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Characterization of strict C^*-algebras

by

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Abstract. A Banach algebra A is called strict if the product morphism is continuous with respect to the weak norm in $A \otimes A$. The following result is proved: A C^*-algebra is strict if and only if all its irreducible representations are finite-dimensional and their dimensions are bounded.

The purpose of this work is to prove a criterion of the strictness of a C^*-algebra in terms of its irreducible representations.

DEFINITION. We say that a Banach algebra A is *strict* if the (product) linear operator $R_A : A \otimes_\omega A \to A$, $a \otimes b \mapsto ab$, is bounded.

Here $A \otimes_\omega A$ is the algebraic tensor product with the weak norm

$$\Big\| \sum_{i=1}^{k} a_i \otimes b_i \Big\|_\omega := \sup \Big\{ \Big| \sum_{i=1}^{k} f(a_i) g(b_i) \Big| : f, g \in A^*, \ \|f\| \le 1, \ \|g\| \le 1 \Big\}.$$

Strict Banach algebras were considered in [6] (where the term "injective" was used) and in [5]. For example, the algebras $C^k[a,b]$ and l_1 are strict, but $L^1(G)$ for an infinite topological group G, the algebra of nuclear operators in an infinite-dimensional Hilbert space and l_p for $p > 1$ are not strict [5]. There are strict algebras among C^*-algebras, namely $C(\Omega)$ where Ω is a locally compact space (this was proved in fact by Grothendieck [2]) and an arbitrary finite-dimensional C^*-algebra, for instance, the algebra M_n of complex $n \times n$ matrices with operator norm. However, $\mathcal{K}(H)$, the algebra of compact operators in an infinite-dimensional Hilbert space H, is not strict [5]. The proof of this fact (see Remark below) is based on the existence in $\mathcal{K}(H)$ of any number of one-dimensional operators with identical initial spaces and pairwise orthogonal ranges. It also

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works, after small modifications, for the algebra $\bigoplus_{n=1}^{\infty} M_n$, the c_0-direct sum of matrix algebras. This proof serves as a pattern for the general case.

Our main result is

THEOREM. *A C^*-algebra is strict if and only if all its irreducible representations are finite-dimensional and their dimensions are bounded.*

PROPOSITION 1. *Let $v_1, \ldots, v_n$ be elements of a C^*-algebra A such that*

(A) $\|\sum v_i^* v_i\| \geq n$,

(B) $\|\sum \lambda_i v_i\| = (\sum |\lambda_i|^2)^{1/2}$ *for any complex numbers* λ_i.

Then $\|R_A\| \geq n$.

Proof. Denote by K the linear span of $\{v_1^*, \ldots, v_n^*\}$. If $f \in A^*$ and $\|f\| \leq 1$, then $\|f|_K\| \leq 1$. Condition (B) implies that K is isometrically isomorphic to an n-dimensional Hilbert space. Hence, $(\sum |f(v_i^*)|^2)^{1/2} \leq 1$. Similarly we have $(\sum |g(v_i)|^2)^{1/2} \leq 1$.

Let $u := \sum v_i^* \otimes v_i \in A \otimes_\omega A$. It is obvious that $\|R_A(u)\| = \|\sum v_i^* v_i\| \geq n$. On the other hand, we have

$$\|u\|_\omega = \sup\Big\{\Big|\sum f(v_i^*)g(v_i)\Big| : f, g \in A^*,\ \|f\| \leq 1,\ \|g\| \leq 1\Big\}$$
$$\leq \sup\Big\{\Big(\sum |f(v_i^*)|^2\Big)^{1/2}\Big(\sum |g(v_i)|^2\Big)^{1/2} : f, g \in A^*,\ \|f\| \leq 1,\ \|g\| \leq 1\Big\} \leq 1. \ \blacksquare$$

Let H be a Hilbert space. We write e_{ij} for the operator $x \mapsto (x, e_j)e_i$ where $e_1, e_2, \ldots$ is some (finite or infinite) orthonormal system in H, and $x \in H$.

It is well known that

$$(*) \qquad \Big\|\sum_{i=1}^{n} \lambda_i e_{i1}\Big\| = \Big(\sum_{i=1}^{n} |\lambda_i|^2\Big)^{1/2}$$

for any complex numbers λ_i.

Remark. Suppose that H is infinite-dimensional. Define v_i to be e_{i1} where $\{e_i\}_{i=1}^{\infty}$ is an infinite orthonormal system in H. It is obvious that $v_i \in \mathcal{K}(H)$. Since $e_{i1}^* e_{i1} = e_{11}$, the system $v_1, \ldots, v_n$ satisfies condition (A) of Proposition 1 for all n. Condition (B) holds by $(*)$. Hence $R_{\mathcal{K}(H)} \geq n$ for all n, and $\mathcal{K}(H)$ is not strict.

PROPOSITION 2. *Let π be an irreducible representation of a C^*-algebra A such that $\dim \pi \geq n$. Then there exist $v_1, \ldots, v_n$ in A which satisfy the assumptions of Proposition 1.*

The proof, which is the main part of our arguments, is postponed until the end of the paper. Now we shall prove the main theorem, taking this proof for granted.

Proof of the Theorem. *Sufficiency.* (It was independently obtained by A. Belotskiĭ.) Recall that a representation of a finite dimension n is a homomorphism $\pi : A \to M_n$. Hence, if $x = \sum_{i=1}^{k} a_i \otimes b_i \in A \otimes_\omega A$, then $\sum_{i=1}^{k} \pi(a_i) \otimes \pi(b_i) \in M_n \otimes_\omega M_n$. The operator $R_{M_n} : M_n \otimes_\omega M_n \to M_n$ is bounded, i.e. there exists C_n such that

$$\|R_{M_n}(y)\| \leq C_n \|y\|_\omega \quad \text{for all } y \in M_n \otimes_\omega M_n.$$

Suppose there exists N such that $\dim \pi \leq N$ for all irreducible representations π. Then there exists C such that $C_n \leq C$ for all n. Denote by $\widehat{A}$ the set of all irreducible representations of A. Then

$$(1) \qquad \Big\|\sum_{i=1}^{k} \pi(a_i)\pi(b_i)\Big\| \leq C_n \Big\|\sum_{i=1}^{k} \pi(a_i) \otimes \pi(b_i)\Big\|_\omega \leq C \Big\|\sum_{i=1}^{k} \pi(a_i) \otimes \pi(b_i)\Big\|_\omega$$

for all $\pi \in \widehat{A}$, where $n = \dim \pi$.

It is easy to see that $\|c\|' := \sup_{\pi \in \hat{A}} \|\pi(c)\|$ is a C^*-norm in A. The uniqueness of the norm in a C^*-algebra implies that for all $c \in A$ we have $\|c\| = \|c\|'$, i.e.

$$(2) \qquad \|c\| = \sup_{\pi \in \hat{A}} \|\pi(c)\|.$$

We conclude from (1) and (2) that

$$(3) \qquad \Big\|\sum_{i=1}^{k} a_i b_i\Big\| = \sup_{\pi \in \hat{A}} \Big\|\sum_{i=1}^{k} \pi(a_i)\pi(b_i)\Big\| \leq \sup_{\pi \in \hat{A}} \Big\|\sum_{i=1}^{k} \pi(a_i) \otimes \pi(b_i)\Big\|_\omega.$$

Let $f, g \in M_n^*$. Then the functionals $\widetilde{f}_\pi : a \mapsto f(\pi(a))$ and $\widetilde{g}_\pi : a \mapsto g(\pi(a))$, where π is some representation, belong to A^* and we have $\|\widetilde{f}_\pi\| \leq$

$\|f\|$ and $\|\widetilde{g}_\pi\| \le \|g\|$. Therefore

$$\Big\|\sum_{i=1}^k \pi(a_i)\otimes\pi(b_i)\Big\|_\omega$$
$$= \sup\Big\{\Big|\sum_{i=1}^k f(\pi(a_i))g(\pi(b_i))\Big| : f,g\in M_n^*,\ \|f\|\le 1,\ \|g\|\le 1\Big\}$$
$$\le \sup\Big\{\Big|\sum_{i=1}^k f'(a_i)g'(b_i)\Big| : f',g'\in A^*,\ \|f'\|\le 1,\ \|g'\|\le 1\Big\}$$
$$= \Big\|\sum_{i=1}^k a_i\otimes b_i\Big\|_\omega.$$

Using inequality (3), we thus have

$$\Big\|\sum_{i=1}^k a_ib_i\Big\| \le C\Big\|\sum_{i=1}^k a_i\otimes b_i\Big\|_\omega.$$

Any element of $A\otimes_\omega A$ has the form $\sum_{i=1}^k a_i\otimes b_i$. This implies that $\|R_A(x)\| \le C\|x\|_\omega$ for all $x\in A\otimes_\omega A$.

Necessity. Assume, on the contrary, that for any n there exists $\pi\in\widehat{A}$ with $\dim\pi\ge n$. Then Propositions 1 and 2 yield $\|R_A\|\ge n$ for all n. This means that R_A is not bounded and A is not strict. ∎

Now we start to prove Proposition 2. First, we reduce it to another statement.

PROPOSITION 3. *Let π be a representation of a C^*-algebra A in a Hilbert space H such that $\dim\pi\ge n$. Let $\{e_1,\dots,e_n\}$ be an orthonormal system in H and $v_1,\dots,v_n$ be elements of A such that*

(i) $v_j^*v_i = 0$ *for* $i\ne j$,
(ii) $q\pi(v_i)q = \pi(v_i)q = e_{i1}$ $(i=1,\dots,n)$,
(iii) $\|v_i\| = 1$,

where q is the orthogonal projection onto the linear span of $\{e_1,\dots,e_n\}$ (in other words, $q=\sum_{i=1}^n e_{ii}$). Then the assumptions of Proposition 1 hold.

Proof. Suppose A is faithfully represented in a Hilbert space K. Then

$$\Big\|\sum\lambda_iv_i\Big\| = \sup_{\substack{x\in K\\ \|x\|\le 1}}\Big(\sum_{i,j}\overline{\lambda}_j\lambda_i(v_j^*v_ix,x)\Big)^{1/2}$$
$$= \sup_{\substack{x\in K\\ \|x\|\le 1}}\Big(\sum|\lambda_i|^2(v_ix,v_ix)\Big)^{1/2} \le \Big(\sum|\lambda_i|^2\Big)^{1/2}.$$

On the other hand, $(*)$ implies

$$\Big\|\sum\lambda_iv_i\Big\| \ge \Big\|\pi\Big(\sum\lambda_iv_i\Big)\Big\| \ge \Big\|q\pi\Big(\sum\lambda_iv_i\Big)q\Big\|$$
$$= \Big\|\sum\lambda_iq\pi(v_i)q\Big\| = \Big\|\sum\lambda_ie_{i1}\Big\| = \Big(\sum|\lambda_i|^2\Big)^{1/2}.$$

Therefore $\|\sum\lambda_iv_i\| = (\sum|\lambda_i|^2)^{1/2}$, and condition (A) holds. Further,

$$\Big\|\sum v_i^*v_i\Big\| \ge \Big\|\sum\pi(v_i^*)\pi(v_i)\Big\| = \Big\|\sum q\pi(v_i^*)\pi(v_i)q\Big\|$$
$$= \Big\|\sum e_{1i}e_{i1}\Big\| = \|ne_{11}\| = n,$$

and condition (B) holds. ∎

Thus, to prove Proposition 2 it is sufficient to show the following: if π is irreducible and $\dim\pi\ge n$, then there exist $v_1,\dots,v_n$ in A satisfying conditions (i)–(iii) of Proposition 3. This will be done in four steps.

1. We construct $a_1,\dots,a_n\in A$ such that $q\pi(a_k)q = \pi(a_k)q = p_k$, $a_k\ge 0$, $\|a_k\| = 1$, where $p_k = \sum_{i=1}^k e_{ii}$, and for all k, l with $k\le l$ we have

$$(**)\qquad a_ka_l = a_la_k = a_k.$$

2. We construct $b_1,\dots,b_n\in A$ such that $q\pi(b_k)q = \pi(b_k)q = e_{kk}$, $b_k^* = b_k$, $\|b_k\| = 1$ and $b_kb_l = 0$ when $k\ne l$.

3. We choose $w_1,\dots,w_n\in A$ such that $q\pi(w_k)q = \pi(w_k)q = e_{k1}$, and $\|w_k\| = 1$.

4. We put $v_k := b_kw_k$, $k=1,\dots,n$, and prove that conditions (i)–(iii) of Proposition 3 hold for $v_1,\dots,v_n$.

First, we prove two technical lemmas.

LEMMA 1. *Let $h:\mathbb{R}\to\mathbb{C}$ be a continuous function such that $|h(t)|\le 1$ and $h(0)=0$, π be a representation of a C^*-algebra A, r be a selfadjoint element of A, and q be an orthoprojection such that $q\pi(r)q = \pi(r)q$. Suppose that $\pi(r)q$ is an orthoprojection. Then $\pi(h(r))q = \pi(r)q$, and $\|h(r)\| = 1$.*

Proof. Since $h(0)=0$, we have $h(r)\in A$ [3; Ch. 4, 7.21]. Obviously, $\|h(r)\| = 1$. Suppose f is a polynomial with $f(0)=0$ and $f(1)=1$. Then $\pi(f(r))q = f(\pi(r))q = f(\pi(r)q) = f(1)\pi(r)q = \pi(r)q$. The function h can be approximated by such polynomials in the norm of $C[-\|r\|,\|r\|]$. Therefore $\pi(h(r))q = \pi(r)q$. ∎

LEMMA 2. *Let $h:\mathbb{R}\to\mathbb{C}$ be a continuous function such that $h(t)=0$ for $t\le 1/2$, and x and y be selfadjoint commuting elements of a C^*-algebra A such that $\|x\| = \|y\| = 1$. Then $h(x-y)h(y) = 0$.*

Proof. Let $s,t\in\mathbb{R}$. The inequality $h(s-t)h(t)\neq 0$ obviously implies that $s-t>1/2$ and $t>1/2$; hence, $t>1$. This means that $\overline{h}(s,t):=h(s-t)h(t)=0$ for all $s,t\in[-1,1]$. Since the continuous functional calculus is a homomorphism, $\overline{h}(x,y)=h(x-y)h(y)$. Thus, $h(x-y)h(y)=0$ because the continuous functional calculus is isometric. ■

Proof of Proposition 2. Let f and g be positive continuous functions on $\mathbb{R}$ such that $fg=f$, g satisfies the assumptions of Lemma 1, and f satisfies the assumptions of Lemmas 1 and 2. For example, we could take

$$f(t):=\begin{cases}0, & t\le 1/2,\\ 2t-1, & 1/2\le t\le 1,\\ 1, & t\ge 1,\end{cases}\qquad g(t):=\begin{cases}0, & t\le 0,\\ 2t, & 0\le t\le 1/2,\\ 1, & t\ge 1/2.\end{cases}$$

If $\pi:A\to\mathcal{B}(H)$ is irreducible, $\varphi_1,\ldots,\varphi_n\in H$, and b is a bounded operator on H, then there exists $a\in A$ such that $\pi(a)\varphi_i=b\varphi_i$ and $\|\pi(a)\|=\|b\|$; moreover, if $b^*=b$ we can suppose that a is selfadjoint. (This is Kadison's transitivity theorem [4].)

1. Let π be an irreducible representation with $\dim\pi\ge n$ and $\{e_1,\ldots,e_n\}$ be an orthonormal system in H. By Kadison's transitivity theorem, we can choose selfadjoint $r_1,\ldots,r_n$ such that $\pi(r_k)e_j=p_ke_j$ for $j,k=1,\ldots,n$ where $p_k=\sum_{i=1}^k e_{ii}$. Put $q=\sum_{i=1}^n e_{ii}$. Then $q\pi(r_k)q=\pi(r_k)q=p_k$. Replacing r_k by $|r_k|$, we can suppose that $r_k\ge 0$. Let

$$r_n':=g(r_n),\quad r_{n-1}':=f(r_n)r_{n-1}f(r_n),\quad r_k':=r_k\quad\text{for } 1\le k\le n-2.$$

By Lemma 1, $\pi(r_n')q=\pi(g(r_n)q)=p_n$, $\pi(r_{n-1}')q=p_np_{n-1}p_n=p_{n-1}$ and $\|r_n'\|=\|r_{n-1}'\|=1$. Furthermore, since $fg=f$, we have $r_n'r_{n-1}'=g(r_n)f(r_n)r_{n-1}f(r_n)=f(r_n)r_{n-1}f(r_n)=r_{n-1}'$, and similarly $r_{n-1}'r_n'=r_{n-1}'$. Thus, property (∗∗) holds for r_k' and r_l' whenever $n-1\le k\le l\le n$.

Suppose now that $r_1,\ldots,r_n$ are such that $\pi(r_k)q=p_k$, $\|r_k\|=1$ and property (∗∗) holds for r_k and r_l whenever $m\le k\le l\le n$ $(m>1)$. Define $r_1',\ldots,r_n'$ by

$$r_k':=\begin{cases}r_k, & k<m-1,\\ cr_kc, & k=m-1,\\ g(r_k), & k>m-1,\end{cases}$$

where $c=\prod_{i=m}^n f(r_j)$. It follows from $r_k\ge 0$ and $g\ge 0$ that $r_k'\ge 0$ for $k\neq m-1$. Since $c^*=c$ we have $r_{m-1}'=cr_{m-1}c\ge 0$ ([1; 1.6.8] or [3; Ch. 4, 7.75]). Thus, $r_k\ge 0$ for $k=1,\ldots,n$.

Lemma 1 implies that $\pi(r_k')q=p_k$ and $\|r_k\|=1$. Since functions of commuting elements commute, property (∗∗) holds for r_k' and r_l' whenever $m-1\le k\le l\le n$. After replacing r_k by r_k' we can continue this procedure. The system of elements obtained after $n-1$ steps will be denoted by $a_1,\ldots,a_n$. It is easy to see that $a_ka_l=a_la_k=a_k$ for $1\le k\le l\le n$, $\pi(a_k)q=p_k$, $a_k\ge 0$ and $\|a_k\|=1$, $k=1,\ldots,n$.

2. Put

$$b_1:=f(a_1),\quad b_k:=\prod_{i=1}^{k-1}f(a_k-a_l),\quad k=2,\ldots,n.$$

Let 1_A be the unit in a C^*-unitization of A. Since $a_k\ge 0$, we have $\|1_A-a_k\|\le\|a_k\|=1$ ([3; Ch. 4, 7.73], [1; 1.6.23]). By property (∗∗),

$$\|a_{k+j}-a_k\|=\|a_{k+j}-a_{k+j}a_k\|\le\|a_{k+j}\|\,\|1_A-a_k\|\le 1$$

for $j\le n-k$. At the same time,

$$\|a_{k+j}-a_k\|\ge\|\pi(a_{k+j}-a_k)q\|=\|p_{k+j}-p_k\|=1.$$

Therefore $\|a_{k+j}-a_k\|=1$, and the elements $x:=a_k-a_i$ and $y:=a_l-a_i$ satisfy the assumptions of Lemma 2 whenever $i<l<k$. Hence,

$$\begin{aligned}f(a_k-a_l)f(a_l-a_i)&=f((a_k-a_i)-(a_l-a_i))f(a_l-a_i)\\&=f(x-y)f(y)=0.\end{aligned}$$

Put $x:=a_k$ and $y:=a_1$. We have $f(a_k-a_1)f(a_1)=0$ $(k\ge 1)$. This means that $b_kb_l=0$ provided $k\neq l$. By Lemma 1, $\|b_k\|=1$, $b_k^*=b_k$ and $\pi(f(a_k-a_i))q=p_k-p_i$ for $k\ge i$. Finally, we have $\pi(b_k)q=e_{kk}$.

3. The existence of $w_1,\ldots,w_n$ such that $q\pi(w_k)q=\pi(w_k)q=e_{k1}$ and $\|w_k\|=1$ follows immediately from Kadison's transitivity theorem.

4. Put $v_k:=b_kw_k$. It is obvious that $q\pi(v_k)q=\pi(v_k)=e_{kk}e_{k1}=e_{k1}$. Further, $v_k^*v_l=w_k^*b_kb_lw_l=0$ $(k\neq l)$ and $\|v_k\|\le 1$. At the same time, $\|v_k\|\ge\|e_{k1}\|=1$. Hence, $\|v_k\|=1$. Thus, the system $v_1,\ldots,v_n$ satisfies conditions (A) and (B) of Proposition 1. The proof is complete. ■

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Ideal norms and trigonometric orthonormal systems

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Abstract. We characterize the UMD-property of a Banach space X by sequences of ideal norms associated with trigonometric orthonormal systems. The asymptotic behavior of those numerical parameters can be used to decide whether X is a UMD-space. Moreover, if this is not the case, we obtain a measure that shows how far X is from being a UMD-space. The main result is that all described sequences are not only simultaneously bounded but are also asymptotically equivalent.

1. Introduction. The study of sequences of ideal norms can be used to quantify certain properties of linear operators. In most cases the boundedness of a sequence of ideal norms for a given operator T describes a well-known property, whereas, in the non-bounded case, the growth rate of the sequence describes how much the operator T deviates from this property.

One particularly interesting case is if two sequences of ideal norms are uniformly equivalent. Then the properties given by these sequences are also quantitatively equivalent.

We introduce several sequences of ideal norms related to the trigonometric orthonormal systems. The boundedness of these sequences for the identity map of a Banach space X is equivalent to X being UMD.

All of these sequences turn out to be uniformly equivalent. As a corollary we deduce that a Banach space X is a UMD-space if and only if there exists a constant $c \geq 0$ such that, for all $x_1, \ldots, x_n \in X$, we have

$$\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\left\|\sum_{k=1}^{n} x_k \sin kt\right\|^2 dt\right)^{1/2} \leq c\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\left\|\sum_{k=1}^{n} x_k \cos kt\right\|^2 dt\right)^{1/2}$$

or, what turns out to be equivalent,

$$\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\left\|\sum_{k=1}^{n} x_k \cos kt\right\|^2 dt\right)^{1/2} \leq c\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\left\|\sum_{k=1}^{n} x_k \sin kt\right\|^2 dt\right)^{1/2}.$$

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