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Ideal norms and trigonometric orthonormal systems

by

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Abstract. We characterize the UMD-property of a Banach space X by sequences of ideal norms associated with trigonometric orthonormal systems. The asymptotic behavior of those numerical parameters can be used to decide whether X is a UMD-space. Moreover, if this is not the case, we obtain a measure that shows how far X is from being a UMD-space. The main result is that all described sequences are not only simultaneously bounded but are also asymptotically equivalent.

1. Introduction. The study of sequences of ideal norms can be used to quantify certain properties of linear operators. In most cases the boundedness of a sequence of ideal norms for a given operator T describes a well-known property, whereas, in the non-bounded case, the growth rate of the sequence describes how much the operator T deviates from this property.

One particularly interesting case is if two sequences of ideal norms are uniformly equivalent. Then the properties given by these sequences are also quantitatively equivalent.

We introduce several sequences of ideal norms related to the trigonometric orthonormal systems. The boundedness of these sequences for the identity map of a Banach space X is equivalent to X being UMD.

All of these sequences turn out to be uniformly equivalent. As a corollary we deduce that a Banach space X is a UMD-space if and only if there exists a constant $c \geq 0$ such that, for all $x_1, \ldots, x_n \in X$, we have

$$\left(\frac{1}{\pi} \int_{-\pi}^{\pi} \left\| \sum_{k=1}^{n} x_k \sin kt \right\|^2 dt \right)^{1/2} \leq c \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \left\| \sum_{k=1}^{n} x_k \cos kt \right\|^2 dt \right)^{1/2}$$

or, what turns out to be equivalent,

$$\left(\frac{1}{\pi} \int_{-\pi}^{\pi} \left\| \sum_{k=1}^{n} x_k \cos kt \right\|^2 dt \right)^{1/2} \leq c \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \left\| \sum_{k=1}^{n} x_k \sin kt \right\|^2 dt \right)^{1/2}.$$

2. Ideal norms. Let X and Y be Banach spaces. Since we deal with the exponential system $(\exp(it),\ldots,\exp(int))$, most of the results only make sense in the complex setting. However, they remain true if the exponential system is replaced by its real analogue

$$(1,\sqrt{2}\cos t,\ldots,\sqrt{2}\cos nt,\sqrt{2}\sin t,\ldots,\sqrt{2}\sin nt).$$

Let $\mathfrak{L}$ denote the ideal of all bounded linear operators.

For the theory of ideal norms and operator ideals we refer to the monographs of Pietsch, [5] and [6]. For a more general treatment of ideal norms associated with orthonormal systems, we refer to [7].

DEFINITION. An *ideal norm* α is a function which assigns to every operator T between arbitrary Banach spaces a non-negative number $\alpha(T)$ such that

$$\alpha(S+T)\leq\alpha(S)+\alpha(T)\quad\text{for all }S,T\in\mathfrak{L}(X,Y),$$
$$\alpha(BTA)\leq\|B\|\,\alpha(T)\,\|A\|$$
$$\text{for all }A\in\mathfrak{L}(X_0,X),T\in\mathfrak{L}(X,Y),B\in\mathfrak{L}(Y,Y_0),$$
$$\alpha(T)=0\quad\text{implies}\quad T=0.$$

We write $\alpha(X)$ instead of $\alpha(I_X)$, where I_X denotes the identity map of the Banach space X.

If we additionally assume that $\alpha(\mathbb{K})\geq 1$, where $\mathbb{K}$ is the scalar field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, then $\alpha(T)\geq\|T\|$ for all operators $T\in\mathfrak{L}$. This assumption is in particular satisfied by all ideal norms considered in this article.

If α is an ideal norm then its *dual ideal norm* α' is defined by $\alpha'(T):=\alpha(T')$.

The ideal norm α is said to be *injective* if $\alpha(JT)=\alpha(T)$ for all $T\in\mathfrak{L}(X,Y)$ and any metric injection $J\in\mathfrak{L}(Y,Y_0)$. A *metric injection* J is a linear map such that $\|Jy\|=\|y\|$ for all $y\in Y$.

Let α be an ideal norm and let $c>0$ be a constant. We write $\alpha\leq c$ if $\alpha(X)\leq c$ for all Banach spaces X. It then follows that for all $T\in\mathfrak{L}$,

$$\alpha(T)\leq\|T\|\,\alpha(X)\leq c\,\|T\|.$$

Given ideal norms α, β and γ, we write $\alpha\leq\beta\circ\gamma$ if

$$\alpha(ST)\leq\beta(S)\,\gamma(T)\quad\text{for all }T\in\mathfrak{L}(X,Y)\text{ and }S\in\mathfrak{L}(Y,Z).$$

The following concept is essential for the further considerations.

DEFINITION. Two sequences of ideal norms (α_n) and (β_n) are said to be *uniformly equivalent* if there exists a constant $c>0$ such that

$$\frac{1}{c}\,\alpha_n(T)\leq\beta_n(T)\leq c\,\alpha_n(T)$$

for all $T\in\mathfrak{L}$.

3. Orthonormal systems. Given any Banach space X and a measure space (M,μ), let $L_2^X(M,\mu)$ denote the Banach space of all μ-measurable functions $f:M\to X$ for which

$$\|f|L_2\|:=\left(\int_M\|f(t)\|^2\,d\mu(t)\right)^{1/2}$$

is finite.

In the following, let

$$\mathcal{A}_n=(a_1,\ldots,a_n)\quad\text{and}\quad\mathcal{B}_n=(b_1,\ldots,b_n)$$

be orthonormal systems in some Hilbert spaces $L_2(M,\mu)$ and $L_2(N,\nu)$, respectively.

For every orthonormal system $\mathcal{A}_n$, we also consider the complex conjugate orthonormal system $\bar{\mathcal{A}}_n$, which consists of the functions $\bar{a}_1,\ldots,\bar{a}_n\in L_2(M,\mu)$.

For $x_1,\ldots,x_n\in X$, we write

$$(1)\qquad\|(x_k)|\mathcal{A}_n\|:=\left(\int_M\left\|\sum_{k=1}^n x_k a_k(s)\right\|^2 d\mu(s)\right)^{1/2}.$$

This expression yields a norm on the nth Cartesian power of X.

PROPOSITION 3.1. $\|x_h\|\leq\|(x_k)|\mathcal{A}_n\|$ *for all* $h=1,\ldots,n$.

Proof. By the Parseval formula, we have for all $x'\in X'$,

$$\sum_{k=1}^n|\langle x_k,x'\rangle|^2=\int_M\left|\sum_{k=1}^n\langle x_k,x'\rangle a_k(s)\right|^2 d\mu(s)$$
$$\leq\int_M\left\|\sum_{k=1}^n x_k a_k(s)\right\|^2 d\mu(s)\,\|x'\|^2.$$

Hence $\|x_h\|=\sup_{\|x'\|\leq 1}|\langle x_h,x'\rangle|\leq\|(x_k)|\mathcal{A}_n\|$. ∎

DEFINITION. For $T\in\mathfrak{L}(X,Y)$ and $n\in\mathbb{N}$ the ideal norm $\varrho(T|\mathcal{B}_n,\mathcal{A}_n)$ is defined as the smallest constant $c\geq 0$ such that

$$(2)\qquad\|(Tx_k)|\mathcal{B}_n\|\leq c\,\|(x_k)|\mathcal{A}_n\|$$

whenever $x_1,\ldots,x_n\in X$.

The ideal norm $\delta(T|\mathcal{B}_n,\mathcal{A}_n)$ is defined as the smallest constant $c\geq 0$ such that

$$(3)\qquad\|(T\langle f,\bar{a}_k\rangle)|\mathcal{B}_n\|\leq c\,\|f|L_2\|$$

whenever $f\in L_2^X(M,\mu)$. Here

$$\langle f,\bar{a}_k\rangle:=\int_M f(s)\overline{a_k(s)}\,d\mu(s)$$

denotes the kth *Fourier coefficient* of f with respect to $\mathcal{A}_n$.

PROPOSITION 3.2. *For any three orthonormal systems* $\mathcal{A}_n$, $\mathcal{B}_n$ *and* $\mathcal{F}_n$, *we have*

(4)
$$\varrho(\mathcal{B}_n, \mathcal{A}_n) \leq \delta(\mathcal{B}_n, \mathcal{A}_n),$$

(5)
$$\delta(\mathcal{B}_n, \mathcal{A}_n) \leq \varrho(\mathcal{B}_n, \mathcal{F}_n) \circ \delta(\mathcal{F}_n, \mathcal{A}_n),$$

(6)
$$\varrho(\mathcal{B}_n, \mathcal{A}_n) \leq \varrho(\mathcal{B}_n, \mathcal{F}_n) \circ \varrho(\mathcal{F}_n, \mathcal{A}_n).$$

P r o o f. The first inequality follows by taking $f = \sum_{k=1}^{n} x_k a_k$ in (3). The other inequalities are trivial. ∎

The next fact is obvious as well.

PROPOSITION 3.3. *The ideal norms* $\delta(\mathcal{B}_n, \mathcal{A}_n)$ *are injective.*

The ideal norms $\delta(\mathcal{B}_n, \mathcal{A}_n)$ enjoy the following duality property.

PROPOSITION 3.4. $\delta(\mathcal{B}_n, \mathcal{A}_n) = \delta'(\overline{\mathcal{A}}_n, \overline{\mathcal{B}}_n).$

P r o o f. For $T \in \mathfrak{L}(X, Y)$ and $g \in L_2^{Y'}(N, \nu)$, we let

$$c := \left(\int_M \left\| \sum_{k=1}^{n} T'\langle g, b_k\rangle \overline{a_k(s)} \right\|^2 d\mu(s) \right)^{1/2} = \|(T'\langle g, b_k\rangle)|\overline{\mathcal{A}}_n\|.$$

Given $\varepsilon > 0$, by [4, p. 232] there exists $f \in L_2^X(M, \mu)$ such that

$$c = \int_M \left\langle f(s), \sum_{k=1}^{n} T'\langle g, b_k\rangle \overline{a_k(s)} \right\rangle d\mu(s)$$

and

$$\|f|L_2\| \leq 1 + \varepsilon.$$

We now obtain

$$c = \int_M \int_N \sum_{k=1}^{n} \langle Tf(s), g(t)\rangle \overline{a_k(s)} b_k(t)\, d\mu(s)\, d\nu(t)$$

$$= \int_N \left\langle \sum_{k=1}^{n} T\langle f, \overline{a}_k\rangle b_k(t), g(t) \right\rangle d\nu(t)$$

$$\leq \left(\int_N \left\| \sum_{k=1}^{n} T\langle f, \overline{a}_k\rangle b_k(t) \right\|^2 d\nu(t) \right)^{1/2} \left(\int_N \|g(t)\|^2 d\nu(t) \right)^{1/2}$$

$$\leq (1 + \varepsilon)\, \delta(T|\mathcal{B}_n, \mathcal{A}_n)\, \|g|L_2\|.$$

Letting ε tend to 0 yields $\|(T'\langle g, b_k\rangle)|\overline{\mathcal{A}}_n\| \leq \delta(T|\mathcal{B}_n, \mathcal{A}_n)\, \|g|L_2\|.$ This proves that

$$\delta(T'|\overline{\mathcal{A}}_n, \overline{\mathcal{B}}_n) \leq \delta(T|\mathcal{B}_n, \mathcal{A}_n).$$

Note that $T''K_X = K_Y T$, where K_X and K_Y denote the canonical embeddings of X in X'' and of Y in Y'', respectively. Using the injectivity of $\delta(\mathcal{B}_n, \mathcal{A}_n)$ and $\|K_X\| \leq 1$, we finally conclude that

$$\delta(T|\mathcal{B}_n, \mathcal{A}_n) = \delta(K_Y T|\mathcal{B}_n, \mathcal{A}_n) = \delta(T''K_X|\mathcal{B}_n, \mathcal{A}_n)$$
$$\leq \delta(T''|\mathcal{B}_n, \mathcal{A}_n) \leq \delta(T'|\overline{\mathcal{A}}_n, \overline{\mathcal{B}}_n) \leq \delta(T|\mathcal{B}_n, \mathcal{A}_n). \ ∎$$

From the duality property of the ideal norms $\delta(\mathcal{B}_n, \mathcal{A}_n)$ and (5), we get the following result.

PROPOSITION 3.5. *Let* $\mathcal{A}_n$ *and* $\mathcal{B}_n$ *as well as* $\mathcal{F}_n$ *and* $\mathcal{G}_n$ *be orthonormal systems. Then*

$$\delta(\mathcal{B}_n, \mathcal{A}_n) \leq \varrho(\mathcal{B}_n, \mathcal{G}_n) \circ \delta(\mathcal{G}_n, \mathcal{F}_n) \circ \varrho'(\overline{\mathcal{A}}_n, \overline{\mathcal{F}}_n).$$

We denote by $\mathcal{A}_n \otimes \mathcal{B}_n$ the orthonormal system in $L_2(M \times N, \mu \times \nu)$ consisting of the functions $a_k \otimes b_k : (s,t) \to a_k(s)b_k(t)$ with $k = 1, \ldots, n$. Note that

(7)
$$\|(x_k)|\mathcal{A}_n \otimes \mathcal{B}_n\| = \|(x_k)|\mathcal{B}_n \otimes \mathcal{A}_n\|$$
$$= \left(\int_M \int_N \left\| \sum_{k=1}^{n} x_k a_k(s) b_k(t) \right\|^2 d\mu(s)\, d\nu(t) \right)^{1/2}.$$

The following fact turns out to be very useful to formulate various proofs.

PROPOSITION 3.6. *Let* $\mathcal{F}_n = (f_1, \ldots, f_n)$ *be another orthonormal system in* $L_2(R, \varrho)$. *Then*

$$\varrho(\mathcal{B}_n \otimes \mathcal{F}_n, \mathcal{A}_n \otimes \mathcal{F}_n) \leq \varrho(\mathcal{B}_n, \mathcal{A}_n).$$

P r o o f. Substituting $(x_k f_k(r))$ with $r \in R$ in (2), we obtain

$$\int_N \left\| \sum_{k=1}^{n} Tx_k b_k(t) f_k(r) \right\|^2 d\nu(t)$$
$$\leq \varrho(T|\mathcal{B}_n, \mathcal{A}_n)^2 \int_M \left\| \sum_{k=1}^{n} x_k a_k(s) f_k(r) \right\|^2 d\mu(s).$$

Integration over $r \in R$ and taking square roots yields

$$\|(Tx_k)|\mathcal{B}_n \otimes \mathcal{F}_n\| \leq \varrho(T|\mathcal{B}_n, \mathcal{A}_n)\, \|(x_k)|\mathcal{A}_n \otimes \mathcal{F}_n\|,$$

which proves the desired result. ∎

4. Trigonometric orthonormal systems. We write

$$c_k(t) := \exp(ikt) \quad \text{for } k \in \mathbb{Z},$$
$$c_k(t) := \sqrt{2}\cos kt, \quad s_k(t) := \sqrt{2}\sin kt \quad \text{for } k \in \mathbb{N}.$$

Note that

$$\mathcal{E}_n := (c_1, \ldots, c_n), \quad \mathcal{C}_n := (c_1, \ldots, c_n) \quad \text{and} \quad \mathcal{S}_n := (s_1, \ldots, s_n)$$

are orthonormal systems in $L_2(-\pi, \pi)$ equipped with the scalar product

$$(f, g) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt.$$

Moreover, we have

$$\|(x_k)|\mathcal{C}_n\| = \left(\frac{2}{\pi}\int_0^\pi \left\|\sum_{k=1}^n x_k \cos kt\right\|^2 dt\right)^{1/2},$$

$$\|(x_k)|\mathcal{S}_n\| = \left(\frac{2}{\pi}\int_0^\pi \left\|\sum_{k=1}^n x_k \sin kt\right\|^2 dt\right)^{1/2}.$$

Note that

$$\|(x_k)|\mathcal{E}_n\| = \left(\frac{1}{2\pi}\int_{-\pi}^\pi \left\|\sum_{k=1}^n x_k \exp(ikt)\right\|^2 dt\right)^{1/2}.$$

Hence the substitution $t \to -t$ yields

(8) $$\|(x_k)|\overline{\mathcal{E}}_n\| = \|(x_k)|\mathcal{E}_n\|.$$

5. Main result. We are now ready to state the main result.

THEOREM. *The following sequences of ideal norms are uniformly equivalent:*

$$\delta(\mathcal{E}_n, \mathcal{E}_n), \quad \delta(\mathcal{S}_n, \mathcal{C}_n), \quad \delta(\mathcal{C}_n, \mathcal{S}_n), \quad \varrho(\mathcal{S}_n, \mathcal{C}_n), \quad \varrho(\mathcal{C}_n, \mathcal{S}_n).$$

DEFINITION. A Banach space X has the *UMD-property* if there exists a constant $c \geq 0$ such that

$$\left\|\sum_{k=0}^n \varepsilon_k\, dM_k \,\Big|\, L_2\right\| \leq c \left\|\sum_{k=0}^n dM_k \,\Big|\, L_2\right\|$$

for all martingales $(M_0, M_1, \ldots)$ with values in X, all $n \in \mathbb{N}$ and all sequences of signs $(\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n$; see [1].

It is known (see [1]–[3]) that a Banach space X is a UMD-space if the sequence of ideal norms $\delta(\mathcal{E}_n, \mathcal{E}_n)$ is bounded. Hence we get the following corollary.

COROLLARY. *Let X be a Banach space. The following conditions are equivalent:*

(1) X *is a UMD-space.*

(2) *There exists a constant $c \geq 0$ such that, for all $f \in L_2^X(-\pi, \pi)$,*

$$\left(\frac{1}{2\pi}\int_{-\pi}^\pi \left\|\sum_{k=1}^n \langle f, \overline{e}_k\rangle \exp(ikt)\right\|^2 dt\right)^{1/2} \leq c\left(\frac{1}{2\pi}\int_{-\pi}^\pi \|f(t)\|^2 dt\right)^{1/2}.$$

(3) *There exists a constant $c \geq 0$ such that, for all $x_1, \ldots, x_n \in X$,*

$$\left\|\sum_{k=1}^n x_k \sin kt \,\Big|\, L_2\right\| \leq c\left\|\sum_{k=1}^n x_k \cos kt \,\Big|\, L_2\right\|.$$

(4) *There exists a constant $c \geq 0$ such that, for all $x_1, \ldots, x_n \in X$,*

$$\left\|\sum_{k=1}^n x_k \cos kt \,\Big|\, L_2\right\| \leq c\left\|\sum_{k=1}^n x_k \sin kt \,\Big|\, L_2\right\|.$$

6. Proof of the main result

LEMMA 6.1. $\varrho(\mathcal{S}_n, \mathcal{S}_n \otimes \mathcal{C}_n) \leq \sqrt{2}$.

Proof. It follows from

$$\sin k(t - s) = \sin kt \cos ks - \cos kt \sin ks$$

and the translation invariance of the Lebesgue measure that for all $s \in \mathbb{R}$ and $x_1, \ldots, x_n \in X$,

$$\|(x_k)|\mathcal{S}_n\|^2 = \frac{1}{\pi}\int_{-\pi}^\pi \left\|\sum_{k=1}^n x_k \sin k(t-s)\right\|^2 dt$$

$$\leq \frac{2}{\pi}\int_{-\pi}^\pi \left\|\sum_{k=1}^n x_k \sin kt \cos ks\right\|^2 dt + \frac{2}{\pi}\int_{-\pi}^\pi \left\|\sum_{k=1}^n x_k \cos kt \sin ks\right\|^2 dt.$$

Integrating this inequality over $s \in [-\pi, \pi]$, we get

$$\|(x_k)|\mathcal{S}_n\|^2 \leq \|(x_k)|\mathcal{S}_n \otimes \mathcal{C}_n\|^2 + \|(x_k)|\mathcal{C}_n \otimes \mathcal{S}_n\|^2.$$

This proves the assertion since $\|(x_k)|\mathcal{S}_n \otimes \mathcal{C}_n\| = \|(x_k)|\mathcal{C}_n \otimes \mathcal{S}_n\|$. ∎

LEMMA 6.2. *For $s \in \mathbb{R}$ and $x_1, \ldots, x_n \in X$, we have*

$$\|(x_k \cos ks)|\mathcal{C}_n\| \leq \|(x_k)|\mathcal{C}_n\|, \quad \|(x_k \sin ks)|\mathcal{S}_n\| \leq \|(x_k)|\mathcal{C}_n\|,$$

$$\|(x_k \cos ks)|\mathcal{S}_n\| \leq \|(x_k)|\mathcal{S}_n\|, \quad \|(x_k \sin ks)|\mathcal{C}_n\| \leq \|(x_k)|\mathcal{S}_n\|.$$

Proof. It follows from

$$2\cos ks \cos kt = \cos k(s+t) + \cos k(s-t)$$

and the translation invariance of the Lebesgue measure that

$$2\|(x_k \cos ks)|\mathcal{C}_n\| = \left(\frac{1}{\pi}\int_{-\pi}^\pi \left\|\sum_{k=1}^n x_k\, 2\cos ks \cos kt\right\|^2 dt\right)^{1/2}$$

$$\leq \left(\frac{1}{\pi}\int_{-\pi}^\pi \left\|\sum_{k=1}^n x_k \cos k(t+s)\right\|^2 dt\right)^{1/2}$$

$$+ \left(\frac{1}{\pi}\int_{-\pi}^\pi \left\|\sum_{k=1}^n x_k \cos k(t-s)\right\|^2 dt\right)^{1/2}$$

$$= 2\|(x_k)|\mathcal{C}_n\|.$$

This proves the first inequality. The others can be proved in the same way. ∎

LEMMA 6.3. $\varrho(\mathcal{S}_n \otimes \mathcal{S}_n, \mathcal{C}_n) \leq \sqrt{2}$.

Proof. Squaring the inequality

$$\|(x_k \sin ks)|\mathcal{S}_n\| \leq \|(x_k)|\mathcal{C}_n\|$$

from Lemma 6.2 and integrating over $s \in [-\pi, \pi]$ yields

$$\|(x_k)|\mathcal{S}_n \otimes \mathcal{S}_n\|^2 \leq 2\|(x_k)|\mathcal{C}_n\|^2. \quad \blacksquare$$

We are now ready to prove our first result.

PROPOSITION 6.1. $\varrho(\mathcal{S}_n, \mathcal{C}_n) \leq 2\varrho(\mathcal{C}_n, \mathcal{S}_n)$.

Proof. By Lemmas 6.1 and 6.3 as well as Proposition 3.6, we get

$$\varrho(\mathcal{S}_n, \mathcal{C}_n) \leq \varrho(\mathcal{S}_n, \mathcal{S}_n \otimes \mathcal{C}_n) \circ \varrho(\mathcal{S}_n \otimes \mathcal{C}_n, \mathcal{S}_n \otimes \mathcal{S}_n) \circ \varrho(\mathcal{S}_n \otimes \mathcal{S}_n, \mathcal{C}_n)$$
$$\leq \sqrt{2}\, \varrho(\mathcal{C}_n, \mathcal{S}_n)\, \sqrt{2}. \quad \blacksquare$$

To prove the converse of Proposition 6.1 we show the following lemma.

LEMMA 6.4. Let $x_1, \ldots, x_n \in X$ and set $x_{-1} = x_0 = x_{n+1} = x_{n+2} := 0$. Then for $t \in \mathbb{R}$, we have

$$2 \sin t \sum_{k=1}^{n} x_k \sin kt = \sum_{k=0}^{n+1} (x_{k+1} - x_{k-1}) \cos kt,$$

$$2 \sin t \sum_{k=1}^{n} x_k \cos kt = \sum_{k=1}^{n+1} (x_{k-1} - x_{k+1}) \sin kt.$$

Proof. The equations above follow from

$$2 \sin t \sin kt = \cos(k-1)t - \cos(k+1)t,$$
$$2 \sin t \cos kt = \sin(k+1)t - \sin(k-1)t$$

by rearranging the summation. $\quad \blacksquare$

LEMMA 6.5. For $x_0, \ldots, x_{n+1} \in X$, we have

$$\left(\frac{2}{\pi} \int_0^\pi \Big\| \sum_{k=1}^{n} x_k \cos kt \Big\|^2 dt \right)^{1/2} \leq \left(\frac{2}{\pi} \int_0^\pi \Big\| \sum_{k=0}^{n+1} x_k \cos kt \Big\|^2 dt \right)^{1/2}$$
$$+ \sqrt{2}\|x_0\| + \|x_{n+1}\|,$$

$$\left(\frac{2}{\pi} \int_0^\pi \Big\| \sum_{k=1}^{n+1} x_k \sin kt \Big\|^2 dt \right)^{1/2} \leq \left(\frac{2}{\pi} \int_0^\pi \Big\| \sum_{k=1}^{n} x_k \sin kt \Big\|^2 dt \right)^{1/2} + \|x_{n+1}\|.$$

Proof. We have

$$\left(\frac{2}{\pi} \int_0^\pi \Big\| \sum_{k=1}^{n} x_k \cos kt \Big\|^2 dt \right)^{1/2}$$
$$\leq \left(\frac{2}{\pi} \int_0^\pi \Big\| \sum_{k=0}^{n+1} x_k \cos kt \Big\|^2 dt \right)^{1/2} + \left(\frac{2}{\pi} \int_0^\pi \|x_0\|^2 dt \right)^{1/2}$$
$$+ \left(\frac{2}{\pi} \int_0^\pi \|x_{n+1} \cos(n+1)t\|^2 dt \right)^{1/2}$$
$$= \left(\frac{2}{\pi} \int_0^\pi \Big\| \sum_{k=0}^{n+1} x_k \cos kt \Big\|^2 dt \right)^{1/2} + \sqrt{2}\|x_0\| + \|x_{n+1}\|.$$

The second inequality can be proved in the same way. $\quad \blacksquare$

LEMMA 6.6. Let $\Delta_0 := [\pi/3, 2\pi/3)$. Then for $x_1, \ldots, x_n \in X$ and $T \in \mathcal{L}(X, Y)$, we have

$$\left(\frac{2}{\pi} \int_{\Delta_0} \Big\| \sum_{k=1}^{n} T x_k \cos kt \Big\|^2 \right)^{1/2} \leq 4\varrho(T|\mathcal{S}_n, \mathcal{C}_n)\|(x_k)|\mathcal{S}_n\|.$$

Proof. If $t \in \Delta_0$, then $\sin(\pi/3) \leq \sin t$. Moreover, by Proposition 3.1 we have $\|x_1\| \leq \|(x_k)|\mathcal{S}_n\|$ and $\|x_n\| \leq \|(x_k)|\mathcal{S}_n\|$. Applying Lemmas 6.4 and 6.5, we obtain

$$2 \sin \frac{\pi}{3} \left(\frac{2}{\pi} \int_{\Delta_0} \Big\| \sum_{k=1}^{n} T x_k \cos kt \Big\|^2 \right)^{1/2}$$
$$\leq \left(\frac{2}{\pi} \int_{\Delta_0} \Big\| 2 \sin t \sum_{k=1}^{n} T x_k \cos kt \Big\|^2 \right)^{1/2}$$
$$\leq \left(\frac{2}{\pi} \int_0^\pi \Big\| \sum_{k=1}^{n+1} T(x_{k-1} - x_{k+1}) \sin kt \Big\|^2 \right)^{1/2}$$
$$\leq \left(\frac{2}{\pi} \int_0^\pi \Big\| \sum_{k=1}^{n} T(x_{k-1} - x_{k+1}) \sin kt \Big\|^2 \right)^{1/2} + \|T x_n\|$$
$$\leq \varrho(T|\mathcal{S}_n, \mathcal{C}_n) \left[\left(\frac{2}{\pi} \int_0^\pi \Big\| \sum_{k=1}^{n} (x_{k-1} - x_{k+1}) \cos kt \Big\|^2 dt \right)^{1/2} + \|x_n\| \right]$$
$$\leq \varrho(T|\mathcal{S}_n, \mathcal{C}_n)$$
$$\times \left[\left(\frac{2}{\pi} \int_0^\pi \Big\| \sum_{k=0}^{n+1} (x_{k-1} - x_{k+1}) \cos kt \Big\|^2 dt \right)^{1/2} + \sqrt{2}\|x_1\| + 2\|x_n\| \right]$$

$$= \varrho(T|\mathcal{S}_n, \mathcal{C}_n)\left[2\left(\frac{2}{\pi}\int_0^\pi\left\|\sin t\sum_{k=1}^n x_k\sin kt\right\|^2 dt\right)^{1/2} + \sqrt{2}\|x_1\| + 2\|x_n\|\right]$$

$$\leq \varrho(T|\mathcal{S}_n, \mathcal{C}_n)[2\|(x_k)|\mathcal{S}_n\| + \sqrt{2}\|x_1\| + 2\|x_n\|]$$

$$\leq (4+\sqrt{2})\varrho(T|\mathcal{S}_n, \mathcal{C}_n)\|(x_k)|\mathcal{S}_n\|.$$

This yields the assertion, since

$$\frac{4+\sqrt{2}}{2\sin\frac{\pi}{3}} = 3.1258\ldots < 4. \blacksquare$$

PROPOSITION 6.2. $\varrho(\mathcal{C}_n, \mathcal{S}_n) \leq 9\varrho(\mathcal{S}_n, \mathcal{C}_n)$.

Proof. Obviously,

$$(9)\quad \|(Tx_k)|\mathcal{C}_n\| = \left(\frac{2}{\pi}\int_0^\pi\left\|\sum_{k=1}^n Tx_k\cos kt\right\|^2 dt\right)^{1/2} = (I_{-1}^2 + I_0^2 + I_1^2)^{1/2},$$

where

$$I_\alpha := \left(\frac{2}{\pi}\int_{\Delta_\alpha}\left\|\sum_{k=1}^n Tx_k\cos kt\right\|^2 dt\right)^{1/2} \quad\text{and}\quad \Delta_\alpha := [\tfrac{1}{3}\pi, \tfrac{2}{3}\pi) + \tfrac{1}{3}\alpha\pi.$$

We know from Lemma 6.6 that

$$(10)\qquad\qquad I_0 \leq 4\varrho(T|\mathcal{S}_n, \mathcal{C}_n)\|(x_k)|\mathcal{S}_n\|.$$

In order to estimate I_α with $\alpha = \pm 1$, we substitute $s := t \mp \pi/3$. Then

$$I_\alpha = \left(\frac{2}{\pi}\int_{\Delta_0}\left\|\sum_{k=1}^n Tx_k\left(\cos k\tfrac{\pi}{3}\cos ks \mp \sin k\tfrac{\pi}{3}\sin ks\right)\right\|^2 ds\right)^{1/2}$$

$$\leq \left(\frac{2}{\pi}\int_{\Delta_0}\left\|\sum_{k=1}^n T\left(x_k\cos k\tfrac{\pi}{3}\right)\cos ks\right\|^2 ds\right)^{1/2}$$

$$+ \left(\frac{2}{\pi}\int_{\Delta_0}\left\|\sum_{k=1}^n T\left(x_k\sin k\tfrac{\pi}{3}\right)\sin ks\right\|^2 ds\right)^{1/2}.$$

We now estimate the first summand by Lemma 6.6 and the second summand by applying the defining inequality (2) of $\varrho(T|\mathcal{S}_n, \mathcal{C}_n)$. This yields

$$I_\alpha \leq \varrho(T|\mathcal{S}_n, \mathcal{C}_n)\left(4\|(x_k\cos k\tfrac{\pi}{3})|\mathcal{S}_n\| + \|(x_k\sin k\tfrac{\pi}{3})|\mathcal{C}_n\|\right).$$

Hence, in view of Lemma 6.2, we arrive at

$$(11)\qquad\qquad I_\alpha \leq 5\,\varrho(T|\mathcal{S}_n, \mathcal{C}_n)\|(x_k)|\mathcal{S}_n\|.$$

Combining (9), (10) and (11) yields

$$\|(Tx_k)|\mathcal{C}_n\| \leq (25 + 16 + 25)^{1/2}\varrho(T|\mathcal{S}_n, \mathcal{C}_n)\|(x_k)|\mathcal{S}_n\|.$$

In view of $\sqrt{66} = 8.1240\ldots < 9$, this completes the proof. $\blacksquare$

Remark. By using the exact value of $3.1258\ldots$ for the constant appearing in Lemma 6.6, we can even obtain a value of $6.6194\ldots$ for the constant in the previous proposition.

We now deal with the ideal norms $\delta(\mathcal{E}_n, \mathcal{E}_n)$.

LEMMA 6.7. For $m, n \in \mathbb{N}$ with $n < m$, we have

$$\delta(\mathcal{E}_{m+n}, \mathcal{E}_{m+n}) \leq \delta(\mathcal{E}_m, \mathcal{E}_m) + \delta(\mathcal{E}_n, \mathcal{E}_n).$$

Proof. We have

$$\|(T\langle f, \bar{e}_k\rangle)|\mathcal{E}_{m+n}\| \leq \|(T\langle f, \bar{e}_k\rangle)|\mathcal{E}_m\| + \|(T\langle f, \bar{e}_{m+k}\rangle)|\mathcal{E}_n\|$$

$$= \|(T\langle f, \bar{e}_k\rangle)|\mathcal{E}_m\| + \|(T\langle fe_{-m}, \bar{e}_k\rangle)|\mathcal{E}_n\|$$

$$\leq \delta(T|\mathcal{E}_m, \mathcal{E}_m)\|f|L_2\| + \delta(T|\mathcal{E}_n, \mathcal{E}_n)\|fe_{-m}|L_2\|$$

$$\leq (\delta(T|\mathcal{E}_m, \mathcal{E}_m) + \delta(T|\mathcal{E}_n, \mathcal{E}_n))\|f|L_2\|.$$

Similarly,

$$\|(T\langle f, \bar{e}_k\rangle)|\mathcal{E}_{m-n}\| \leq \|(T\langle f, \bar{e}_k\rangle)|\mathcal{E}_m\| + \|(T\langle f, \bar{e}_{m-n+k}\rangle)|\mathcal{E}_n\|$$

$$= \|(T\langle f, \bar{e}_k\rangle)|\mathcal{E}_m\| + \|(T\langle fe_{-m+n}, \bar{e}_k\rangle)|\mathcal{E}_n\|$$

$$\leq \delta(T|\mathcal{E}_m, \mathcal{E}_m)\|f|L_2\| + \delta(T|\mathcal{E}_n, \mathcal{E}_n)\|fe_{-m+n}|L_2\|$$

$$\leq (\delta(T|\mathcal{E}_m, \mathcal{E}_m) + \delta(T|\mathcal{E}_n, \mathcal{E}_n))\|f|L_2\|. \blacksquare$$

In the following, we write

$$\mu_n(T) := \max\{\varrho(T|\mathcal{C}_n, \mathcal{S}_n), \varrho(T|\mathcal{S}_n, \mathcal{C}_n)\}.$$

LEMMA 6.8. For $x_{-n}, \ldots, x_0, \ldots, x_n \in X$, we have

$$\left(\frac{1}{2\pi}\int_{-\pi}^\pi\left\|\sum_{k=1}^n Tx_k\exp(ikt)\right\|^2 dt\right)^{1/2}$$

$$\leq 4\mu_n(T)\left(\frac{1}{2\pi}\int_{-\pi}^\pi\left\|\sum_{|k|\leq n} x_k\exp(ikt)\right\|^2 dt\right)^{1/2}.$$

Proof. For $k = 1, \ldots, n$, we let

$$u_k := x_k + x_{-k} \quad\text{and}\quad v_k := x_k - x_{-k}.$$

It follows from $u_k + v_k = 2x_k$ and Euler's formula that

$$x_k\exp(ikt) = \tfrac{1}{2}(u_k\cos kt + v_k\cos kt + iu_k\sin kt + iv_k\sin kt).$$

Hence

$$\|(Tx_k)|\mathcal{E}_n\| = \left(\frac{1}{2\pi}\int_{-\pi}^\pi\left\|\sum_{k=1}^n Tx_k\exp(ikt)\right\|^2 dt\right)^{1/2}$$

$$\leq \frac{1}{2}\left\{\left(\frac{1}{2\pi}\int_{-\pi}^\pi\left\|\sum_{k=1}^n Tu_k\cos kt\right\|^2 dt\right)^{1/2}\right.$$

$$+ \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} T v_k \cos kt \Big\|^2 dt \right)^{1/2}$$

$$+ \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} T u_k \sin kt \Big\|^2 dt \right)^{1/2}$$

$$+ \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} T v_k \sin kt \Big\|^2 dt \right)^{1/2} \bigg\}$$

$$\leq \frac{1}{2} \bigg\{ \|T\| \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} u_k \cos kt \Big\|^2 dt \right)^{1/2}$$

$$+ \varrho(T|\mathcal{C}_n, \mathcal{S}_n) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} v_k \sin kt \Big\|^2 dt \right)^{1/2}$$

$$+ \varrho(T|\mathcal{S}_n, \mathcal{C}_n) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} u_k \cos kt \Big\|^2 dt \right)^{1/2}$$

$$+ \|T\| \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} v_k \sin kt \Big\|^2 dt \right)^{1/2} \bigg\}$$

$$\leq \mu_n(T) \bigg[\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} u_k \cos kt \Big\|^2 dt \right)^{1/2}$$

$$+ \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} v_k \sin kt \Big\|^2 dt \right)^{1/2} \bigg].$$

By the obvious fact that

$$\|u\|, \|v\| \leq \tfrac{1}{2}(\|u+iv\| + \|u-iv\|) \quad \text{for } u, v \in X,$$

we obtain

$$\|(T x_k)|\mathcal{E}_n\| \leq \mu_n(T) \bigg\{ \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} (u_k \cos kt + i v_k \sin kt) \Big\|^2 dt \right)^{1/2}$$

$$+ \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} (u_k \cos kt - i v_k \sin kt) \Big\|^2 dt \right)^{1/2} \bigg\}.$$

Substituting $-t$ for t in the last term on the right-hand side yields

$$\|(T x_k)|\mathcal{E}_n\| \leq 2 \mu_n(T) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} (u_k \cos kt + i v_k \sin kt) \Big\|^2 dt \right)^{1/2}.$$

Finally, we conclude from

$$\sum_{|k| \leq n} x_k \exp(ikt) - x_0 = \sum_{k=1}^{n} (u_k \cos kt + i v_k \sin kt)$$

and Proposition 3.1 that

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{k=1}^{n} (u_k \cos kt + i v_k \sin kt) \Big\|^2 dt \right)^{1/2}$$

$$\leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{|k| \leq n} x_k \exp(ikt) \Big\|^2 dt \right)^{1/2} + \|x_0\|$$

$$\leq 2 \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\| \sum_{|k| \leq n} x_k \exp(ikt) \Big\|^2 dt \right)^{1/2}.$$

This proves the desired result. ∎

The basic trick in the following proof is due to M. Junge.

PROPOSITION 6.3. $\delta(\mathcal{E}_n, \mathcal{E}_n) \leq 96\, \mu_n$.

Proof. The mth de la Vallée Poussin kernel V_m is defined as

$$V_m(t) := \frac{1}{m} \sum_{k=m}^{2m-1} D_k(t),$$

where $D_k(t) := \sum_{|l| \leq k} \exp(ilt)$ is the kth Dirichlet kernel. It is known that

$$(12) \qquad \langle V_m, \bar{e}_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} V_m(t) \overline{e_k(t)}\, dt$$

$$= \begin{cases} 1 & \text{for } |k| \leq m, \\ (2m - |k|)/m & \text{for } m < |k| < 2m, \\ 0 & \text{for } |k| \geq 2m, \end{cases}$$

and

$$(13) \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} |V_m(t)|\, dt \leq 3;$$

see e.g. Zygmund [8].

On $L_2^X(-\pi, \pi)$ we consider the mth de la Vallée Poussin operator

$$V_m : f(t) \longrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} V_m(s - t) f(t)\, dt.$$

It follows from (13) that $\|V_m f | L_2\| \leq 3\, \|f | L_2\|$.

For $f \in L_2^X(-\pi, \pi)$, we let

$$x_k^{(m)} := \langle V_m f, \overline{e}_k \rangle = \langle V_m, \overline{e}_k \rangle \langle f, \overline{e}_k \rangle.$$

Hence, by (12),

$$V_m f = \sum_{|k| \leq 2m-1} e_k \otimes x_k^{(m)}.$$

The triangle inequality implies that

$$\|(T\langle f, \overline{e}_k \rangle) | \mathcal{E}_m\| = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| \sum_{k=1}^{m} T x_k^{(m)} e_k(t) \right\|^2 dt \right)^{1/2} \leq I_1 + I_2,$$

where

$$I_1 := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| \sum_{k=1}^{2m-1} T x_k^{(m)} e_k(t) \right\|^2 dt \right)^{1/2},$$

$$I_2 := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| \sum_{k=m+1}^{2m-1} T x_k^{(m)} e_k(t) \right\|^2 dt \right)^{1/2}.$$

Lemma 6.8 implies that

$$I_1 \leq 4\,\mu_{2m-1}(T) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| \sum_{|k| \leq 2m-1} x_k^{(m)} \exp(ikt) \right\|^2 dt \right)^{1/2}$$

$$= 4\,\mu_{2m-1}(T) \|V_m f | L_2\| \leq 12\,\mu_{2m-1}(T) \|f | L_2\|.$$

To estimate the second term, we recall that $x_k^{(m)} = 0$ if $|k| \geq 2m$. Therefore

$$I_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| \sum_{k=m+1}^{2m-1} T x_k^{(m)} \exp(ikt) \right\|^2 dt \right)^{1/2}$$

$$= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| \sum_{k=1}^{3m-1} T x_{k+m}^{(m)} \exp(ikt) \right\|^2 dt \right)^{1/2}$$

$$\leq 4\,\mu_{3m-1}(T) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| \sum_{|k| \leq 3m-1} x_{k+m}^{(m)} \exp(ikt) \right\|^2 dt \right)^{1/2}$$

$$= 4\,\mu_{3m-1}(T) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| \sum_{|k| \leq 2m-1} x_k^{(m)} \exp(ikt) \right\|^2 dt \right)^{1/2}$$

$$= 4\,\mu_{3m-1}(T) \|V_m f | L_2\| \leq 12\,\mu_{3m-1}(T) \|f | L_2\|.$$

Combining the preceding estimates and taking into account the monotonicity of $\mu_n(T)$, we arrive at

$$\delta(T | \mathcal{E}_m, \mathcal{E}_m) \leq 24\,\mu_{3m-1}(T).$$

To complete the proof, for given $n \in \mathbb{N}$, we choose m such that $3m - 1 \leq n \leq 3m + 1$. Then it follows from Lemma 6.7 that

$$\delta(\mathcal{E}_n, \mathcal{E}_n) = \left\{ \begin{array}{l} \delta(\mathcal{E}_{3m-1}, \mathcal{E}_{3m-1}) \leq 2\,\delta(\mathcal{E}_m, \mathcal{E}_m) + \delta(\mathcal{E}_{m-1}, \mathcal{E}_{m-1}) \\ \delta(\mathcal{E}_{3m}, \mathcal{E}_{3m}) \leq 3\,\delta(\mathcal{E}_m, \mathcal{E}_m) \\ \delta(\mathcal{E}_{3m+1}, \mathcal{E}_{3m+1}) \leq 2\,\delta(\mathcal{E}_m, \mathcal{E}_m) + \delta(\mathcal{E}_{m+1}, \mathcal{E}_{m+1}) \end{array} \right\}$$

$$\leq 4\,\delta(\mathcal{E}_m, \mathcal{E}_m).$$

Hence $\delta(\mathcal{E}_n, \mathcal{E}_n) \leq 4\,\delta(\mathcal{E}_m, \mathcal{E}_m) \leq 96\,\mu_{3m-1} \leq 96\,\mu_n.$ ∎

The next proposition is a special case of (4).

PROPOSITION 6.4. $\varrho(\mathcal{S}_n, \mathcal{C}_n) \leq \delta(\mathcal{S}_n, \mathcal{C}_n)$ *and* $\varrho(\mathcal{C}_n, \mathcal{S}_n) \leq \delta(\mathcal{C}_n, \mathcal{S}_n)$.

To estimate the ideal norms $\delta(\mathcal{S}_n, \mathcal{C}_n)$ and $\delta(\mathcal{C}_n, \mathcal{S}_n)$ by $\delta(\mathcal{E}_n, \mathcal{E}_n)$, we need one more lemma.

LEMMA 6.9. $\varrho(\mathcal{C}_n, \mathcal{E}_n) \leq \sqrt{2}$ *and* $\varrho(\mathcal{S}_n, \mathcal{E}_n) \leq \sqrt{2}$.

P r o o f. By Euler's formula, we have $c_k = \frac{1}{\sqrt{2}}(e_k + \overline{e}_k)$. Hence by (8),

$$\|(x_k) | \mathcal{C}_n\| \leq \frac{1}{\sqrt{2}} (\|(x_k) | \mathcal{E}_n\| + \|(x_k) | \overline{\mathcal{E}}_n\|) = \sqrt{2}\,\|(x_k) | \mathcal{E}_n\|.$$

This proves the left-hand inequality. The right-hand inequality can be obtained in the same way. ∎

The next proposition follows immediately from Proposition 3.5 and Lemma 6.9.

PROPOSITION 6.5. $\delta(\mathcal{S}_n, \mathcal{C}_n) \leq 2\delta(\mathcal{E}_n, \mathcal{E}_n)$ *and* $\delta(\mathcal{C}_n, \mathcal{S}_n) \leq 2\delta(\mathcal{E}_n, \mathcal{E}_n)$.

We now combine Propositions 6.1 through 6.5 to complete the proof of the theorem.

P r o o f o f t h e t h e o r e m. Proposition 6.3 states that $\delta(\mathcal{E}_n, \mathcal{E}_n)$ lies below $\varrho(\mathcal{S}_n, \mathcal{C}_n)$ and $\varrho(\mathcal{C}_n, \mathcal{S}_n)$.

Proposition 6.4 implies that the sequences $\varrho(\mathcal{S}_n, \mathcal{C}_n)$ and $\varrho(\mathcal{C}_n, \mathcal{S}_n)$, respectively, lie below $\delta(\mathcal{S}_n, \mathcal{C}_n)$ and $\delta(\mathcal{C}_n, \mathcal{S}_n)$.

Finally, it follows from Proposition 6.5 that $\delta(\mathcal{S}_n, \mathcal{C}_n)$ and $\delta(\mathcal{C}_n, \mathcal{S}_n)$ lie below $\delta(\mathcal{E}_n, \mathcal{E}_n)$.

This proves the uniform equivalence of all five sequences of ideal norms and thus completes the proof of the theorem. ∎

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On unbounded hyponormal operators III

by

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Abstract. The paper deals mostly with spectral properties of unbounded hyponormal operators. Some nontrivial examples of such operators are given.

I. Introduction. In this work we continue our previous study of unbounded hyponormal operators, [1], [2]. We concentrate on some of their basic spectral properties, and on their polar factors. We also find when the square of a hyponormal operator is the generator of a holomorphic semigroup. The paper ends up with two examples of new classes of unbounded hyponormal operators.

Let H be a complex Hilbert space and let T be a densely defined linear operator in H with domain $D(T)$.

We say that T is *hyponormal* if $D(T) \subset D(T^*)$ and $\|T^*f\| \leq \|Tf\|$, $f \in D(T)$. We refer to [1] for basic facts concerning unbounded hyponormal operators. Throughout the paper $\sigma(T)$, $W(T)$ and $R(\lambda, T)$ denote the spectrum, the numerical range and the resolvent of T, respectively. For a set $A \subset \mathbb{C}$ its closure is denoted by $\operatorname{cl} A$, $\bar{A}$ stands for $\{\bar{\lambda} : \lambda \in A\}$, and $\operatorname{conv} A$ denotes the closed convex hull of A.

II. A few spectral relations. Though some elementary facts about unbounded hyponormal operators were proved in our earlier works [1], [2], the following lemmas seem to be useful, and were not stated there.

LEMMA 2.1. *Let T be a closed hyponormal operator in H. Then $W(T) \subset \operatorname{conv} \sigma(T)$.*

Proof. There are two possibilities.

1) $\operatorname{conv} \sigma(T) = \mathbb{C}$. Then the inclusion is trivial.

2) $\operatorname{conv} \sigma(T) \neq \mathbb{C}$. Since $\alpha T + \beta I$ is hyponormal for any $\alpha, \beta \in \mathbb{C}$, we may assume without loss of generality that $\operatorname{conv} \sigma(T) \subset \mathbb{C}^+ = \{\lambda : \operatorname{Re} \lambda \geq 0\}$. It remains to prove that $W(T) \subset \mathbb{C}^+$.