respectively. For a set $A \subset \mathbb{C}$ its closure is denoted by $\operatorname{cl} A$, $\bar{A}$ stands for $\{\bar{\lambda} : \lambda \in A\}$, and $\operatorname{conv} A$ denotes the closed convex hull of A.

II. A few spectral relations. Though some elementary facts about unbounded hyponormal operators were proved in our earlier works [1], [2], the following lemmas seem to be useful, and were not stated there.

LEMMA 2.1. *Let T be a closed hyponormal operator in H. Then $W(T) \subset \operatorname{conv} \sigma(T)$.*

Proof. There are two possibilities.

1) $\operatorname{conv} \sigma(T) = \mathbb{C}$. Then the inclusion is trivial.

2) $\operatorname{conv} \sigma(T) \ne \mathbb{C}$. Since $\alpha T + \beta I$ is hyponormal for any $\alpha, \beta \in \mathbb{C}$, we may assume without loss of generality that $\operatorname{conv} \sigma(T) \subset \mathbb{C}^+ = \{\lambda : \operatorname{Re} \lambda \ge 0\}$. It remains to prove that $W(T) \subset \mathbb{C}^+$.

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Suppose that $0 \notin \sigma(T)$. Then for $y \in D(T)$, $\|y\| = 1$, we have $y = T^{-1}x$ for some $x \neq 0$. Hence

$$(Ty, y) = (x, T^{-1}x) = \|x\|^2(x_1, T^{-1}x_1),$$

where $x_1 = x\|x\|^{-1}$. Since $\operatorname{cl} W(T^{-1}) = \operatorname{conv} \sigma(T^{-1})$ and $z \to 1/z$ maps $\mathbb{C}^+$ onto $\mathbb{C}^+$, we have $(Ty, y) \in \mathbb{C}^+$.

If $0 \in \sigma(T)$ and $\operatorname{conv} \sigma(T) \subset \mathbb{C}^+$, then $T + \varepsilon I$ is invertible for any $\varepsilon > 0$. Therefore $((T + \varepsilon I)y, y) \in \mathbb{C}^+$ by the above argument and we get the desired inclusion by letting $\varepsilon \to 0$. ∎

As a consequence of Lemma 1 we have

LEMMA 2.2. *If T is a closed hyponormal operator and $\operatorname{cl} W(T) \cup \operatorname{cl} \overline{W(T^*)}$ is convex, then*

$$\operatorname{conv} \sigma(T) = \operatorname{cl} W(T) \cup \operatorname{cl} \overline{W(T^*)}.$$

Proof. We may assume that $\operatorname{conv}(T) \neq \mathbb{C}$. By repeating the reasoning given in the proof of Lemma 1 we check that $\operatorname{cl} \overline{W(T^*)} \subset \operatorname{conv} \sigma(T)$. On the other hand, for any closed operator A,

$$\sigma(A) \subset \operatorname{cl} W(A) \cup \operatorname{cl} \overline{W(A^*)}$$

(see [6]). It follows that

$$\operatorname{conv} \sigma(T) \subset \operatorname{cl} W(T) \cup \operatorname{cl} \overline{W(T^*)}$$

and this completes the proof. ∎

COROLLARY 2.3. *If T is a closed hyponormal operator and $D(T) = D(T^*)$, then $\operatorname{conv} \sigma(T) = \operatorname{cl} W(T)$.*

Proof. The equality $D(T) = D(T^*)$ implies that $\overline{W(T^*)} = W(T)$. Hence $\operatorname{cl} W(T) \cup \operatorname{cl} \overline{W(T^*)} = \operatorname{cl} W(T)$ is convex and the result follows from Lemma 2. ∎

We conclude this section with a useful theorem concerning the problem of computing $\sigma(T)$. It is well known that for a sequence T_n of bounded hyponormal operators which is uniformly convergent to T one has the equality

$$\text{(R)} \qquad \sigma(T) = \bigcap_{n=1}^{\infty} \operatorname{cl} \bigcup_{m=n}^{\infty} \sigma(T_m).$$

An extension of this result to the unbounded case has been given in [1]. The theorem we are going to prove below gives another useful extension of (R) in the unbounded case. Before we state this extension let us introduce the following notation.

For a sequence σ_k of closed sets in $\mathbb{C}$ we define

$$\lim \sigma_k = \{\lambda \in \mathbb{C} : \text{there exists a sequence } \lambda_k \in \sigma_k \text{ such that } \lambda = \lim_k \lambda_k\}.$$

Recall that a linear subspace $D \subset D(T)$ is a *core* for a closed operator T if $\overline{T_{|D}} = T$, the closure in the graph norm.

THEOREM 2.4. *Let T_k be a sequence of closed hyponormal operators. Suppose that T is a closed hyponormal operator such that $D(T) \subset D(T_k)$ and $D(T^*) \subset D(T_k^*)$. Assume that there exist sequences of positive numbers a_k, b_k, c_k, d_k, all converging to zero, and satisfying the inequalities*

(a) $\|(T^* - T_k^*)h\| \leq a_k\|h\| + b_k\|T^*h\|$, $h \in D^*$,

(b) $\|(T - T_k)f\| \leq c_k\|f\| + d_k\|Tf\|$, $f \in D$,

where D and D^* are the cores of T and T^*, respectively. If $T - T_k$ considered on $D(T)$ is closed for $k \geq k_0$, then

$$\lim_k \sigma(T_k) = \sigma(T).$$

Proof. (i) We first prove the inclusion $\sigma(T) \subset \lim_k \sigma(T_k)$. Suppose that $0 \in \sigma(T)$. Then there exists a sequence $f_n \in D^*$, $\|f_n\| = 1$, for which $\lim_n \|T^* f_n\| = 0$. Hence

$$\begin{aligned} \operatorname{dist}(0, \sigma(T_m^*)) &\leq \|T_m^* f_m\| \\ &\leq \|(T_m^* - T^*)f_m\| + \|T^* f_m\| \\ &\leq (1 + b_m)\|T^* f_m\| + a_m \to 0 \end{aligned}$$

as $m \to \infty$. This proves the inclusion.

(ii) To prove the opposite inclusion suppose that $0 \notin \sigma(T)$. Then for any λ in the open disc $K(0, (2\|T^{-1}\|)^{-1})$ and g such that $T^{-1}g \in D$ we have

$$\begin{aligned} \|(T - T_n + \lambda I)T^{-1}g\| &\leq \|(T - T_n)T^{-1}g\| + |\lambda|\,\|T^{-1}g\| \\ &\leq c_n\|T^{-1}g\| + d_n\|g\| + |\lambda|\,\|T^{-1}g\|. \end{aligned}$$

Since the set of all g such that $T^{-1}g \in D$ is dense in H it follows that $\|(T - T_n - \lambda I)T^{-1}\| < 1$, for n sufficiently large and $|\lambda| < (2\|T^{-1}\|)^{-1}$. Theorem 5.11 of [8] now implies that $\lambda I - T_n$ is a bijection for λ and n as above. Therefore $\sigma(T_n) \cap K(0, (2\|T^{-1}\|)^{-1}) = \emptyset$, for n sufficiently large. Thus $0 \notin \lim_n \sigma(T_n)$. This completes the proof. ∎

Remark 2.1. Note that for bounded T_k the operators $T - T_k$ are always closed on $D(T)$.

Remark 2.2. The equality $\lim_n \sigma(T_n) = \sigma(T)$ is stronger than the equality (R), in general.

III. A class of Putnam. Following the work of Putnam [5] we now consider a special class of hyponormal operators, namely those hyponormal operators T which can be written in the form

$$(1) \qquad T = K + iL,$$

where K and L are selfadjoint and L is bounded. Note that $D(T) = D(K)$ and $T^* = K - iL$. By careful repetition of the reasoning given in [5] one can prove the following

THEOREM 3.1. *Let T be a hyponormal operator which has a Cartesian decomposition* (1). *Assume there is no nonzero $f \in D(T)$ such that $\|Tf\| = \|T^*f\|$. Then*

(i) *K is absolutely continuous*,

(ii) *$\sigma(K) = \Pi_x\sigma(T)$, where Π_x denotes the projection of $\mathbb{C}$ onto the real axis*,

(iii) *if $\sigma(K) \neq \mathbb{R}$ then $\sigma(L) = \mathrm{cl}(\Pi_y\sigma(T))$, where Π_y is the projection onto the imaginary axis, and L is absolutely continuous*,

(iv) *if the planar measure $m(\sigma(T))$ is finite then $LD(K) \subset D(K)$ and the commutator $KL - LK$ is bounded.*

In Section IV we shall give some examples of hyponormal T which have a Cartesian decomposition of type (1).

IV. When does T^2 generate a semigroup? In our earlier work [1] we gave a sufficient condition for a hyponormal operator A to generate a hyponormal semigroup. Since the square A^2 is not hyponormal in general, the following result seems to be interesting.

THEOREM 4.1. *Let A be a closed hyponormal operator. Assume that there exists $\varepsilon > 0$ for which*

$$\sigma(A) \subset \{w : \pi/2 \le \arg w \le 3\pi/4 - \varepsilon\} \cup \{w : -3\pi/4 + \varepsilon \le \arg w \le -\pi/2\}.$$

Then A^2 generates a holomorphic semigroup in the set

$$\{v : |\arg v| < \varrho,\ 0 < \varrho < 2\varepsilon\}.$$

Proof. It is enough to check that

$$(*)\qquad \|R(\lambda, A^2)\| \le M|\lambda|^{-1},$$

for any λ such that $|\arg\lambda| < \pi/2 + \delta$, $0 < \delta < 2\varepsilon$. Write $\lambda = v^2$ for some v with $|\arg v| < \pi/4 + \delta/2$. This choice of v implies that

$$\mathrm{dist}(v, \sigma(A)) \ge \mathrm{Re}\, v \ge |v|\cos(\pi/4 + \delta/2)$$

and

$$\mathrm{dist}(v, \sigma(-A)) \ge |v|\sin(\varepsilon - \delta/2).$$

Hence

$$\begin{aligned}\|R(\lambda, A^2)\| &= \|R(v^2, A^2)\| = \left\|\frac{1}{2v}[R(v, A) + R(v, -A)]\right\| \\ &\le \frac{1}{2|v|}[\mathrm{dist}(v, \sigma(A))^{-1} + \mathrm{dist}(v, \sigma(-A))^{-1}] \\ &\le \frac{1}{2|v|^2}\left[\left(\cos\left(\frac{\pi}{4} + \frac{\delta}{2}\right)\right)^{-1} + \left(\sin\left(\varepsilon - \frac{\delta}{2}\right)\right)^{-1}\right] = M|\lambda|^{-1}.\end{aligned}$$

This ends the proof of (*) and completes the proof. ∎

V. Spectral properties of polar factors. Let T be a hyponormal operator with polar decomposition

$$(2)\qquad T = U|T|.$$

Below we shall discuss some spectral properties of the polar factors $|T|$ and U. In what follows we always assume that U is unitary. It is easy to check that this is equivalent to $0 \notin \sigma_p(T^*)$. As in the bounded case we have

PROPOSITION 5.1. *If $\sigma(U) \neq \mathbb{T}$ (the unit circle) and $M_\lambda = \{x : |T|x = \lambda x\}$ then $UM_\lambda \subseteq M_\lambda$.*

Proof. Applying the Cayley transform (see [9, p. 6]) the proof is reduced to the following property of a hyponormal operator S. If $S = X + iY$, where $X = X^*$ is bounded and $Y^* = Y$ is positive, and $M_a = \{h : Yh = ah\}$, then $SM_a \subseteq M_a$. ∎

In order to state the next result we extend Lemma 2.1 of [9] to the unbounded case. Let $z = \varrho e^{i\theta} \neq 0$ be a complex number. Write $T_z = T - zI$. Since U is unitary a direct computation gives

$$(\mathrm{r})\qquad \begin{aligned}\|T_z f\|^2 &= \|(|T| - \varrho)f\|^2 + \varrho\| \,|T|^{1/2}(U - e^{i\theta})f\|^2 \\ &\quad + \varrho((|T| - |T^*|)f, f),\end{aligned}$$

where $f \in D(T^*T)$. But $D(|T|) = D(T)$ and $D(T^*T)$ is the core for T, hence (r) holds for $f \in D(T)$.

Let $\sigma_\pi(T)$ (respectively $\sigma_e(T)$) stand for the approximate point spectrum (respectively essential spectrum) of T.

COROLLARY 5.2. *Let T be a closed hyponormal operator.*

(i) *If $z \in \sigma_\pi(T)$ $(z \neq 0)$, then $|z| \in \sigma(|T|)$.*

(ii) *If $w \in \sigma(T)$ $(w \neq 0)$ and $\overline{w} \notin \sigma_p(T^*)$, then $|w| \in \sigma_e(|T|)$.*

The next result is based on the paper [4] of Putnam.

PROPOSITION 5.3. *Assume that there is no $f \neq 0$ for which $\|Tf\| = \|T^*f\|$. Suppose that $0 \notin \sigma(T)$ and there exists a wedge $W = \{z : a <$

$\arg z < b\}$ *which has an empty intersection with* $\sigma(T)$. *Then* U *and* $|T|$ *are absolutely continuous.*

Proof. It is easy to check that T^{-1} is purely hyponormal, i.e. without normal part. Note that

$$T^{-1} = U^*|T^*|^{-1},$$

and $\sigma(T^{-1}) \cap \{w : -b < \arg w < -a\} = \emptyset$. From Theorem 6 of [4] we obtain absolute continuity of U^* and $|T^*|^{-1}$. Since $U|T| = |T^*|U$, this completes the proof. ■

VI. Some new examples. As noticed in our earlier works [1], [2] the question whether a given operator is hyponormal is not easy. Therefore the results and examples we shall give below deserve to be stated. We begin with a general lemma.

LEMMA 6.1. *Suppose we are given hyponormal operators* A *and* B *in* H. *Assume that there exists a dense subspace* $D \subset D(A) \cap D(B)$ *such that* $AD \subset D(B)$, $BD \subset D(A)$ *and*

$$ABx = BAx, \quad x \in D.$$

If $|\lambda| \le 1$ *and* $\|Bx\|^2 - \|B^*x\|^2 \le \|Ax\|^2 - \|A^*x\|^2$, $x \in D$, *then the operator* $A + \lambda B^*$ *defined on* D *is hyponormal in* H.

Proof. Define $Sx = (A+\lambda B^*)x$, $x \in D$. Direct computation shows that

$$\begin{aligned}\|Sx\|^2 - \|S^*x\|^2 &= \|Ax\|^2 - \|A^*x\|^2 - |\lambda|^2(\|Bx\|^2 - \|B^*x\|^2)\\ &\ge (1-|\lambda|^2)(\|Bx\|^2 - \|B^*x\|^2) \ge 0, \quad x \in D. \ ■\end{aligned}$$

EXAMPLE 6.2. The last lemma has the following application. Let T be an operator satisfying the canonical commutation relation

$$[T^*, T]f = f,$$

for f in some dense linear subspace M such that $M \subset D(T) \cap D(T^*)$, $TM \subset M$ and $T^*M \subset M$. Suppose that T has a total set of quasianalytic vectors. Then T must be subnormal (see [7]).

Let $A = T^{k+1}$ and $B = T^k$, $k \in \mathbb{N}$. Since A and B are subnormal they are both hyponormal. We claim that A and B satisfy the assumptions of Lemma 6.1. In fact, we take $D = M$ and applying Lemma of [7] we have

$$(\alpha) \qquad \|T^{k+1}f\|^2 = \sum_{s=0}^{k+1}[(k+1)\dots(k+2-s)]^2\frac{\|T^{*k+1-s}f\|^2}{s!}, \qquad f \in M.$$

It follows that

$$(\beta) \qquad \|Af\|^2 - \|A^*f\|^2 \ge \|Bf\|^2 - \|B^*f\|^2, \qquad f \in M.$$

Indeed, using (α) and comparing the $(s+1)$-term of the left hand side of (β) to the s-term of the right hand side of (β) we obtain

$$\begin{aligned}&[(k+1)\dots(k+1-s)]^2\frac{\|T^{*k-s}f\|^2}{(s+1)!}\\ &\qquad \ge [k\dots(k-s+1)]^2\frac{\|T^{*k-s}f\|^2}{s!}, \qquad s = 0,\dots,k.\end{aligned}$$

Therefore (β) holds and $S = A + \lambda B^*$ is hyponormal.

EXAMPLE 6.3. In a recent paper of Kato [3] a class of bounded hyponormal operators has been found. Namely he considered the following problem. Let $P = \frac{1}{i}\frac{d}{dx}$, $Q = M_x$ be the canonical pair of differentiation and multiplication by x operators in $L^2(\mathbb{R})$. If f and g are real-valued functions such that $f', g' \in L^1(\mathbb{R})$, then the problem is to find a sufficient condition on f and g which guarantees the positivity of the commutator $i[f(P), g(Q)]$. Since f and g are bounded the operator $T = f(P) + ig(Q)$ is bounded and $i[f(P), g(Q)]$ is positive if and only if T is hyponormal.

Let $f(x) = \tanh(ax)$, $g(x) = \tanh(bx)$, where $a > 0$, $b > 0$ and $ab = \pi/2$. Take a finite positive measure μ on $\mathbb{R}$ and a σ-finite positive measure ν on $\mathbb{R}$. For each $N > 0$ define a measure on $\mathbb{R}$ by

$$\nu_N(E) = \nu([-N, N] \cap E).$$

Following Kato [3] we consider the functions

$$u = f * \mu, \qquad v_N = g * \nu_N.$$

Note that u and v_N are bounded on $\mathbb{R}$. Let $T_N = u(P) + iv_N(Q)$. Applying Theorem III of [3] we know that T_N is hyponormal in $L^2(\mathbb{R})$.

Suppose, additionally, that ν is symmetric, i.e. $\nu(E) = \nu(-E)$. Then for any $h \in C_0^\infty(\mathbb{R})$ the sequence $v_N(Q)h$ is strongly convergent. Indeed, for $M < N$ we have

$$\begin{aligned}&\|(v_N(Q) - v_M(Q))h\|^2\\ &\qquad = \int\Big|\int_{[-N,-M]} g(x-t)\,d\nu(t) + \int_{[M,N]} g(x-t)\,d\nu(t)\Big|^2|h(x)|^2\,dx.\end{aligned}$$

Since $\operatorname{supp} h$ is compact and $g(x-t) \to -1$ as $t \to +\infty$ and $g(x-t) \to 1$ as $t \to -\infty$, the last integral is arbitrarily small for M sufficiently large. Hence $T_N h$ is also convergent and the limit $Th = \lim_{N\to\infty} T_N h$ defines an unbounded hyponormal operator in $L^2(\mathbb{R})$. Note, in passing, that T has the Cartesian decomposition $T = K + iL$, where $K = u(P)$ is bounded and $L = L^*$ is unbounded if $\nu(\mathbb{R}) = +\infty$.

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On topologization of countably generated algebras

by

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Abstract. We prove that any real or complex countably generated algebra has a complete locally convex topology making it a topological algebra. Assuming the continuum hypothesis, it is the best possible result expressed in terms of the cardinality of a set of generators. This result is a corollary to a theorem stating that a free algebra provided with the maximal locally convex topology is a topological algebra if and only if the number of variables is at most countable. As a byproduct we obtain an example of a semitopological (non-topological) algebra with every commutative subalgebra topological.

A *topological algebra* is a real or complex algebra whose underlying vector space is a topological (Hausdorff) vector space and whose multiplication is jointly continuous. In the case when the multiplication is merely separately continuous, we say that the algebra in question is *semitopological*. A *locally convex algebra* is a topological algebra whose underlying vector space is a locally convex space. Its topology can be given by means of a family $(\|x\|_\alpha)_{\alpha\in\mathfrak{a}}$ of seminorms such that for each index α there is a $\beta\in\mathfrak{a}$ so that

$$\|xy\|_\alpha \le \|x\|_\beta \|y\|_\beta \tag{1}$$

for all x and y in the algebra in question. For more information on locally convex algebras the reader is referred to [1] and [4].

The *maximal locally convex topology* on a vector space X is the topology given by means of all seminorms. It is clearly a Hausdorff topology. This topology will be denoted by $\tau_{\max}^{\mathrm{LC}}$. It is known (see [3], Example on p. 56) that $(X, \tau_{\max}^{\mathrm{LC}})$ is a complete locally convex space. In [5] we observed that any real or complex algebra topologized with $\tau_{\max}^{\mathrm{LC}}$ is a semitopological algebra. On the other hand, we proved in [6] that for any real or complex vector space X the algebra $L(X)$ of all its endomorphisms has a locally convex topology making it a locally convex algebra if and only if the dimension of X is finite. A natural question arises: under what conditions a given algebra is topologizable as a locally convex algebra? In this paper we give an answer

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