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Spectral decompositions and harmonic analysis on UMD spaces

by

EARL BERKSON (Urbana, Ill.) and T. A. GILLESPIE (Edinburgh)

Abstract. We develop a spectral-theoretic harmonic analysis for an arbitrary UMD space X. Our approach utilizes the spectral decomposability of X and the multiplier theory for L^p_X to provide on the space X itself analogues of the classical themes embodied in the Littlewood–Paley Theorem, the Strong Marcinkiewicz Multiplier Theorem, and the M. Riesz Property. In particular, it is shown by spectral integration that classical Marcinkiewicz multipliers have associated transforms acting on X.

1. Introduction. The aim of this article is to extend to spectral theory on Banach spaces the themes of classical harmonic analysis embodied in the M. Riesz Conjugacy Theorem, the Littlewood–Paley Decomposition Theorem, and the Strong Marcinkiewicz Multiplier Theorem, each of which has versions applying to the p-integrable complex-valued functions on the circle $\mathbb{T}$, the real line $\mathbb{R}$, and the integers $\mathbb{Z}$, for p in the range $1 < p < \infty$ ([14, Theorem 6.7.4, Theorem 7.2.1, and Chapter 8]). The Banach spaces which we consider are those possessing the unconditionality property for martingale differences, the so-called UMD spaces. The UMD spaces form a natural medium for our considerations because of their Hilbert transform characterization: a Banach space X is UMD if and only if for $1 < p < \infty$ the classical Hilbert kernels for $\mathbb{T}$, $\mathbb{R}$, and $\mathbb{Z}$ define bounded convolution operators on the corresponding L^p-spaces of X-valued functions (see [10], [12], and [8, §2]).

The basic spectral-theoretic tool for our considerations is the notion of "spectral family of projections," together with its allied notion of spectral integration.

DEFINITION. Let $\mathfrak{B}(\mathfrak{X})$ denote the Banach algebra of all bounded linear mappings of a Banach space $\mathfrak{X}$ into itself, and let I be the identity operator on $\mathfrak{X}$. A *spectral family of projections* in $\mathfrak{X}$ is a projection-valued function $E(\cdot) : \mathbb{R} \to \mathfrak{B}(\mathfrak{X})$ satisfying:

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(i) $\sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\} < \infty$;

(ii) $E(\lambda)E(\tau) = E(\tau)E(\lambda) = E(\lambda)$ whenever $\lambda \leq \tau$;

(iii) $E(\cdot)$ is right-continuous on $\mathbb{R}$ with respect to the strong operator topology of $\mathfrak{B}(\mathfrak{X})$;

(iv) at each $\lambda \in \mathbb{R}$, $E(\cdot)$ has a left-hand limit $E(\lambda^-)$ in the strong operator topology of $\mathfrak{B}(\mathfrak{X})$;

(v) with respect to the strong operator topology of $\mathfrak{B}(\mathfrak{X})$, $E(\lambda) \to I$ as $\lambda \to \infty$, and $E(\lambda) \to 0$ as $\lambda \to -\infty$.

If there is a compact interval $[a,b]$ such that $E(\lambda) = 0$ for $\lambda < a$ and $E(\lambda) = I$ for $\lambda \geq b$, then $E(\cdot)$ is said to be *concentrated* on $[a,b]$.

A Riemann–Stieltjes notion of integration with respect to an arbitrary spectral family $E(\cdot)$ of projections in $\mathfrak{X}$ can be defined as follows. Given a bounded, complex-valued function f on a compact interval $J = [\alpha,\beta]$ of $\mathbb{R}$, for each partition $\mathcal{P} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ of J put

$$\mathcal{S}(\mathcal{P}; f, E) = \sum_{k=1}^{n} f(\lambda_k)\{E(\lambda_k) - E(\lambda_{k-1})\}.$$

If the net $\{\mathcal{S}(\mathcal{P}; f, E)\}$ converges in the strong operator topology of $\mathfrak{B}(\mathfrak{X})$ as $\mathcal{P}$ increases through the partitions of J directed by refinement, then we denote the strong limit by $\int_{[\alpha,\beta]} f\,dE$, and further define $\int^{\oplus}_{[\alpha,\beta]} f\,dE$ by writing

$$\int^{\oplus}_{[\alpha,\beta]} f\,dE = f(\alpha)E(\alpha) + \int_{[\alpha,\beta]} f\,dE.$$

In the setting of the arbitrary Banach space $\mathfrak{X}$, the most general class of functions known to be integrable with respect to $E(\cdot)$ over the compact interval J is the algebra $\mathrm{BV}(J)$ consisting of all complex-valued functions ϕ on J whose total variation $\mathrm{var}(\phi, J)$ is finite (see [13, Chapter 17] or the abbreviated account of spectral integration in [5, §2]). In fact, the mapping $\mathrm{BV}(J) \ni \phi \mapsto \int^{\oplus}_{[\alpha,\beta]} \phi\,dE$ is an algebra homomorphism of $\mathrm{BV}(J)$ into $\mathfrak{B}(\mathfrak{X})$ satisfying

$$\Big\| \int^{\oplus}_{[\alpha,\beta]} \phi\,dE \Big\| \leq \|\phi\|_{\mathrm{BV}(J)} \sup\{\|E(\lambda)\| : \lambda \in J\},$$

where $\|\cdot\|_{\mathrm{BV}(J)}$ denotes the Banach algebra norm on $\mathrm{BV}(J)$ specified by

$$\|\phi\|_{\mathrm{BV}(J)} = |\phi(\beta)| + \mathrm{var}(\phi, J).$$

If $\phi \in \mathrm{BV}(J)$ is a continuous function, then for $k = 1, \ldots, n$, $\phi(\lambda_k)$ can be replaced in the expression for the approximating sum $\mathcal{S}(\mathcal{P}; \phi, E)$ by $\phi(\lambda'_k)$, where $\lambda'_k \in [\lambda_{k-1}, \lambda_k]$ is chosen arbitrarily. The corresponding assertions for spectral integration with respect to $E(\cdot)$ over $\mathbb{R}$ likewise obtain when $\mathrm{BV}(J)$ is replaced by $\mathrm{BV}(\mathbb{R})$ (see [20, Proposition 2.1.11 and Theorem 2.1.14]). In particular, for $\phi \in \mathrm{BV}(\mathbb{R})$, the corresponding Riemann–Stieltjes approximating sums $\mathcal{S}(\mathcal{P}; \phi, E)$ are taken on partitions $\mathcal{P}$ of the extended real number system $[-\infty, \infty]$, and have a limit in the strong operator topology of $\mathfrak{B}(\mathfrak{X})$ denoted by $\int_{\mathbb{R}} \phi\,dE$. In this situation the functions $E(\cdot)$ and ϕ are extended from $\mathbb{R}$ to $[-\infty,\infty]$ by defining $E(-\infty) = 0$, $E(\infty) = I$, and $\phi(\pm\infty) = \lim_{\lambda\to\pm\infty} \phi(\lambda)$, while the corresponding algebra homomorphism $\mathrm{BV}(\mathbb{R}) \ni \phi \mapsto \int_{\mathbb{R}} \phi\,dE \in \mathfrak{B}(\mathfrak{X})$ satisfies

$$\Big\| \int_{\mathbb{R}} \phi\,dE \Big\| \leq \|\phi\|_{\mathrm{BV}(\mathbb{R})} \sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\},$$

in terms of the Banach algebra norm $\|\cdot\|_{\mathrm{BV}(\mathbb{R})}$ on $\mathrm{BV}(\mathbb{R})$ specified by

$$\|\phi\|_{\mathrm{BV}(\mathbb{R})} = |\phi(\infty)| + \sup\{\mathrm{var}(\phi, [-N,N]) : N = 1,2,\ldots\}.$$

The relationship between spectral integration over compact intervals and over $\mathbb{R}$ is expressed by the fact that for $\phi \in \mathrm{BV}(\mathbb{R})$, $\int_{[-a,a]} \phi\,dE$ converges in the strong operator topology of $\mathfrak{B}(\mathfrak{X})$ to $\int_{\mathbb{R}} \phi\,dE$, as $a \to \infty$.

Having attended to the preliminaries, we can now describe the framework to be considered. We begin by recalling the following three results concerning spectral decomposability in UMD spaces. Henceforth the upper case letter "C" with a (possibly empty) set of subscripts will denote a positive real constant depending only on its subscripts, and may change from one occurrence to another.

THEOREM A ([8, Theorem (4.5)]). *Let X be a UMD space, and let $U : X \to X$ be an invertible bounded linear operator such that $c \equiv \sup\{\|U^n\| : n \in \mathbb{Z}\} < \infty$. Then there is a unique spectral family of projections $E(\cdot)$ in X such that $E(\cdot)$ is concentrated on $[0, 2\pi]$, $E((2\pi)^-) = I$, and*

$$U = \int^{\oplus}_{[0,2\pi]} e^{i\lambda}\,dE(\lambda).$$

Moreover, $\sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\} \leq c^2 C_X$.

THEOREM B ([8, Theorem (5.5), Corollary (5.8), Theorem (5.12)(ii)]). *Let $\{U_t : t \in \mathbb{R}\}$ be a strongly continuous one-parameter group of operators on a UMD space X such that $c \equiv \sup\{\|U_t\| : t \in \mathbb{R}\} < \infty$. Then there is a unique spectral family $E(\cdot)$ of projections in X such that*

$$U_t x = \lim_{a\to\infty} \int_{[-a,a]} e^{i\lambda t}\,dE(\lambda)x \quad \textit{for all } t \in \mathbb{R},\ x \in X.$$

Moreover, $\sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\} \leq c^2 C_X$.

THEOREM C ([4, Theorem (4.2)], [9, Theorem (3.3)]). *Let $\omega \mapsto R_\omega$ be a strongly continuous representation of $\mathbb{T}$ in a UMD space X, and let*

$\{P_n\}_{n=-\infty}^{\infty}$ be the associated sequence of spectral projections defined by

$$P_n x = \int_{\mathbb{T}} \omega^{-n} R_\omega x \, d\lambda(\omega) \quad \text{for all } n \in \mathbb{Z},\ x \in X,$$

where $d\lambda$ denotes normalized Haar measure on $\mathbb{T}$. Then for every $\omega \in \mathbb{T}$,

$$R_\omega = \sum_{n=0}^{\infty} \omega^n P_n + \sum_{n=1}^{\infty} \omega^{-n} P_{-n},$$

where each of the series on the right converges in the strong operator topology of $\mathfrak{B}(X)$. Moreover,

$$\sup\Big\{\Big\|\sum_{n=L}^{M} P_n\Big\| : L \in \mathbb{Z},\ M \in \mathbb{Z},\ L \le M\Big\} \le c^2 C_X,$$

where $c \equiv \sup\{\|R_\omega\| : \omega \in \mathbb{T}\} < \infty$.

When Theorem C is specialized to the regular representation of $\mathbb{T}$ (that is, representation of $\mathbb{T}$ by translation operators) on $X = L^p(\mathbb{T})$, where $1 < p < \infty$, the operator P_n projects each $f \in L^p(\mathbb{T})$ onto the nth term of its Fourier series, and so the result of Theorem C in this special case is the version of the M. Riesz Conjugacy Theorem asserting the convergence of Fourier series in the norm topology of $L^p(\mathbb{T})$. Likewise, when the one-parameter group $\{U_t : t \in \mathbb{R}\}$ in Theorem B is specialized to be the translation group acting on $X = L^p(\mathbb{R})$, $1 < p < \infty$, the projection $E(\lambda)$ is the multiplier transform corresponding to the characteristic function $\chi_{(-\infty,\lambda]}$; the existence of which stems from the boundedness of the Hilbert transform on $L^p(\mathbb{R})$. In a similar vein, the classical antecedent of Theorem A concerns the left shift U on $X = L^p(\mathbb{Z})$, $1 < p < \infty$. Here the projection $E(\lambda)$, $0 \le \lambda \le 2\pi$, is the multiplier transform corresponding to the characteristic function of the arc of $\mathbb{T}$ given by $\{e^{it} : 0 \le t \le \lambda\}$. In this case, the existence and uniform boundedness of $E(\cdot)$ come from the boundedness of the discrete Hilbert transform on $L^p(\mathbb{Z})$.

The preceding comments show that the results of Theorems A, B, and C can be viewed as abstractions to the UMD space setting of the M. Riesz Conjugacy Theorem. In the present article we shall discuss related abstractions of the Littlewood–Paley and Strong Marcinkiewicz Multiplier Theorems for $\mathbb{T}$, $\mathbb{R}$, and $\mathbb{Z}$.

To be more precise, we introduce some further notation. Let $\{s_n\}_{n=-\infty}^{\infty}$ be the dyadic points of $\mathbb{R}$ specified by: $s_n = 2^{n-1}$ for $n > 0$; $s_n = -2^{-n}$ for $n \le 0$. Let $\{\Lambda_n\}_{n=-\infty}^{\infty}$ and $\{\widetilde{\Lambda}_n\}_{n=-\infty}^{\infty}$ be the dyadic decompositions of $\mathbb{Z}$ and $\mathbb{R}$, respectively, which are given by

$$\widetilde{\Lambda}_n = [s_n, s_{n+1}) \quad \text{if } n > 0, \qquad \widetilde{\Lambda}_n = (s_n, s_{n+1}] \quad \text{if } n < 0, \qquad \widetilde{\Lambda}_0 = (s_0, s_1);$$

and

$$\Lambda_n = \widetilde{\Lambda}_n \cap \mathbb{Z} \quad \text{for } n \in \mathbb{Z}.$$

Further, let $\{t_n\}_{n=-\infty}^{\infty}$ denote the sequence of dyadic points in $(0, 2\pi)$ defined by

$$t_n = 2^{n-1}\pi \quad \text{if } n \le 0, \qquad t_n = 2\pi - 2^{-n}\pi \quad \text{if } n > 0.$$

For $n \in \mathbb{Z}$, we shall write

$$\omega_n = e^{it_n}, \quad \Gamma_n = \{e^{it} : t_n < t < t_{n+1}\}, \quad \Delta_n = \{e^{it} : t_n \le t \le t_{n+1}\}.$$

As discussed in [1] and [3], the singleton sets $\{1\}$ and $\{\omega_n\}$, for $n \in \mathbb{Z}$, together with the sets Γ_n, for $n \in \mathbb{Z}$, generate the dyadic sigma-algebra $\Sigma_{\mathrm{d}}(\mathbb{T})$ on $\mathbb{T}$, while the dyadic sigma-algebra $\Sigma_{\mathrm{d}}(\mathbb{R})$ is generated on $\mathbb{R}$ by the sets $\widetilde{\Lambda}_n$ for $n \in \mathbb{Z}$ and the singleton sets $\{s_n\}$ for $n \in \mathbb{Z}$.

Let $\mathfrak{M}(\mathbb{R})$ denote the set of all functions $\phi : \mathbb{R} \to \mathbb{C}$ such that $\|\phi\|_{\mathfrak{M}(\mathbb{R})} < \infty$, where

$$\|\phi\|_{\mathfrak{M}(\mathbb{R})} = \sup\{|\phi(s)| : s \in \mathbb{R}\} + \sup\{\operatorname{var}(\phi, [s_n, s_{n+1}]) : n \in \mathbb{Z}\}.$$

Elements of $\mathfrak{M}(\mathbb{R})$ are called *Marcinkiewicz multipliers*, since the Strong Marcinkiewicz Multiplier Theorem for $\mathbb{R}$ ([14, Theorem 8.3.1]) asserts that, for $1 < p < \infty$, each $\phi \in \mathfrak{M}(\mathbb{R})$ is a Fourier multiplier for $L^p(\mathbb{R})$ with p-multiplier norm not exceeding $C_p\|\phi\|_{\mathfrak{M}(\mathbb{R})}$. With pointwise operations on $\mathbb{R}$, $\mathfrak{M}(\mathbb{R})$ is a unital Banach algebra under the norm $\|\cdot\|_{\mathfrak{M}(\mathbb{R})}$. The unital Banach algebras $\mathfrak{M}(\mathbb{Z})$ and $\mathfrak{M}(\mathbb{T})$ are defined analogously after writing (for $f : \mathbb{Z} \to \mathbb{C}$ and $g : \mathbb{T} \to \mathbb{C}$)

$$\|f\|_{\mathfrak{M}(\mathbb{Z})} = \sup_{n\in\mathbb{Z}} |f(n)| + \sup_{n\in\mathbb{Z}} \sum_{j=s_n+1}^{s_{n+1}} |f(j) - f(j-1)|;$$

$$\|g\|_{\mathfrak{M}(\mathbb{T})} = \sup_{z\in\mathbb{T}} |g(z)| + \sup_{n\in\mathbb{Z}} \operatorname{var}(g, \Delta_n).$$

The Strong Marcinkiewicz Multiplier Theorem can be phrased for either of the groups $\mathbb{T}$, $\mathbb{Z}$ in complete analogy with its preceding statement for $\mathbb{R}$ upon replacing $\mathfrak{M}(\mathbb{R})$ by $\mathfrak{M}(\mathbb{Z})$ and $\mathfrak{M}(\mathbb{T})$, respectively [14, Theorems 8.2.1 and 8.4.2]. In terms of the foregoing notation, our main results are stated as follows.

(1.1) THEOREM. *Let X, U, $E(\cdot)$, and c be as in Theorem A above. Then the following assertions are valid.*

(i) *There is a (necessarily unique) strongly countably additive spectral measure $\mathcal{E}(\cdot)$ defined on the dyadic sigma-algebra $\Sigma_{\mathrm{d}}(\mathbb{T})$ such that $\mathcal{E}(\{1\}) = E(0)$, and, for $n \in \mathbb{Z}$,*

$$\mathcal{E}(\{\omega_n\}) = E(t_n) - E(t_n^-), \qquad \mathcal{E}(\Gamma_n) = E(t_{n+1}^-) - E(t_n).$$

Furthermore, $\sup\{\|\mathcal{E}(\sigma)\| : \sigma \in \Sigma_{\mathrm{d}}(\mathbb{T})\} \le c^2 C_X$.

(ii) *For each* $\phi \in \mathfrak{M}(\mathbb{T})$, $\int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda)$ *exists in the strong operator topology, and the mapping*

$$\phi \mapsto \int_{[0,2\pi]}^{\oplus} \phi(e^{i\lambda})\, dE(\lambda)$$

is an identity-preserving algebra homomorphism of $\mathfrak{M}(\mathbb{T})$ *into* $\mathfrak{B}(X)$ *satisfying*

$$\Big\| \int_{[0,2\pi]}^{\oplus} \phi(e^{i\lambda})\, dE(\lambda) \Big\| \le c^2 C_X \|\phi\|_{\mathfrak{M}(\mathbb{T})}.$$

For the analogue of (1.1)(ii) in the setting of Theorem B, we define the Riemann–Stieltjes sums in the case of $\phi \in \mathfrak{M}(\mathbb{R})$ to be of the form

$$\sum_{k=1}^{n} \phi(\lambda_k)\{E(\lambda_k) - E(\lambda_{k-1})\},$$

where $n \ge 1$, $-\infty < \lambda_0 < \lambda_1 < \ldots < \lambda_n < \infty$, and the strictly increasing finite sequences in $\mathbb{R}$ are directed to increase by set inclusion $\supseteq$.

(1.2) THEOREM. *Let* X, $\{U_t : t \in \mathbb{R}\}$, $E(\cdot)$, *and* c *be as in Theorem* B *above. Then the following assertions are valid.*

(i) *There is a (necessarily unique) strongly countably additive spectral measure* $\mathcal{E}(\cdot)$ *defined on the dyadic sigma-algebra* $\Sigma_{\mathrm{d}}(\mathbb{R})$ *such that, for* $n \in \mathbb{Z}$,

$$\mathcal{E}(\{s_n\}) = E(s_n) - E(s_n^-), \qquad \mathcal{E}((s_n, s_{n+1})) = E(s_{n+1}^-) - E(s_n).$$

Furthermore, $\sup\{\|\mathcal{E}(\sigma)\| : \sigma \in \Sigma_{\mathrm{d}}(\mathbb{R})\} \le c^2 C_X$.

(ii) *For each* $\phi \in \mathfrak{M}(\mathbb{R})$, $\int_{\mathbb{R}} \phi(\lambda)\, dE(\lambda)$ *exists as the strong limit of Riemann–Stieltjes sums, and the mapping*

$$\phi \mapsto \int_{\mathbb{R}} \phi(\lambda)\, dE(\lambda)$$

is an identity-preserving algebra homomorphism of $\mathfrak{M}(\mathbb{R})$ *into* $\mathfrak{B}(X)$ *satisfying*

$$\Big\| \int_{\mathbb{R}} \phi(\lambda)\, dE(\lambda) \Big\| \le c^2 C_X \|\phi\|_{\mathfrak{M}(\mathbb{R})}.$$

Remark. In the set-up of (1.2)(ii), it is easy to see from the existence of $\int_{\mathbb{R}} \phi(\lambda)\, dE(\lambda)$ that for $\phi \in \mathfrak{M}(\mathbb{R})$ and $-\infty < \alpha < \beta < \infty$,

$$\int_{[\alpha,\beta]} \phi(\lambda)\, dE(\lambda) = \int_{\mathbb{R}} \phi(\lambda)\, dE(\lambda)\{E(\beta) - E(\alpha)\},$$

and consequently for $\phi \in \mathfrak{M}(\mathbb{R})$, $\int_{[-a,a]} \phi(\lambda)\, dE(\lambda)$ tends to $\int_{\mathbb{R}} \phi(\lambda)\, dE(\lambda)$ in the strong operator topology as $a \to \infty$.

(1.3) THEOREM. *Let* X, R, $\{P_n\}_{n=-\infty}^{\infty}$, *and* c *be as in Theorem* C *above. For* $n \in \mathbb{Z}$, *let* $Q_n = \sum_{k \in \Lambda_n} P_k$. *Then the following assertions are valid.*

(i) *For each sequence* $\{\varepsilon_n\}_{n=-\infty}^{\infty}$ *such that* $\varepsilon_n = \pm 1$ *for all* $n \in \mathbb{Z}$, *the partial sums* $\sum_{n=-N}^{N} \varepsilon_n Q_n$ *of the series* $\sum_{n=-\infty}^{\infty} \varepsilon_n Q_n$ *converge in the strong operator topology, as* $N \to \infty$, *and* $\|\sum_{n=-\infty}^{\infty} \varepsilon_n Q_n\| \le c^2 C_X$. *Furthermore*, $I = \sum_{n=-\infty}^{\infty} Q_n$.

(ii) *For each* $\phi \in \mathfrak{M}(\mathbb{Z})$, *the series* $\sum_{n=1}^{\infty} \phi(-n) P_{-n}$ *and the series* $\sum_{n=0}^{\infty} \phi(n) P_n$ *converge in the strong operator topology, and the mapping* $\phi \mapsto \sum_{n=-\infty}^{\infty} \phi(n) P_n$ *is an identity-preserving algebra homomorphism of* $\mathfrak{M}(\mathbb{Z})$ *into* $\mathfrak{B}(X)$ *satisfying*

$$\Big\| \sum_{n=-\infty}^{\infty} \phi(n) P_n \Big\| \le c^2 C_X \|\phi\|_{\mathfrak{M}(\mathbb{Z})}. \tag{1.4}$$

Part (i) in each of Theorems (1.1), (1.2), and (1.3) can be viewed as an abstract analogue of the Littlewood–Paley Theorem for $L^p(G)$, where $1 < p < \infty$, with $G = \mathbb{Z}$ in Theorem (1.1), $G = \mathbb{R}$ in Theorem (1.2), and $G = \mathbb{T}$ in Theorem (1.3). To be more precise, consider first the result of Theorem (1.3) in the context of the regular representation of $\mathbb{T}$ in $X = L^p(\mathbb{T})$, where $1 < p < \infty$. In this situation, for each $f \in X$, $Q_n f$ is the sum of the terms in the Fourier series of f corresponding to the dyadic block Λ_n in $\mathbb{Z}$. The usual version of the Littlewood–Paley Theorem for $L^p(\mathbb{T})$ ([14, §7.2]) gives the existence of a positive constant α_p, depending on p, such that

$$\alpha_p^{-1} \|f\|_p \le \Big\| \Big\{ \sum_{n=-\infty}^{\infty} |Q_n f|^2 \Big\}^{1/2} \Big\|_p \le \alpha_p \|f\|_p, \tag{1.5}$$

for all $f \in L^p(\mathbb{T})$. However, by making use of Khinchin's Inequality [17, Theorem 2.b.3], it is readily seen that the validity of (1.5) for all $f \in L^p(\mathbb{T})$ is equivalent to the unconditional convergence of the series $\sum_{n=-\infty}^{\infty} Q_n f$ to f in the $L^p(\mathbb{T})$-norm for all $f \in L^p(\mathbb{T})$—a fact which (1.3)(i) asserts for this context.

Similar remarks apply to Theorem (1.1) when U is taken to be the left shift on $X = L^p(\mathbb{Z})$, with $1 < p < \infty$. In this context, $\mathcal{E}(\Gamma_n)$ is the multiplier transform on $L^p(\mathbb{Z})$ corresponding to the characteristic function $\chi_{\Gamma_n} : \mathbb{T} \to \mathbb{C}$, the projections $\mathcal{E}(\{\omega_n\})$ and $\mathcal{E}(\{1\})$ are both 0, and the result of Theorem (1.1)(i) provides, in particular, the unconditional convergence of the series $\sum_{n=-\infty}^{\infty} \mathcal{E}(\Gamma_n) f$ to f in $L^p(\mathbb{Z})$-norm for each $f \in L^p(\mathbb{Z})$. Again, by use of Khinchin's Inequality, this amounts to the Littlewood–Paley inequalities

$$\beta_p^{-1}\|f\|_p \le \Big\|\Big\{\sum_{n=-\infty}^{\infty} |\mathcal{E}(\Gamma_n)f|^2\Big\}^{1/2}\Big\|_p \le \beta_p\|f\|_p,$$

for every $f \in L^p(\mathbb{Z})$, where β_p is a positive constant depending only on p. In much the same way, when it is applied to the regular representation of $\mathbb{R}$ defined by the translations on $L^p(\mathbb{R})$, where $1 < p < \infty$, Theorem (1.2)(i) reduces to the classical Littlewood–Paley Theorem for $L^p(\mathbb{R})$.

Similarly, the second parts of Theorems (1.1), (1.2), and (1.3) are closely related to the Strong Marcinkiewicz Multiplier Theorem for the groups $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{T}$, respectively. Indeed, when Theorem (1.3) is applied to the regular representation of $\mathbb{T}$ in $L^p(\mathbb{T})$, $1 < p < \infty$, the strong convergence of $\sum_{n=-\infty}^{\infty}\phi(n)P_n$ in $\mathfrak{B}(L^p(\mathbb{T}))$ for $\phi \in \mathfrak{M}(\mathbb{Z})$ and the inequality in (1.4) amount to the Strong Marcinkiewicz Multiplier Theorem for $\mathbb{T}$ ([14, Theorem 8.2.1]). Likewise, when Theorem (1.1)(ii) is specialized to the left shift U on $X = L^p(\mathbb{Z})$, $1 < p < \infty$, as above, the existence of $\int_{[0,2\pi]}\phi(e^{i\lambda})\,dE(\lambda)$ on $L^p(\mathbb{Z})$ for $\phi \in \mathfrak{M}(\mathbb{T})$ shows that ϕ is a p-multiplier with corresponding multiplier transform $\int_{[0,2\pi]}^{\oplus}\phi(e^{i\lambda})\,dE(\lambda)$. This, together with the estimate in (1.1)(ii), provides the Strong Marcinkiewicz Multiplier Theorem for $\mathbb{Z}$. Similar remarks include the Strong Marcinkiewicz Multiplier Theorem for $\mathbb{R}$ under the result of Theorem (1.2)(ii) taken in the context of the regular representation of $\mathbb{R}$ in $L^p(\mathbb{R})$, $1 < p < \infty$.

Theorem (1.3) also specializes to include results established more recently by J. Bourgain [11] which will play an important part in the proofs of our present results. Let $\mathcal{Y}$ be a UMD space, suppose that $1 < p < \infty$, and let $X = L^p(\mathbb{T}, \mathcal{Y})$, the space of $\mathcal{Y}$-valued functions on $\mathbb{T}$, p-integrable in Bochner's sense. Then X is itself a UMD space. When R in Theorem (1.3) is specialized to be the regular representation of $\mathbb{T}$ in $L^p(\mathbb{T}, \mathcal{Y})$, the projection P_n, for $n \in \mathbb{Z}$, acts, as in the scalar case, by taking each function in $L^p(\mathbb{T}, \mathcal{Y})$ to the nth term of its Fourier series. The result of Theorem (1.3)(i) in this case reduces to [11, Theorem 3], while Theorem (1.3)(ii) gives [11, Theorem 4].

Finally, it should be mentioned that versions of Theorem (1.1) and Theorem (1.2)(i) have already been considered when X is a subspace of $L^p(\mu)$, where $1 < p < \infty$ and μ is an arbitrary measure ([1]–[3]). A number of new techniques are required when working in the Banach space setting we shall treat here.

2. Background results. We now outline several general results which will be used to establish the main results described above. Firstly, we shall make substantial use of averages over finite Cantor groups, such averages playing a role similar to that of square functions in classical Littlewood–Paley theory. To be more precise, let N belong to the set $\mathbb{N}$ of positive

integers, let $d\varepsilon$ denote Haar measure of total mass 1 on the product D^N of N copies of the multiplicative group $D = \{-1, 1\}$, and let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$ be a typical element of D^N. We shall need the standard results concerning averages listed in the following two theorems, the first of these being the statement of the Khinchin–Kahane Inequality.

(2.1) THEOREM ([18, Theorem 1.e.13]). *Suppose that $1 < p < \infty$. There is a constant A_p depending only on p such that for any $N \in \mathbb{N}$ and any elements $x_1, \ldots, x_N$ in an arbitrary Banach space, we have*

$$\int_{D^N}\Big\|\sum_{n=1}^{N}\varepsilon_n x_n\Big\|\,d\varepsilon \le \Big\{\int_{D^N}\Big\|\sum_{n=1}^{N}\varepsilon_n x_n\Big\|^p d\varepsilon\Big\}^{1/p} \le A_p\int_{D^N}\Big\|\sum_{n=1}^{N}\varepsilon_n x_n\Big\|\,d\varepsilon.$$

(2.2) THEOREM. *Suppose that $\mathfrak{X}$ is an arbitrary complex Banach space. The following assertions hold.*

(i) *Let $N \in \mathbb{N}$, let $x_1, \ldots, x_N \in \mathfrak{X}$, let $a_1, \ldots, a_N \in \mathbb{C}$, and suppose that $1 \le p < \infty$. Then*

$$\Big\{\int_{D^N}\Big\|\sum_{n=1}^{N}\varepsilon_n a_n x_n\Big\|^p d\varepsilon\Big\}^{1/p} \le 2\|a\|_\infty\Big\{\int_{D^N}\Big\|\sum_{n=1}^{N}\varepsilon_n x_n\Big\|^p d\varepsilon\Big\}^{1/p},$$

where $\|a\|_\infty = \sup\{|a_n| : 1 \le n \le N\}$. If $\{a_n\}_{n=1}^N \subseteq \mathbb{R}$, then the constant 2 can be deleted from the majorant in this inequality.

(ii) *Let Ω be a collection of bounded linear operators mapping $\mathfrak{X}$ into $\mathfrak{X}$, with the property that, for some p such that $1 \le p < \infty$ and for some constant $K_{p,\Omega}$,*

$$\Big\{\int_{D^N}\Big\|\sum_{n=1}^{N}\varepsilon_n T_n x_n\Big\|^p d\varepsilon\Big\}^{1/p} \le K_{p,\Omega}\Big\{\int_{D^N}\Big\|\sum_{n=1}^{N}\varepsilon_n x_n\Big\|^p d\varepsilon\Big\}^{1/p} \tag{2.3}$$

for all $N \in \mathbb{N}$, all $T_1, \ldots, T_N \in \Omega$, and all $x_1, \ldots, x_N \in \mathfrak{X}$. Then

$$\Big\{\int_{D^N}\Big\|\sum_{n=1}^{N}\varepsilon_n W_n x_n\Big\|^p d\varepsilon\Big\}^{1/p} \le 2K_{p,\Omega}\Big\{\int_{D^N}\Big\|\sum_{n=1}^{N}\varepsilon_n x_n\Big\|^p d\varepsilon\Big\}^{1/p}$$

for all $N \in \mathbb{N}$, all $x_1, \ldots, x_N \in \mathfrak{X}$, and all $W_1, \ldots, W_N$ in $\mathrm{aco}(\Omega)$, *where* $\mathrm{aco}(\Omega)$ *denotes the absolutely convex hull of Ω, defined by*

$$\mathrm{aco}(\Omega) = \Big\{\sum_{k=1}^{r}\alpha_k T_k : r \in \mathbb{N},\ \{\alpha_k\}_{k=1}^r \subseteq \mathbb{C},\ \{T_k\}_{k=1}^r \subseteq \Omega,\ \sum_{k=1}^{r}|\alpha_k| \le 1\Big\}.$$

Comments. (a) The result of Theorem (2.1), which is due to J.-P. Kahane [16], shows that the existence of an inequality of the form (2.3) for some p in the range $1 \le p < \infty$ implies a similar inequality for every p in this range.

(b) Theorem (2.2)(i) is also due to J.-P. Kahane (see [19, pp. 45–46]), while Theorem (2.2)(ii) is proved by a straightforward argument involving rational convex combinations of the elements of Ω (as observed in [11, proof of Lemma 7]).

Motivated by the hypothesis in Theorem (2.2)(ii), we make the following definition.

(2.4) DEFINITION. A collection Ω of bounded linear operators mapping a Banach space $\mathfrak{X}$ into $\mathfrak{X}$ is said to have the *R-property* if for some (and hence every) p in the range $1 \le p < \infty$ there is a corresponding constant $K_{p,\Omega}$ such that (2.3) holds for all $N \in \mathbb{N}$, all $T_1, \ldots, T_N \in \Omega$, and all $x_1, \ldots, x_N \in \mathfrak{X}$. (This is an operator-theoretic analogue of the Riesz Property discussed in [14, 1.2.12].)

The second technique on which we shall rely is the following vector-valued version of the Coifman–Weiss Transference Theorem.

(2.5) THEOREM ([9, Theorem (2.8)]) *Let $u \mapsto \Theta_u$ be a strongly continuous representation of a locally compact abelian group G in a Banach space $\mathfrak{X}$ such that $c \equiv \sup\{\|\Theta_u\| : u \in G\} < \infty$. Let $k \in L^1(G)$, and let H_k denote the operator defined on $\mathfrak{X}$ (with the aid of Bochner integration with respect to Haar measure du on G) by*

$$H_k x = \int_G k(u)\Theta_{-u} x\, du \quad \text{for all } x \in \mathfrak{X}.$$

Then for $1 \le p < \infty$, $\|H_k\| \le c^2 N_{p,\mathfrak{X}}(k)$, where $N_{p,\mathfrak{X}}(k)$ denotes the norm of convolution by k on $L^p(G, \mathfrak{X})$.

We shall also use the following extension of Theorem (2.5), the proof of which is a simple modification of the proof given in [9] and cited above.

(2.6) THEOREM. *Suppose $1 \le p < \infty$. Let G, $\mathfrak{X}$, Θ, and c be as in Theorem (2.5). Let $N \in \mathbb{N}$, and $k_1, \ldots, k_N \in L^1(G)$. Suppose that K is a non-negative constant such that*

$$\Big\{\int_{D^N} \Big\|\sum_{n=1}^N \varepsilon_n k_n * f_n\Big\|^p_{L^p(G,\mathfrak{X})} d\varepsilon\Big\}^{1/p} \le K\Big\{\int_{D^N} \Big\|\sum_{n=1}^N \varepsilon_n f_n\Big\|^p_{L^p(G,\mathfrak{X})} d\varepsilon\Big\}^{1/p}$$

for all $f_1, \ldots, f_N \in L^p(G, \mathfrak{X})$. Then

$$\Big\{\int_{D^N} \Big\|\sum_{n=1}^N \varepsilon_n H_{k_n} x_n\Big\|^p_{\mathfrak{X}} d\varepsilon\Big\}^{1/p} \le c^2 K\Big\{\int_{D^N} \Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|^p_{\mathfrak{X}} d\varepsilon\Big\}^{1/p}$$

for all $x_1, \ldots, x_N \in \mathfrak{X}$.

For convenience, we record here the following result concerning convolution norms in the vector-valued setting.

(2.7) LEMMA. *Let $(\mathcal{N}, \mu)$ be any measure space containing a set of finite positive measure. Let G be a locally compact abelian group, let $k \in L^1(G)$, and suppose that $1 \le p < \infty$. Then for any Banach space $\mathfrak{X}$,*

$$N_{p,\mathfrak{X}}(k) = N_{p,Y}(k),$$

where $Y = L^p(\mu, \mathfrak{X})$.

Proof. Let σ_0 be a subset of $\mathcal{N}$ such that $0 < \mu(\sigma_0) < \infty$, and denote by λ_{σ_0} the characteristic function of σ_0. Suppose that f is a Haar-integrable $\mathfrak{X}$-valued simple function on G. Define $\widetilde{f} \in L^p(G, Y)$ by putting $\widetilde{f}(s) = \lambda_{\sigma_0} f(s) \in Y$ for all $s \in G$. In fact, it is easy to see by direct calculation that

$$\|\widetilde{f}\|_{L^p(G,Y)} = \mu(\sigma_0)^{1/p}\|f\|_{L^p(G,\mathfrak{X})}.$$

Easy direct calculations also show that

$$\|k * \widetilde{f}\|_{L^p(G,Y)} = \mu(\sigma_0)^{1/p}\|k * f\|_{L^p(G,\mathfrak{X})}.$$

The inequality $N_{p,\mathfrak{X}}(k) \le N_{p,Y}(k)$ follows immediately. The reverse inequality is readily established by making obvious modifications in the method of proof for [6, Lemma (4.2)]. ■

3. Harmonic analysis on UMD spaces. Let G be a locally compact abelian group with dual group $\widehat{G}$, and let $\mathfrak{X}$ be a Banach space. Given a linear space $\mathcal{L}$ of scalar-valued functions on G (or of equivalence classes of Haar-measurable scalar-valued functions modulo equality almost everywhere on G), a typical element $f_1 \otimes x_1 + \ldots + f_n \otimes x_n$ of the algebraic tensor product $\mathcal{L} \otimes \mathfrak{X}$ may be considered as the function $f : G \to \mathfrak{X}$ defined pointwise (or pointwise almost everywhere) on G by

$$f(u) = \sum_{k=1}^n f_k(u) x_k.$$

Let $\mathfrak{S}_0(G, \mathfrak{X})$ denote the space $[L^1(G) \cap L^\infty(G)] \otimes \mathfrak{X}$. Notice that for $1 \le p < \infty$, $\mathfrak{S}_0(G, \mathfrak{X})$ is dense in $L^p(G, \mathfrak{X})$. Suppose that ϕ is a complex-valued, bounded, measurable function on $\widehat{G}$, and denote the corresponding multiplier transform on $L^2(G)$ by S_ϕ. Thus $S_\phi F \equiv (\phi \widehat{F})^\vee$. For $f = \sum_{k=1}^n f_k \otimes x_k \in \mathfrak{S}_0(G, \mathfrak{X})$, let $T_\phi f \in L^2(G, \mathfrak{X})$ be given by

$$(T_\phi f)(u) = \sum_{k=1}^n ((S_\phi f_k)(u)) x_k.$$

It is easy to show that T_ϕ is a well-defined linear mapping of $\mathfrak{S}_0(G, \mathfrak{X})$ into $L^2(G, \mathfrak{X})$. For $1 \le p < \infty$, let $M_{p,\mathfrak{X}}(\widehat{G})$ be the space of all bounded measurable $\phi : \widehat{G} \to \mathbb{C}$ such that T_ϕ extends from $\mathfrak{S}_0(G, \mathfrak{X})$ to a bounded linear mapping of $L^p(G, \mathfrak{X})$ into $L^p(G, \mathfrak{X})$. In this case the continuous extension to

$L^p(G,\mathfrak{X})$ is unique, and will also be denoted by T_ϕ, and we define $\|\phi\|_{M_{p,\mathfrak{X}}(\widehat{G})}$ to be the norm of T_ϕ on $L^p(G,\mathfrak{X})$. The usual space of Fourier multipliers for $L^p(G)$ will be denoted by $M_p(\widehat{G})$, and the norm in $M_p(\widehat{G})$ by $\|\cdot\|_{M_p(\widehat{G})}$. Thus, $M_{p,\mathbb{C}}(\widehat{G})$ and $\|\cdot\|_{M_{p,\mathbb{C}}(\widehat{G})}$ in the foregoing notation coincide with $M_p(\widehat{G})$ and $\|\cdot\|_{M_p(\widehat{G})}$. It is apparent that if $\mathfrak{X}\neq\{0\}$, then $M_{p,\mathfrak{X}}(\widehat{G})\subseteq M_p(\widehat{G})$, with $\|\phi\|_{M_p(\widehat{G})}\le\|\phi\|_{M_{p,\mathfrak{X}}(\widehat{G})}$ for all $\phi\in M_{p,\mathfrak{X}}(\widehat{G})$. However, $M_{p,\mathfrak{X}}(\widehat{G})$ and $M_p(\widehat{G})$ need not coincide—in particular, if $\mathfrak{X}$ is not a UMD space, then for $1<p<\infty$, the signum function on $\mathbb{R}$ belongs to $M_p(\mathbb{R})$ but not to $M_{p,\mathfrak{X}}(\mathbb{R})$. Notice that if $k\in L^1(G)$, then $\widehat{k}\in M_{p,\mathfrak{X}}(\widehat{G})$ for $1\le p<\infty$, the operator $T_{\widehat{k}}$ being convolution by k on $L^p(G,\mathfrak{X})$ (consequently $\|\widehat{k}\|_{M_{p,\mathfrak{X}}(\widehat{G})}=N_{p,\mathfrak{X}}(k)$). As in the scalar case, $M_{p,\mathfrak{X}}(\widehat{G})$ is an algebra under pointwise operations, and the mapping $\phi\mapsto T_\phi$ is an identity-preserving algebra homomorphism of $M_{p,\mathfrak{X}}(\widehat{G})$ into $\mathfrak{B}(L^p(G,\mathfrak{X}))$. Moreover, if $\mathfrak{X}\neq\{0\}$, then after we identify elements of $M_{p,\mathfrak{X}}(\widehat{G})$ modulo equality locally almost everywhere, $M_{p,\mathfrak{X}}(\widehat{G})$ becomes a commutative unital Banach algebra under $\|\cdot\|_{M_{p,\mathfrak{X}}(\widehat{G})}$.

For later use, it will be convenient to record here the following two propositions which draw further parallels between $M_{p,\mathfrak{X}}(\widehat{G})$ and $M_p(\widehat{G})$. The proof of the first proposition is straightforward, and its Bochner integral formula readily provides the second proposition. For a bounded function $\psi:\widehat{G}\to\mathbb{C}$ and $y\in\widehat{G}$, we shall denote by ψ_y the corresponding translated function defined on $\widehat{G}$ by $\psi_y(x)=\psi(x+y)$.

PROPOSITION A. *Let G be a locally compact abelian group with dual group $\widehat{G}$, and let τ be a Haar measure in $\widehat{G}$. Suppose that $\mathfrak{X}$ is a Banach space, and $1\le p<\infty$. If $k\in L^1(\widehat{G})$ and $\psi\in M_{p,\mathfrak{X}}(\widehat{G})$, then $k*\psi\in M_{p,\mathfrak{X}}(\widehat{G})$, and*

$$\|k*\psi\|_{M_{p,\mathfrak{X}}(\widehat{G})}\le\|k\|_{L^1(\widehat{G})}\|\psi\|_{M_{p,\mathfrak{X}}(\widehat{G})}.$$

The multiplier transform $T_{k*\psi}$ on $L^p(G,\mathfrak{X})$ corresponding to $k*\psi$ is given by

$$T_{k*\psi}f=\int_{\widehat{G}}k(y)T_{\psi_{-y}}f\,d\tau(y)\quad\textit{for all } f\in L^p(G,\mathfrak{X}),$$

where the integral on the right is an $L^p(G,\mathfrak{X})$-valued Bochner integral.

PROPOSITION B. *Under the hypotheses of Proposition A, suppose that $N\in\mathbb{N}$, $\{\psi_j\}_{j=1}^N\subseteq M_{p,\mathfrak{X}}(\widehat{G})$, and $1\le r<\infty$. Let A be a constant such that*

$$\Big\{\int_{D^N}\Big\|\sum_{j=1}^N\varepsilon_jT_{\psi_j}f_j\Big\|_p^r\,d\varepsilon\Big\}^{1/r}\le A\Big\{\int_{D^N}\Big\|\sum_{j=1}^N\varepsilon_jf_j\Big\|_p^r\,d\varepsilon\Big\}^{1/r}$$

for all $\{f_j\}_{j=1}^N\subseteq L^p(G,\mathfrak{X})$. Then

$$\Big\{\int_{D^N}\Big\|\sum_{j=1}^N\varepsilon_jT_{k*\psi_j}f_j\Big\|_p^r\,d\varepsilon\Big\}^{1/r}\le A\|k\|_{L^1(\widehat{G})}\Big\{\int_{D^N}\Big\|\sum_{j=1}^N\varepsilon_jf_j\Big\|_p^r\,d\varepsilon\Big\}^{1/r}$$

for all $\{f_j\}_{j=1}^N\subseteq L^p(G,\mathfrak{X})$.

We now pass to the setting of an arbitrary UMD space X. The characterization of the UMD property for X in terms of the boundedness of the Hilbert transform on $L^p(G,X)$, where $1<p<\infty$, and $G=\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{T}$, together with a routine argument involving translations, gives, as in the classical case when $X=\mathbb{C}$, the following scholium. For brevity, an arc on the circle $\mathbb{T}$ will be referred to as an interval.

(3.1) SCHOLIUM. *Suppose that $1<p<\infty$. Let $G=\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{T}$, and let X be a UMD space. Then for each interval J in G, the characteristic function $\chi_J\in M_{p,X}(G)$. Furthermore,*

$$\|\chi_J\|_{M_{p,X}(G)}\le C_{p,X}. \tag{3.2}$$

Remark. In the setting of Scholium (3.1), if J is compact, then T_{χ_J} is convolution on $L^p(\widehat{G},X)$ by the inverse Fourier transform $(\chi_J)^\vee$.

The classical multiplier results of Stechkin for $M_p(\mathbb{Z})$, $M_p(\mathbb{R})$, and $M_p(\mathbb{T})$, $1<p<\infty$ ([14, Theorems 6.2.5, 6.3.5, and 6.4.4]), also extend to the UMD setting by adapting their classical proofs. We record this fact in the next theorem. Recall that for $G=\mathbb{Z}$ (respectively, $G=\mathbb{R}$, $G=\mathbb{T}$), $\mathrm{BV}(G)$ consists of all complex-valued functions f on G such that $\mathrm{var}(f,\mathbb{Z})\equiv\sum_{n=-\infty}^{\infty}|f(n)-f(n-1)|<\infty$ (respectively, $\mathrm{var}(f,\mathbb{R})\equiv\sup_{N\in\mathbb{N}}\mathrm{var}(f,[-N,N])<\infty$, $\mathrm{var}(f,\mathbb{T})\equiv\mathrm{var}(f(e^{i(\cdot)}),[0,2\pi])<\infty$). $\mathrm{BV}(G)$ is a unital Banach algebra under pointwise operations and the norm $\|\cdot\|_{\mathrm{BV}(G)}$ given, respectively, by

$$\|f\|_{\mathrm{BV}(\mathbb{Z})}=|\lim_{n\to\infty}f(n)|+\mathrm{var}(f,\mathbb{Z});$$
$$\|f\|_{\mathrm{BV}(\mathbb{R})}=|\lim_{t\to\infty}f(t)|+\mathrm{var}(f,\mathbb{R});$$
$$\|f\|_{\mathrm{BV}(\mathbb{T})}=|f(1)|+\mathrm{var}(f,\mathbb{T}).$$

(3.3) THEOREM. *Assume the hypotheses of Scholium (3.1). Then $\mathrm{BV}(G)\subseteq M_{p,X}(G)$. Furthermore,*

$$\|\phi\|_{M_{p,X}(G)}\le C_{p,X}\|\phi\|_{\mathrm{BV}(G)}\quad\textit{for all } \phi\in\mathrm{BV}(G). \tag{3.4}$$

Proof. For $G=\mathbb{T}$ or $G=\mathbb{R}$, the respective methods of proof in parts (i) and (ii) of [6, Theorem (3.4)] show the desired conclusions directly from the boundedness of the Hilbert transform on $L^p(\widehat{G},X)$. The situation here when $G=\mathbb{R}$ requires slight elaboration, since the method of proof for [6, Theorem (3.4)(ii)] actually shows that when $\phi\in\mathrm{BV}(\mathbb{R})\cap L^1(\mathbb{R})$, convolution by $\phi^\vee$ is a bounded linear mapping of $L^p(\mathbb{R},X)$ into $L^p(\mathbb{R},X)$ with norm not exceeding $C_{p,X}\|\phi\|_{\mathrm{BV}(\mathbb{R})}$. Theorem (3.3) for arbitrary $\phi \in \mathrm{BV}(\mathbb{R})$ follows readily by applying this special case to the functions $\widehat{\kappa}_n\phi$, $n \in \mathbb{N}$, where κ_n is the Fejér kernel of order n for $\mathbb{R}$:

$$\kappa_n(t) \equiv \frac{n}{2\pi}\left(\frac{\sin(2^{-1}nt)}{2^{-1}nt}\right)^2.$$

In order to complete the proof of Theorem (3.3) it remains to establish the desired conclusions when $G = \mathbb{Z}$, and we now suppose that $\phi \in \mathrm{BV}(\mathbb{Z})$. For $n \in \mathbb{Z}$ let $\mathfrak{e}_n : \mathbb{T} \to \mathbb{C}$ be defined by $\mathfrak{e}_n(z) \equiv z^n$, and let $\mathcal{Q}(X)$ be the class of all X-valued trigonometric polynomials defined on $\mathbb{T}$. For $f = \sum_{j=-N}^{N} \mathfrak{e}_j \otimes x_j \in \mathcal{Q}(X)$, we have $T_\phi f = \sum_{j=-N}^{N} \phi(j)\mathfrak{e}_j \otimes x_j$. For $-N \le n \le N$, put $\mathfrak{s}_n = \sum_{j=-N}^{n} \mathfrak{e}_j \otimes x_j$, and let λ_n be the characteristic function, defined on $\mathbb{Z}$, of $\{j \in \mathbb{Z} : -N \le j \le n\}$. Thus, $T_{\lambda_n} f = \mathfrak{s}_n$, and so by Scholium (3.1),

$$\|\mathfrak{s}_n\|_p \le C_{p,X}\|f\|_p \quad \text{for each } n.$$

A summation by parts gives

$$T_\phi f = \sum_{j=-N}^{N-1} (\phi(j) - \phi(j+1))\mathfrak{s}_j + \phi(N)\mathfrak{s}_N,$$

and we can now infer that $\|T_\phi f\|_p \le C_{p,X}\|f\|_p\|\phi\|_{\mathrm{BV}(\mathbb{Z})}$. The desired conclusions for the case when $G = \mathbb{Z}$ are now readily obtained, and the proof of Theorem (3.3) is complete. ∎

The result of Scholium (3.1) can be strengthened as stated in Lemma (3.5) below. This was established for $G = \mathbb{Z}$ as the first step in the proof of [11, Lemma 7], but the argument there adapts to the cases $G = \mathbb{R}$ and $\mathbb{T}$ with the aid of Theorem (2.1) and Theorem (2.2)(i) (for each group the reasoning reduces to the boundedness of an appropriate "Riesz projection," in analogy with the scalar-valued treatment in [14, pp. 107, 112, 115–116]).

(3.5) LEMMA. *Suppose that $1 < p < \infty$. Let $G = \mathbb{Z}$, $\mathbb{R}$, or $\mathbb{T}$, and let X be a UMD space. Then the set $\{T_{\chi_J} : J$ is an interval in $G\}$ of bounded operators on $L^p(\widehat{G}, X)$ has the R-property.*

Fix $X \in UMD$ and $p \in (1, \infty)$. We now consider the following analogue of the Littlewood–Paley property for the UMD space $L^p(G, X)$, where $G = \mathbb{T}$, $\mathbb{R}$, or $\mathbb{Z}$. For $n \in \mathbb{Z}$, let S_n (respectively, $\widetilde{S}_n$, V_n) denote the multiplier transform on $L^p(\mathbb{T}, X)$ (respectively, $L^p(\mathbb{R}, X)$, $L^p(\mathbb{Z}, X)$) associated with the characteristic function of the dyadic interval Λ_n (respectively, $\widetilde{\Lambda}_n$, Γ_n) considered as an element of $M_{p,X}(\mathbb{Z})$ (respectively, $M_{p,X}(\mathbb{R})$, $M_{p,X}(\mathbb{T})$).

(3.6) THEOREM. *Let X be a UMD space, and suppose that $1 < p < \infty$. The following assertions hold.*

(i) *Given $f \in L^p(\mathbb{T}, X)$, the series $\sum_{n=-\infty}^{\infty} S_n f$ converges unconditionally in $L^p(\mathbb{T}, X)$ to f, and*

$$C_{p,X}^{-1}\|f\|_p \le \Big\|\sum_{n=-\infty}^{\infty} \varepsilon_n S_n f\Big\|_p \le C_{p,X}\|f\|_p$$

for all $f \in L^p(\mathbb{T}, X)$ and for all choices of $\varepsilon_n = \pm 1$.

(ii) *Given $f \in L^p(\mathbb{R}, X)$, the series $\sum_{n=-\infty}^{\infty} \widetilde{S}_n f$ converges unconditionally in $L^p(\mathbb{R}, X)$ to f, and*

$$C_{p,X}^{-1}\|f\|_p \le \Big\|\sum_{n=-\infty}^{\infty} \varepsilon_n \widetilde{S}_n f\Big\|_p \le C_{p,X}\|f\|_p$$

for all $f \in L^p(\mathbb{R}, X)$ and for all choices of $\varepsilon_n = \pm 1$.

(iii) *Given $f \in L^p(\mathbb{Z}, X)$ the series $\sum_{n=-\infty}^{\infty} V_n f$ converges unconditionally in $L^p(\mathbb{Z}, X)$ to f, and*

$$C_{p,X}^{-1}\|f\|_p \le \Big\|\sum_{n=-\infty}^{\infty} \varepsilon_n V_n f\Big\|_p \le C_{p,X}\|f\|_p$$

for all $f \in L^p(\mathbb{Z}, X)$ and for all choices of $\varepsilon_n = \pm 1$.

The conclusions in (3.6)(i) were established by J. Bourgain in [11, Theorem 3], where it is indicated that a similar approach may be used in other situations. Rather than pursue this course in detail, we shall deduce (3.6)(ii) and (3.6)(iii) from (3.6)(i) by using vector-valued transference. This approach will be facilitated by the following lemma concerning multiplier extensions, which adapts and generalizes the reasoning for [7, Theorem (2.1)] from the scalar to the UMD setting.

LEMMA. *Let X be a UMD space, and suppose that $1 < p < \infty$. Given $\phi : \mathbb{Z} \to \mathbb{C}$, let $\phi^\# : \mathbb{R} \to \mathbb{C}$ be defined pointwise by*

$$\phi^\# = \sum_{n=-\infty}^{\infty} \phi(n)\chi_n,$$

where χ_n is the characteristic function, defined on $\mathbb{R}$, of the interval $[n, n+1)$. Then if $\phi \in M_{p,X}(\mathbb{Z})$, we have $\phi^\# \in M_{p,X}(\mathbb{R})$, and

$$\|\phi^\#\|_{M_{p,X}(\mathbb{R})} \le C_{p,X}\|\phi\|_{M_{p,X}(\mathbb{Z})}.$$

Proof. Let $Y = L^p(\mathbb{R}, X)$, and for each $t \in \mathbb{R}$, let $U_t : Y \to Y$ be translation by t: $(U_t f)(s) = f(s+t)$. Thus $\{U_t : t \in \mathbb{R}\}$ is a strongly continuous one-parameter group of isometries on the UMD space Y. It follows, in particular, that $U_{2\pi}$ has an associated spectral family of projections $E(\cdot)$ in Y, as described in Theorem A (§1). Putting $Q = \int_{[0,2\pi]}^{\oplus} \lambda\, dE(\lambda)$, we have $U_{2\pi} = e^{iQ}$. By [5, Theorem (3.11) and Theorem (3.14)(i)], we see that for all $f \in Y$,

$$Qf = \pi f + i \lim_{N\to\infty} \sum_{0<|n|\le N} n^{-1} U_{2n\pi} f.$$

Notice that Q commutes with U_t for all $t \in \mathbb{R}$. For each $t \in \mathbb{R}$, let $[t]$ denote the largest integer not exceeding t. For $f \in Y$, such that $f = g \otimes x$, where $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $x \in X$, we infer from the last equation (also applied to $L^p(\mathbb{R})$) and [5, Proposition (4.23)(i)] that the function Φ defined on $\mathbb{R}$ by writing

$$\Phi(t) = 2\pi(t - [t])$$

belongs to $M_{p,X}(\mathbb{R})$ and its corresponding multiplier transform T_Φ coincides with Q on Y.

For each $t \in \mathbb{R}$, put $\mathbf{V}_t = U_t e^{-i(t/(2\pi))Q}$. Thus $\{\mathbf{V}_t : t \in \mathbb{R}\}$ is a strongly continuous one-parameter group of operators on Y such that $\mathbf{V}_{2\pi} = I$. It follows that we can unambiguously define a strongly continuous representation R of $\mathbb{T}$ in Y by writing $R_z = \mathbf{V}_t$ for $z = e^{it}$. It is straightforward to see from the foregoing considerations that for $z \in \mathbb{T}$, $z^{[\cdot]} \in M_{p,X}(\mathbb{R})$ and $R_z = T_{z^{[\cdot]}}$ on Y. Let $\{P_n\}_{n=-\infty}^{\infty}$ be the sequence of spectral projections associated with the representation R in accordance with Theorem C. We claim that for each $n \in \mathbb{Z}$, $P_n = T_{\chi_n}$ on Y. In order to establish this, notice first that, since the functions in $L^1(\mathbb{R})$ with compactly supported Fourier transforms are dense in $L^p(\mathbb{R})$, we can use the uniform boundedness assertion in Scholium (3.1) to infer that for every $f \in L^p(\mathbb{R}, X)$, $\sum_{n=-N}^{N} T_{\chi_n} f \to f$ in $L^p(\mathbb{R}, X)$, as $N \to \infty$. So it suffices for the claim to show that for $n \in \mathbb{Z}$, $m \in \mathbb{Z}$, we have $P_n T_{\chi_m} = T_{\chi_n} T_{\chi_m}$. The latter equality is readily obtained, however, since for $z \in \mathbb{T}$,

$$R_z T_{\chi_m} = T_{z^{[\cdot]}} T_{\chi_m} = T_{z^{[\cdot]}\chi_m} = z^m T_{\chi_m},$$

and hence for $f \in L^p(\mathbb{R}, X)$,

$$P_n T_{\chi_m} f = \int_{\mathbb{T}} z^{-n} R_z T_{\chi_m} f \, d\lambda(z) = \int_{\mathbb{T}} z^{-n} z^m T_{\chi_m} f \, d\lambda(z) = T_{\chi_n} T_{\chi_m} f.$$

To complete the proof of the lemma, it now clearly suffices to show that there is a constant $C_{p,X}$ such that if $\phi \in M_{p,X}(\mathbb{Z})$ and $N \in \mathbb{N}$, then

$$\Big\| \sum_{n=-N}^{N} \phi(n) P_n \Big\| \le C_{p,X} \|\phi\|_{M_{p,X}(\mathbb{Z})}.$$

This can be seen as follows. For $N \in \mathbb{N}$, let $\psi_N : \mathbb{T} \to \mathbb{C}$ be the trigonometric polynomial given by

$$\psi_N(z) = \sum_{n=-N}^{N} \phi(n) z^n.$$

Then for $f \in L^p(\mathbb{R}, X)$ we have

$$\sum_{n=-N}^{N} \phi(n) P_n f = \sum_{n=-N}^{N} \phi(n) \int_{\mathbb{T}} z^n R_{z^{-1}} f \, d\lambda(z) = \int_{\mathbb{T}} \psi_N(z) R_{z^{-1}} f \, d\lambda(z).$$

Hence by the vector-valued transference result in Theorem (2.5), we see that

$$\Big\| \sum_{n=-N}^{N} \phi(n) P_n \Big\| \le C_{p,X} N_{p,Y}(\psi_N).$$

However, by Lemma (2.7), $N_{p,X}(\psi_N) = N_{p,Y}(\psi_N)$, and consequently

$$\Big\| \sum_{n=-N}^{N} \phi(n) P_n \Big\| \le C_{p,X} N_{p,X}(\psi_N).$$

Moreover, after denoting by γ_N the characteristic function, defined on $\mathbb{Z}$, of $\{k \in \mathbb{Z} : -N \le k \le N\}$, we have

$$N_{p,X}(\psi_N) = \|\widehat{\psi}_N\|_{M_{p,X}(\mathbb{Z})} = \|\phi\gamma_N\|_{M_{p,X}(\mathbb{Z})} \le C_{p,X} \|\phi\|_{M_{p,X}(\mathbb{Z})},$$

the last step by virtue of Scholium (3.1). Combining the last two inequalities completes the proof of the lemma. ∎

Proof of Theorem (3.6). As noted earlier, (3.6)(i) has been established previously by J. Bourgain in [11, Theorem 3], and we shall demonstrate (3.6)(ii) and (3.6)(iii) with the aid of (3.6)(i) and vector-valued transference. For the proof of (3.6)(ii) notice first that standard considerations based on Scholium (3.1) give

$$\sum_{n=-N}^{N} \widetilde{S}_n f \to f \quad \text{in } L^p(\mathbb{R}, X), \text{ as } N \to \infty,$$

for all $f \in L^p(\mathbb{R}, X)$. Consequently, it suffices for (3.6)(ii) to show the existence of a constant $C_{p,X}$ such that the sequence of disjoint projections $\{\widetilde{S}_n\}_{n=-\infty}^{\infty}$ satisfies

$$\Big\| \sum_{n=-N}^{N} \varepsilon_n \widetilde{S}_n \Big\| \le C_{p,X}$$

for all $N \in \mathbb{N}$ and all $\varepsilon_n = \pm 1$, for $-N \le n \le N$. Since $\widetilde{\Lambda}_{-n} = (-1)\widetilde{\Lambda}_n$ for $n > 0$, we need only show that there is a constant $C_{p,X}$ such that the characteristic functions $\chi_{\widetilde{\Lambda}_n}$ of the dyadic intervals $\widetilde{\Lambda}_n$ in $\mathbb{R}$ satisfy

$$\Big\| \sum_{n=1}^{N} \varepsilon_n \chi_{\widetilde{\Lambda}_n} \Big\|_{M_{p,X}(\mathbb{R})} \le C_{p,X}$$

for all $N \in \mathbb{N}$ and all $\varepsilon_n = \pm 1$, for $1 \le n \le N$. Let $\phi \in M_{p,X}(\mathbb{Z})$ be defined by writing $\phi = \sum_{n=1}^{N} \varepsilon_n \gamma_{\Lambda_n}$, where γ_{Λ_n} denotes the characteristic function, defined on $\mathbb{Z}$, of Λ_n. By (3.6)(i), $\|\phi\|_{M_{p,X}(\mathbb{Z})} \le C_{p,X}$. Using the preceding lemma on multiplier extensions, we see that $\|\phi^{\#}\|_{M_{p,X}(\mathbb{R})} \le C_{p,X}$. Since it is evident that $\phi^{\#} = \sum_{n=1}^{N} \varepsilon_n \chi_{\tilde{\Lambda}_n}$, the proof of (3.6)(ii) is complete.

To prove (3.6)(iii) it suffices to obtain inequalities of the form

$$\Big\| \sum_{n=1}^{N} \varepsilon_n V_n \Big\| \le C_{p,X} \tag{3.7}$$

and

$$\Big\| \sum_{n=-N}^{-1} \varepsilon_n V_n \Big\| \le C_{p,X}, \tag{3.8}$$

valid for all $N \in \mathbb{N}$, and all choices of $\varepsilon_n = \pm 1$. However, since, for $n > 0$, the arc Γ_n is $(\Gamma_{-n})^{-1}$, it is clear that the general validity of (3.7) is equivalent to that of (3.8). So it suffices to establish (3.8).

For $0 \le \lambda \le 2\pi$, let $E(\lambda)$ denote the operator on $W \equiv L^p(\mathbb{Z}, X)$ associated with the characteristic function of the arc $\{e^{it} : 0 \le t \le \lambda\}$, considered as an element of $M_{p,X}(\mathbb{T})$, and define $E(\lambda) = 0$ for $-\infty < \lambda < 0$, $E(\lambda) = I$ for $2\pi < \lambda < \infty$. It is easy to see from Scholium (3.1) that $E(\cdot)$ is a spectral family in W, concentrated on $[0, 2\pi]$, and continuous on $\mathbb{R}$ in the strong operator topology of $\mathfrak{B}(W)$. Direct calculation with Fourier transforms shows that the operator $U = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE(\lambda)$ on W is the left shift $(Uf)(n) = f(n+1)$. Let $B = \int_{[0,2\pi]}^{\oplus} \lambda\, dE(\lambda)$.

Now fix $N \in \mathbb{N}$, and, for $t \in \mathbb{R}$, let

$$Q_t = \exp(i\pi^{-1} 2^{N+1} t B) = \int_{[0,2\pi]}^{\oplus} \exp(i\pi^{-1} 2^{N+1} t\lambda)\, dE(\lambda).$$

Then $\{Q_t : t \in \mathbb{R}\}$ is a one-parameter group in W, continuous in the uniform operator topology of $\mathfrak{B}(W)$, and $\sup\{\|Q_t\|_{\mathfrak{B}(W)} : t \in \mathbb{R}\}$ is a finite constant depending only on X and p. Indeed, since e^{iB} is the invertible isometry U on W, we have

$$\begin{aligned}\sup\{\|Q_t\|_{\mathfrak{B}(W)} : t \in \mathbb{R}\} &= \sup\{\|e^{itB}\|_{\mathfrak{B}(W)} : t \in \mathbb{R}\} \\ &= \sup\{\|e^{itB}\|_{\mathfrak{B}(W)} : 0 \le t \le 1\} < \infty.\end{aligned}$$

For $\mu \in \mathbb{R}$, let $F(\mu) = E(\pi 2^{-N-1}\mu)$. Then $F(\cdot)$ is a spectral family in W, concentrated on $[0, 2^{N+2}]$, continuous in the strong operator topology of $\mathfrak{B}(W)$, and satisfying

$$Q_t = \int_{[0,2^{N+2}]}^{\oplus} e^{it\mu}\, dF(\mu) \quad \text{for all } t \in \mathbb{R}. \tag{3.9}$$

For $m \in \mathbb{N}$, let κ_m be the Fejér kernel of order m for $\mathbb{R}$. If $f \in W$, $m \in \mathbb{N}$, and $1 \le n \le N$, we have from (3.9),

$$\begin{aligned}&\int_{\mathbb{R}} \widehat{\kappa}_m(t)(\chi_{\tilde{\Lambda}_n})^{\vee}(t) Q_{-t} f\, dt \\ &\qquad = \int_{[-m,m]} \widehat{\kappa}_m(t)(\chi_{\tilde{\Lambda}_n})^{\vee}(t) \Big[\int_{[0,2^{N+2}]}^{\oplus} e^{-it\mu}\, dF(\mu) f\Big]\, dt.\end{aligned}$$

Using an integration by parts in the inner integral on the right, and changing the order of integration, we see that

$$\int_{\mathbb{R}} \widehat{\kappa}_m(t)(\chi_{\tilde{\Lambda}_n})^{\vee}(t) Q_{-t} f\, dt = \int_{[0,2^{N+2}]}^{\oplus} (\kappa_m * \chi_{\tilde{\Lambda}_n})(\mu)\, dF(\mu) f. \tag{3.10}$$

Standard considerations show that

$$\operatorname{var}(\kappa_m * \chi_{\tilde{\Lambda}_n}, \mathbb{R}) \le \operatorname{var}(\chi_{\tilde{\Lambda}_n}, \mathbb{R}) = 2,$$

and that for each $\mu \in \mathbb{R}$,

$$\lim_{m\to\infty} (\kappa_m * \chi_{\tilde{\Lambda}_n})(\mu) = 2^{-1}[\lim_{\lambda\to\mu^+} \chi_{\tilde{\Lambda}_n}(\lambda) + \lim_{\lambda\to\mu^-} \chi_{\tilde{\Lambda}_n}(\lambda)].$$

Applying these two facts, together with the limit theorem for spectral integrals described in [5, Proposition (2.10)], to the right-hand side of (3.10), we see that as $m \to \infty$,

$$\int_{\mathbb{R}} \widehat{\kappa}_m(t)(\chi_{\tilde{\Lambda}_n})^{\vee}(t) Q_{-t} f\, dt \to \{F(s_{n+1}) - F(s_n)\} f,$$

in the norm topology of W. In view of the definitions of $F(\cdot)$ and $E(\cdot)$, this can be rewritten

$$\int_{\mathbb{R}} \widehat{\kappa}_m(t)(\chi_{\tilde{\Lambda}_n})^{\vee}(t) Q_{-t} f\, dt \to V_{n-N-1} f, \quad \text{in } W, \text{ as } m \to \infty. \tag{3.11}$$

Now let $\varepsilon_n = \pm 1$ for $1 \le n \le N$, and put $\Psi = \sum_{n=1}^{N} \varepsilon_n (\chi_{\tilde{\Lambda}_n})^{\vee}$ and $\Psi_m = \widehat{\kappa}_m \Psi \in L^1(\mathbb{R})$ for $m \in \mathbb{N}$. Applying the vector-valued transference result in Theorem (2.5) to Ψ_m and the uniformly bounded one-parameter group $\{Q_t : t \in \mathbb{R}\}$, and then invoking Lemma (2.7), we have, in the notation of Theorem (2.5),

$$\|H_{\Psi_m}\| \le C_{p,X} N_{p,X}(\Psi_m) \quad \text{for } m \in \mathbb{N}. \tag{3.12}$$

Next observe that $\widehat{\Psi}_m = \kappa_m * \sum_{n=1}^{N} \varepsilon_n \chi_{\tilde{\Lambda}_n}$, and so, with the aid of Proposition A (§3), we can infer that

$$N_{p,X}(\Psi_m) = \|\widehat{\Psi}_m\|_{M_{p,X}(\mathbb{R})} \le \Big\| \sum_{n=1}^{N} \varepsilon_n \chi_{\tilde{\Lambda}_n} \Big\|_{M_{p,X}(\mathbb{R})}.$$

It now follows by (3.6)(ii) that $N_{p,X}(\Psi_m) \le C_{p,X}$ for all $m \in \mathbb{N}$.

Applying this fact to (3.12), we get

$$\|H_{\Psi_m}\| \le C_{p,X} \quad \text{for all } m \in \mathbb{N}. \tag{3.13}$$

By (3.11), $H_{\Psi_m} \to \sum_{n=1}^{N} \varepsilon_n V_{n-N-1}$ in the strong operator topology of $\mathfrak{B}(W)$ as $m \to \infty$. This fact, together with (3.13), gives (3.8), and the proof of Theorem (3.6) is now complete. ■

4. The spectral integration of Marcinkiewicz multipliers. We now establish several operator-theoretic results which underpin the proofs of the main results described in §1.

(4.1) THEOREM. *Let $E(\cdot)$ be a spectral family of projections in a Banach space $\mathfrak{X}$ such that $\{E(\lambda) : \lambda \in \mathbb{R}\}$ has the R-property, and suppose that*

$$\sup \Big\| \sum_{n=N}^{M} \varepsilon_n \{E(s_{n+1}) - E(s_n)\} \Big\| < \infty,$$

where $\{s_n\}_{n=-\infty}^{\infty}$ is the sequence of dyadic points of $\mathbb{R}$, and the supremum is taken over all $N \in \mathbb{Z}$, $M \in \mathbb{Z}$ such that $N \le M$, and all choices of $\varepsilon_n = \pm 1$ for $N \le n \le M$. Then, given $\phi \in \mathfrak{M}(\mathbb{R})$, the integral $\int_{\mathbb{R}} \phi(\lambda)\, dE(\lambda)$ exists in the strong operator topology (in the sense indicated immediately before the statement of Theorem (1.2)), and satisfies

$$\Big\| \int_{\mathbb{R}} \phi(\lambda)\, dE(\lambda) \Big\| \le C_E \|\phi\|_{\mathfrak{M}(\mathbb{R})}. \tag{4.2}$$

The proof of Theorem (4.1) will be facilitated by the following lemma.

(4.3) LEMMA. *Suppose that $E(\cdot)$ satisfies the hypotheses of Theorem (4.1), and that $\phi \in \mathfrak{M}(\mathbb{R})$. Let $\mathcal{P} = (\lambda_0, \lambda_1, \ldots, \lambda_r)$, where $-\infty < \lambda_0 < \lambda_1 < \ldots < \lambda_r < \infty$, and let*

$$\mathcal{S}(\mathcal{P}; \phi, E) = \sum_{k=1}^{r} \phi(\lambda_k)\{E(\lambda_k) - E(\lambda_{k-1})\}$$

be the Riemann–Stieltjes sum corresponding to $\mathcal{P}$, ϕ, and $E(\cdot)$. Then

$$\|\mathcal{S}(\mathcal{P}; \phi, E)\| \le C_E \|\phi\|_{\mathfrak{M}(\mathbb{R})}.$$

Proof. It is easy to see that if $\mathcal{P}'$ arises by adjoining a new point to $\mathcal{P}$, then

$$\|\mathcal{S}(\mathcal{P}'; \phi, E) - \mathcal{S}(\mathcal{P}; \phi, E)\| \le C_E \sup\{|\phi(y)| : y \in \mathbb{R}\}.$$

It follows that we can assume without loss of generality that $\lambda_0 = s_N$, $\lambda_r = s_M$, where $N \in \mathbb{Z}$, $M \in \mathbb{Z}$, and $N < M$, and that $r \ge 3$.

Now fix $x \in \mathfrak{X}$, and put $x_n = \{E(s_{n+1}) - E(s_n)\}x$ for $N \le n \le M-1$. Consider such an n. If there is a (necessarily unique) k_n such that $1 \le k_n \le r$, and $[s_n, s_{n+1}] \subseteq [\lambda_{k_n-1}, \lambda_{k_n}]$, then

$$\mathcal{S}(\mathcal{P}; \phi, E)x_n = \phi(\lambda_{k_n})x_n.$$

Otherwise,

$$\{k \in \mathbb{N} : k \le r \text{ and } \lambda_k \in (s_n, s_{n+1})\}$$

is not empty. In this case we choose k_n and m_n in $\mathbb{N}$ with $k_n \le m_n \le r-1$, and $\mathcal{P} \cap (s_n, s_{n+1}) = \{\lambda_j : k_n \le j \le m_n\}$. In this case, we further see, with the aid of a summation by parts, that

$$\mathcal{S}(\mathcal{P}; \phi, E)x_n = \sum_{j=k_n}^{m_n} \{\phi(\lambda_j) - \phi(\lambda_{j+1})\}E(\lambda_j)x_n + \phi(\lambda_{m_n+1})x_n.$$

It now follows by Theorem (2.2)(ii) that

$$\int_{D^{M-N}} \Big\| \sum_{n=1}^{M-N} \varepsilon_n \mathcal{S}(\mathcal{P}; \phi, E)x_{n+N-1} \Big\| \, d\varepsilon \le C_E \|\phi\|_{\mathfrak{M}(\mathbb{R})} \int_{D^{M-N}} \Big\| \sum_{n=1}^{M-N} \varepsilon_n x_{n+N-1} \Big\| \, d\varepsilon.$$

Since, from our hypotheses,

$$\Big\| \sum_{n=1}^{M-N} \varepsilon_n x_{n+N-1} \Big\| = \Big\| \sum_{n=1}^{M-N} \varepsilon_n \{E(s_{n+N}) - E(s_{n+N-1})\}x \Big\| \le C_E \|x\|,$$

we see that

$$\int_{D^{M-N}} \Big\| \sum_{n=1}^{M-N} \varepsilon_n \mathcal{S}(\mathcal{P}; \phi, E)x_{n+N-1} \Big\| \, d\varepsilon \le C_E \|\phi\|_{\mathfrak{M}(\mathbb{R})} \|x\|.$$

Moreover,

$$\begin{aligned} &\mathcal{S}(\mathcal{P}; \phi, E)x \\ &= \Big\{ \sum_{n=1}^{M-N} E(s_{n+N}) - E(s_{n+N-1}) \Big\} \mathcal{S}(\mathcal{P}; \phi, E)x \\ &= \Big[\sum_{n=1}^{M-N} \varepsilon_n \{E(s_{n+N}) - E(s_{n+N-1})\} \Big] \sum_{n=1}^{M-N} \varepsilon_n \mathcal{S}(\mathcal{P}; \phi, E)x_{n+N-1}, \end{aligned}$$

and consequently we have

$$\|\mathcal{S}(\mathcal{P}; \phi, E)x\| \le C_E \Big\| \sum_{n=1}^{M-N} \varepsilon_n \mathcal{S}(\mathcal{P}; \phi, E)x_{n+N-1} \Big\|.$$

Combining this with the preceding inequality, we obtain

$$\|\mathcal{S}(\mathcal{P}; \phi, E)x\| \le C_E \|\phi\|_{\mathfrak{M}(\mathbb{R})} \|x\|.$$

This completes the proof of Lemma (4.3). ■

Proof of Theorem (4.1). Let $\phi \in \mathfrak{M}(\mathbb{R})$, $x \in \mathfrak{X}$, and suppose $\delta > 0$ is given. Since $E(s_n)x \to x$ as $n \to \infty$, and $E(s_n)x \to 0$ as $n \to -\infty$, we can choose $N \in \mathbb{N}$ so that $\|E(s_N)x - E(s_{-N})x - x\| < \delta$. Put $x_N = \{E(s_N) - E(s_{-N})\}x$, and let $x = x_N + y_N$. In particular, $\|y_N\| < \delta$. Since $\phi \in \mathrm{BV}([s_{-N}, s_N])$, we can pick a partition $\mathcal{W}_0$ of $[s_{-N}, s_N]$ such that whenever $\mathcal{W}_1$ and $\mathcal{W}_2$ are partitions of $[s_{-N}, s_N]$ refining $\mathcal{W}_0$, then

$$\|\mathcal{S}(\mathcal{W}_1; \phi, E)x_N - \mathcal{S}(\mathcal{W}_2; \phi, E)x_N\| < \delta.$$

If $\mathcal{S}(\mathcal{P}_1; \phi, E)$ and $\mathcal{S}(\mathcal{P}_2; \phi, E)$ are Riemann–Stieltjes approximating sums for $\int_{\mathbb{R}} \phi(\lambda)\, dE(\lambda)$ such that $\mathcal{P}_1 \supseteq \mathcal{W}_0$ and $\mathcal{P}_2 \supseteq \mathcal{W}_0$, we see that

$$\begin{aligned}
&\mathcal{S}(\mathcal{P}_1; \phi, E)x - \mathcal{S}(\mathcal{P}_2; \phi, E)x \\
&\quad = \{\mathcal{S}(\mathcal{P}_1; \phi, E)x_N - \mathcal{S}(\mathcal{P}_2; \phi, E)x_N\} + \{\mathcal{S}(\mathcal{P}_1; \phi, E)y_N - \mathcal{S}(\mathcal{P}_2; \phi, E)y_N\} \\
&\quad = \mathcal{S}(\mathcal{W}_1; \phi, E)x_N - \mathcal{S}(\mathcal{W}_2; \phi, E)x_N + \mathcal{S}(\mathcal{P}_1; \phi, E)y_N - \mathcal{S}(\mathcal{P}_2; \phi, E)y_N,
\end{aligned}$$

where $\mathcal{W}_1$ and $\mathcal{W}_2$ are partitions of $[s_{-N}, s_N]$ refining $\mathcal{W}_0$. It now follows with the aid of Lemma (4.3) that

$$\|\mathcal{S}(\mathcal{P}_1; \phi, E)x - \mathcal{S}(\mathcal{P}_2; \phi, E)x\| \le (1 + C_E\|\phi\|_{\mathfrak{M}(\mathbb{R})})\delta.$$

From this we readily infer that $\lim_{\mathcal{P}} \mathcal{S}(\mathcal{P}; \phi, E)x$ exists in the norm topology of $\mathfrak{X}$. Another application of Lemma (4.3) completes the proof of Theorem (4.1). ∎

Comment. It is worth noting that the results of Theorem (4.1) will be valid if the dyadic points $\{s_n\}_{n=-\infty}^{\infty}$ are replaced by any bilateral strictly increasing sequence $\xi \equiv \{\xi_n\}_{n=-\infty}^{\infty}$ such that $\xi_n \to \pm\infty$ as $n \to \pm\infty$, and the space $\mathfrak{M}(\mathbb{R})$ is replaced by the space of functions $\phi : \mathbb{R} \to \mathbb{C}$ satisfying

$$\|\phi\|_\xi = \sup\{|\phi(s)| : s \in \mathbb{R}\} + \sup\{\mathrm{var}(\phi, [\xi_n, \xi_{n+1}]) : n \in \mathbb{Z}\} < \infty.$$

In this case, $C_E\|\phi\|_{\mathfrak{M}(\mathbb{R})}$ in (4.2) is replaced by $C_{E,\xi}\|\phi\|_\xi$.

The periodic and the discrete analogues of Theorem (4.1) take the following forms and are proved in much the same way.

(4.4) THEOREM. *Let $E(\cdot)$ be a spectral family of projections in a Banach space $\mathfrak{X}$. Suppose that $E(\cdot)$ is concentrated on $[0, 2\pi]$, that $\{E(\lambda) : 0 \le \lambda \le 2\pi\}$ has the R-property, and that*

$$\sup \Big\| \sum_{n=N}^{M} \varepsilon_n \{E(t_{n+1}) - E(t_n)\} \Big\| < \infty,$$

where $\{t_n\}_{n=-\infty}^{\infty}$ is the sequence of dyadic points in $(0, 2\pi)$, and the supremum is taken over all $N \in \mathbb{Z}$, $M \in \mathbb{Z}$ such that $N \le M$, and all choices of $\varepsilon_n = \pm 1$ for $N \le n \le M$. Then, given $\phi \in \mathfrak{M}(\mathbb{T})$, the integral $\int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda)$ exists in the strong operator topology and satisfies

$$\Big\| \int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda) \Big\| \le C_E \|\phi\|_{\mathfrak{M}(\mathbb{T})}.$$

(4.4)′ THEOREM. *Let $\mathfrak{X}$ be a Banach space, and suppose that $P \equiv \{P_n\}_{n=-\infty}^{\infty} \subseteq \mathfrak{B}(\mathfrak{X})$ is a sequence of projections such that: $P_m P_n = 0$ for $m \ne n$; $\sum_{n=-N}^{N} P_n \to I$ in the strong operator topology of $\mathfrak{B}(\mathfrak{X})$ as $N \to \infty$; and*

$$\Big\{ \sum_{n=N}^{M} P_n : N, M \in \mathbb{Z},\ N \le M \Big\}$$

has the R-property. For each $n \in \mathbb{Z}$, define Q_n by writing $Q_n = \sum_{k \in \Lambda_n} P_k$, and suppose further that

$$\sup \Big\{ \Big\| \sum_{n=N}^{M} \varepsilon_n Q_n \Big\| : N, M \in \mathbb{Z},\ N \le M,\ \varepsilon_n = \pm 1 \text{ for } N \le n \le M \Big\} < \infty.$$

Then, given $\phi \in \mathfrak{M}(\mathbb{Z})$, each of the series $\sum_{n=1}^{\infty} \phi(-n) P_{-n}$ and $\sum_{n=0}^{\infty} \phi(n) P_n$ converges in the strong operator topology of $\mathfrak{B}(\mathfrak{X})$, and

$$\Big\| \sum_{n=-\infty}^{\infty} \phi(n) P_n \Big\| \le C_P \|\phi\|_{\mathfrak{M}(\mathbb{Z})}.$$

Comment. As with the comment following the proof of Theorem (4.1), there is a variant of Theorem (4.4) in which the dyadic points $\{t_n\}_{n=-\infty}^{\infty}$ of $(0, 2\pi)$ are replaced by a bilateral strictly increasing sequence $\{\eta_n\}_{n=-\infty}^{\infty}$ in $(0, 2\pi)$ such that $\eta_n \to 2\pi$ (respectively, $\eta_n \to 0$) as $n \to \infty$ (respectively, $n \to -\infty$). In this framework, $\mathfrak{M}(\mathbb{T})$ and $\|\cdot\|_{\mathfrak{M}(\mathbb{T})}$ must be replaced by the analogous space and norm corresponding to the sequence $\eta \equiv \{\eta_n\}_{n=-\infty}^{\infty}$, and the constant C_E in the conclusion of Theorem (4.4) must be replaced by $C_{E,\eta}$.

The results of Theorems (4.1), (4.4), and (4.4)′ include the Strong Marcinkiewicz Multiplier Theorem for the spaces $M_{p,X}(\mathbb{R})$, $M_{p,X}(\mathbb{T})$, and $M_{p,X}(\mathbb{Z})$, respectively, where X is an arbitrary UMD space and $1 < p < \infty$. This generalization of the classical Strong Marcinkiewicz Multiplier Theorem was established by J. Bourgain in [11, Theorem 4] (which treats $M_{p,X}(\mathbb{Z})$ explicitly), and is stated as follows.

(4.5) THEOREM. *Let X be a UMD space, suppose that $1 < p < \infty$, and let $G = \mathbb{R}$, $\mathbb{T}$, or $\mathbb{Z}$. If $\phi \in \mathfrak{M}(G)$, then $\phi \in M_{p,X}(G)$ and*

$$\|\phi\|_{M_{p,X}(G)} \le C_{p,X} \|\phi\|_{\mathfrak{M}(G)}.$$

Proof. Consider the case when $G = \mathbb{R}$. For each $\lambda \in \mathbb{R}$, let $E(\lambda)$ be the multiplier transform on $L^p(\mathbb{R}, X)$ corresponding to the characteristic function of the interval $(-\infty, \lambda]$ in $\mathbb{R}$. It is easy to see that $E(\cdot)$ is a spectral family of projections in $L^p(\mathbb{R}, X)$. By Lemma (3.5) and Theorem (3.6)(ii), $E(\cdot)$ satisfies the hypotheses of Theorem (4.1), and so for each $\phi \in \mathfrak{M}(\mathbb{R})$, $\int_{\mathbb{R}} \phi(\lambda)\, dE(\lambda)$ exists and satisfies (4.2). Straightforward calculations with Fourier transforms now show that $\phi \in M_{p,X}(\mathbb{R})$ with corresponding multiplier transform $\int_{\mathbb{R}} \phi(\lambda)\, dE(\lambda)$.

In the case when $G = \mathbb{T}$, we use Lemma (3.5) together with Theorems (3.6)(iii) and (4.4) to reason analogously with the spectral family $E(\cdot)$ in $L^p(\mathbb{Z}, X)$ which was defined in the proof of Theorem (3.6)(iii). In the case when $G = \mathbb{Z}$, we define $P_n f$ to be the nth term of the Fourier series of f, for $f \in L^p(\mathbb{T}, X)$, and $n \in \mathbb{Z}$. Thus P_n is the multiplier transform corresponding to the characteristic function of the singleton set consisting of n. In this situation we obtain the desired conclusions by applying Scholium (3.1) and Lemma (3.5), together with Theorems (3.6)(i) and (4.4)$'$. ∎

5. Proofs of the main results. We now turn our attention towards establishing the main results, which are stated in Theorems (1.1), (1.2), and (1.3). Throughout the ensuing discussion, X will be a fixed UMD space, and p will be a fixed real number such that $1 < p < \infty$. We shall take up Theorem (1.3) first, since its focus on sums in place of spectral integrals specializes and simplifies its treatment. In fact, the following somewhat stronger assertion than (1.3)(ii) holds (compare [7, Theorem (1.1)]).

(5.1) THEOREM. *Assume the context and notation of Theorem* C, *and suppose* $1 < p < \infty$. *Let* $\phi \in M_{p,X}(\mathbb{Z})$. *Then each of the series* $\sum_{n=1}^{\infty} \phi(-n) P_{-n}$ *and* $\sum_{n=0}^{\infty} \phi(n) P_n$ *converges in the strong operator topology of* $\mathfrak{B}(X)$. *Furthermore, the mapping* $\phi \mapsto \sum_{n=-\infty}^{\infty} \phi(n) P_n$ *is an identity-preserving algebra homomorphism of* $M_{p,X}(\mathbb{Z})$ *into* $\mathfrak{B}(X)$ *such that for all* $\phi \in M_{p,X}(\mathbb{Z})$,

$$\Big\| \sum_{n=-\infty}^{\infty} \phi(n) P_n \Big\| \le c^2 C_{p,X} \|\phi\|_{M_{p,X}(\mathbb{Z})}.$$

Proof. Since the ranges of the projections P_n, $n \in \mathbb{Z}$, span a dense linear manifold in X, while $P_m P_n = 0$ for $m \neq n$, and the two series $\sum_{n=1}^{\infty} P_{-n}$ and $\sum_{n=0}^{\infty} P_n$ converge separately in the strong operator topology, it suffices to show that for $\phi \in M_{p,X}(\mathbb{Z})$ and $N \in \mathbb{N}$ we have

$$\Big\| \sum_{n=-N}^{N} \phi(n) P_n \Big\| \le c^2 C_{p,X} \|\phi\|_{M_{p,X}(\mathbb{Z})}.$$

This follows readily from Theorem (2.5), Scholium (3.1), and the definition of the sequence $\{P_n\}_{n=-\infty}^{\infty}$ by observing that

$$\sum_{n=-N}^{N} \phi(n) P_n = H_{q_N},$$

where $q_N(z) \equiv \sum_{n=-N}^{N} \phi(n) z^n$ for all $z \in \mathbb{T}$, and consequently

$$\Big\| \sum_{n=-N}^{N} \phi(n) P_n \Big\| \le c^2 N_{p,X}(q_N) = c^2 \|\phi \chi_{[-N,N]}\|_{M_{p,X}(\mathbb{Z})} \le c^2 C_{p,X} \|\phi\|_{M_{p,X}(\mathbb{Z})}. \ ∎$$

In view of Theorem (4.5), Theorem (1.3)(ii) is an immediate consequence of Theorem (5.1). Moreover, Theorem (1.3)(i) becomes apparent upon applying (1.3)(ii) to functions $\phi \in \mathfrak{M}(\mathbb{Z})$ having the form $\phi = \sum_{n=-N}^{N} \varepsilon_n \chi_{\Lambda_n}$, where $N \in \mathbb{N}$ and $\varepsilon_n = \pm 1$ for $-N \le n \le N$.

The stage is now set for consideration of Theorems (1.1) and (1.2). In essence, the strategy we shall follow consists of transferring the R-property for the family of multiplier transforms defined by intervals (Lemma (3.5)) and the Littlewood–Paley property for L_X^p in Theorem (3.6) to the spectral decompositions occurring in Theorems (1.1) and (1.2). In view of Theorems (4.1) and (4.4), it will then remain only to establish a bound of the form $c^2 C_X$ for the homomorphism which assigns each Marcinkiewicz multiplier its spectral integral.

The transferred R-property just mentioned will be deduced by establishing more general facts concerning transferred bounds for the averages of multiplier transforms on $L^p(G, X)$, where $G = \mathbb{Z}$ or $\mathbb{R}$, and the relevant multiplier transforms correspond (by Theorem (3.3)) to arbitrary functions in $\mathrm{BV}(\widehat{G})$. In order to describe the latter type of transference phenomena, we begin with some additional items of notation. Suppose that $\phi \in \mathrm{BV}(\mathbb{T})$. Define $\Phi : \mathbb{R} \to \mathbb{C}$ by writing

$$\Phi(t) = 2^{-1}\{\lim_{s \to t^+} \phi(e^{is}) + \lim_{s \to t^-} \phi(e^{is})\}. \tag{5.2}$$

Observe that Φ is 2π-periodic, and that $\mathrm{var}(\Phi, [0, 2\pi]) \le \mathrm{var}(\phi, \mathbb{T})$. We shall also write

$$\mathfrak{T}_\phi = \int_{[0,2\pi]}^{\oplus} \Phi(\lambda)\, dE(\lambda), \tag{5.3}$$

where $E(\cdot)$ is the spectral family in X corresponding to the power-bounded operator U in accordance with Theorem A. Similarly, if $\psi \in \mathrm{BV}(\mathbb{R})$, define

$\varPsi : \mathbb{R} \to \mathbb{C}$ by putting

$$\varPsi(t) = 2^{-1}\Big\{ \lim_{s\to t^+} \psi(s) + \lim_{s\to t^-} \psi(s)\Big\}. \tag{5.4}$$

In particular, $\operatorname{var}(\varPsi, \mathbb{R}) \le \operatorname{var}(\psi, \mathbb{R})$, and we shall also write

$$\mathfrak{T}_\psi = \int_{\mathbb{R}} \varPsi(\lambda)\, dE(\lambda), \tag{5.5}$$

where $E(\cdot)$ is the spectral family in X corresponding to the uniformly bounded, strongly continuous one-parameter group $\{U_t : t \in \mathbb{R}\}$ in accordance with Theorem B. In terms of the foregoing notation, our result for transferring averages of multiplier transforms on $L^p(\mathbb{Z}, X)$ takes the following form.

(5.6) THEOREM. *Assume the hypotheses and notation of Theorem* A, *and suppose that* $1 < p < \infty$. *Let* $N \in \mathbb{N}$, *and suppose that* $\{\phi_j\}_{j=1}^N \subseteq \mathrm{BV}(\mathbb{T})$. *Let* K *be a constant such that the multiplier transforms* $\{T_{\phi_j}\}_{j=1}^N$ *on* $L^p(\mathbb{Z}, X)$ *satisfy*

$$\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j T_{\phi_j} f_j \Big\|_p^p d\varepsilon \Big\}^{1/p} \le K\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j f_j \Big\|_p^p d\varepsilon \Big\}^{1/p} \tag{5.7}$$

for all $\{f_j\}_{j=1}^N \subseteq L^p(\mathbb{Z}, X)$. *Then, in the notation of* (5.2) *and* (5.3), *we have*

$$\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j \mathfrak{T}_{\phi_j} x_j \Big\|_X^p d\varepsilon \Big\}^{1/p} \le c^2 K\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j x_j \Big\|_X^p d\varepsilon \Big\}^{1/p}$$

for all $\{x_j\}_{j=1}^N \subseteq X$.

Proof. For $n \ge 0$ and $1 \le j \le N$, define $\eta_{n,j} \in L^1(\mathbb{Z})$ by putting

$$\eta_{n,j}(m) = \widehat{\kappa}_n(m)\widehat{\phi}_j(-m) \quad \text{for all } m \in \mathbb{Z},$$

where κ_n is the Fejér kernel of order n for $\mathbb{T}$. In particular, $\widehat{\eta}_{n,j} = \kappa_n * \phi_j$. Using (5.7) and Proposition B (in §3), we see that for all $\{f_j\}_{j=1}^N \subseteq L^p(\mathbb{Z}, X)$,

$$\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j \eta_{n,j} * f_j \Big\|_p^p d\varepsilon \Big\}^{1/p} = \Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j T_{\kappa_n * \phi_j} f_j \Big\|_p^p d\varepsilon \Big\}^{1/p} \le K\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j f_j \Big\|_p^p d\varepsilon \Big\}^{1/p}.$$

Specializing the representation R in Theorem (2.6) to be the mapping $\mathbb{Z} \ni m \mapsto U^m$, and applying Theorem (2.6) to the last inequality, we see that

$$\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j H_{\eta_{n,j}} x_j \Big\|_X^p d\varepsilon \Big\}^{1/p} \le c^2 K\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j x_j \Big\|_X^p d\varepsilon \Big\}^{1/p} \tag{5.8}$$

for all $\{x_j\}_{j=1}^N \subseteq X$. Since

$$H_{\eta_{n,j}} = \sum_{m=-n}^{n} \widehat{\kappa}_n(m)\widehat{\phi}_j(m) U^m = \int_{[0,2\pi]}^{\oplus} (\kappa_n * \phi_j)(e^{i\lambda})\, dE(\lambda),$$

while $\operatorname{var}(\kappa_n * \phi_j, \mathbb{T}) \le \operatorname{var}(\phi_j, \mathbb{T})$, and, in the notation of (5.2),

$$\lim_{n\to\infty} (\kappa_n * \phi_j)(e^{i\lambda}) = \varPhi_j(\lambda) \quad \text{for all } \lambda \in \mathbb{R},$$

we see from the limit theorem for spectral integrals [5, Proposition (2.10)] that, for $1 \le j \le N$, $H_{\eta_{n,j}} \to \mathfrak{T}_{\phi_j}$ in the strong operator topology of $\mathfrak{B}(X)$ as $n \to \infty$. Applying this fact to (5.8) completes the proof of Theorem (5.6). ∎

(5.9) COROLLARY. *Assume the hypotheses and notation of Theorem* A. *If* $\phi \in \mathrm{BV}(\mathbb{T})$ *and* $1 < p < \infty$, *then, in the notation of* (5.3), *we have*

$$\|\mathfrak{T}_\phi\| \le c^2 \|\phi\|_{M_{p,X}(\mathbb{T})}.$$

Proof. The assertion is an immediate consequence of taking $N = 1$ in Theorem (5.6). ∎

(5.10) COROLLARY. *Assume the hypotheses and notation of Theorem* A. *Then* $\{E(\lambda) : 0 \le \lambda \le 2\pi\}$ *has the R-property. In fact, if* $1 < p < \infty$, *then for* $N \in \mathbb{N}$, $\{\lambda_j\}_{j=1}^N \subseteq [0, 2\pi]$, *and* $\{x_j\}_{j=1}^N \subseteq X$, *we have*

$$\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j E(\lambda_j) x_j \Big\|_X^p d\varepsilon \Big\}^{1/p} \le c^2 C_{p,X}\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j x_j \Big\|_X^p d\varepsilon \Big\}^{1/p}.$$

Proof. Since $E(2\pi) = E((2\pi)^-)$, and $E(0) = E(0^+)$ in the strong operator topology, it is enough to show the desired inequality when $\{\lambda_j\}_{j=1}^N \subseteq (0, 2\pi)$, which we now suppose. For $1 \le j \le N$, pick $\delta_j \in (0, 2\pi)$ so that $\lambda_j < \delta_j$, and let $\phi_j \in \mathrm{BV}(\mathbb{T})$ be the characteristic function of the arc $\{e^{it} : 0 \le t \le \delta_j\}$. It is easy to see directly that

$$\mathfrak{T}_{\phi_j} = 2^{-1}\{E(\delta_j) + E(\delta_j^-)\} - 2^{-1}E(0). \tag{5.11}$$

By Lemma (3.5),

$$\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j T_{\phi_j} f_j \Big\|_p^p d\varepsilon \Big\}^{1/p} \le C_{p,X}\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j f_j \Big\|_p^p d\varepsilon \Big\}^{1/p}$$

for all $\{f_j\}_{j=1}^N \subseteq L^p(\mathbb{Z}, X)$. It follows by Theorem (5.6) that

$$\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j \mathfrak{T}_{\phi_j} x_j \Big\|_X^p d\varepsilon \Big\}^{1/p} \le c^2 C_{p,X}\Big\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j x_j \Big\|_X^p d\varepsilon \Big\}^{1/p}$$

for all $\{x_j\}_{j=1}^N \subseteq X$. The desired conclusion becomes evident after using (5.11) in this inequality, and letting δ_j run through a sequence approaching λ_j, for each j. ■

Reasoning similar to the foregoing yields companion results in the one-parameter group setting of Theorem B, and we describe this state of affairs next. However, the reasoning requires some extra details occasioned by the fact that $\mathrm{BV}(\mathbb{R})$ is not a subset of $L^1(\mathbb{R})$.

(5.12) THEOREM. *Assume the hypotheses and notation of Theorem* B, *and suppose that* $1 < p < \infty$. *Let* $N \in \mathbb{N}$, *suppose* $\{\psi_j\}_{j=1}^N \subseteq \mathrm{BV}(\mathbb{R})$, *and let* K *be a constant such that*

$$\text{(5.13)}\qquad \Big\{\int\limits_{D^N}\Big\|\sum_{j=1}^N \varepsilon_j T_{\psi_j} f_j\Big\|_p^p\,d\varepsilon\Big\}^{1/p} \le K\Big\{\int\limits_{D^N}\Big\|\sum_{j=1}^N \varepsilon_j f_j\Big\|_p^p\,d\varepsilon\Big\}^{1/p}$$

for all $\{f_j\}_{j=1}^N \subseteq L^p(\mathbb{R},X)$. *Then, in the notation of* (5.4) *and* (5.5), *we have*

$$\Big\{\int\limits_{D^N}\Big\|\sum_{j=1}^N \varepsilon_j \mathfrak{T}_{\psi_j} x_j\Big\|_X^p\,d\varepsilon\Big\}^{1/p} \le c^2K\Big\{\int\limits_{D^N}\Big\|\sum_{j=1}^N \varepsilon_j x_j\Big\|_X^p\,d\varepsilon\Big\}^{1/p}$$

for all $\{x_j\}_{j=1}^N \subseteq X$.

Proof. Since $\{E(a)-E(-a)\} \to I$ in the strong operator topology as $a \to \infty$, it suffices to show that for $a > 0$ and $\{x_j\}_{j=1}^N \subseteq \{E(a)-E(-a)\}X$,

$$\text{(5.14)}\qquad \Big\{\int\limits_{D^N}\Big\|\sum_{j=1}^N \varepsilon_j \int\limits_{[-a,a]} \boldsymbol{\Psi}_j\,dE\,x_j\Big\|_X^p\,d\varepsilon\Big\}^{1/p} \le c^2K\Big\{\int\limits_{D^N}\Big\|\sum_{j=1}^N \varepsilon_j x_j\Big\|_X^p\,d\varepsilon\Big\}^{1/p}.$$

When $\psi_j \in L^1(\mathbb{R})$ for $1 \le j \le N$, the proof of (5.14) can be handled in analogy with the demonstration of Theorem (5.6). Specifically, under the additional assumption $\{\psi_j\}_{j=1}^N \subseteq L^1(\mathbb{R})$, we define $\gamma_{n,j} \in L^1(\mathbb{R})$ for $n \ge 1$ and $1 \le j \le N$ by writing

$$\gamma_{n,j}(t) = (2\pi)^{-1}\widehat{\kappa}_n(t)\widehat{\psi}_j(-t) = \widehat{\kappa}_n(t)(\psi_j)^{\vee}(t),$$

where κ_n now denotes the Fejér kernel of order n for $\mathbb{R}$. In particular, we have $\widehat{\gamma}_{n,j} = \kappa_n * \psi_j$. Notice that $\mathrm{var}(\kappa_n * \psi_j, \mathbb{R}) \le \mathrm{var}(\psi_j, \mathbb{R})$. An argument like that used to establish (3.10) shows that

$$\int\limits_{[-a,a]} (\kappa_n * \psi_j)(\lambda)\,dE(\lambda)x_j = \int\limits_{[-n,n]} \gamma_{n,j}(t)U_{-t}x_j\,dt.$$

The proof of (5.14) for the special case $\{\psi_j\}_{j=1}^N \subseteq L^1(\mathbb{R})$ can now be carried out with the same kind of reasoning used to establish Theorem (5.6) by using $\gamma_{n,j}$ in place of $\eta_{n,j}$.

In the general case, put $\delta_{n,j} = \widehat{\kappa}_n\psi_j$. Then $\delta_{n,j} \in \mathrm{BV}(\mathbb{R}) \cap L^1(\mathbb{R})$, and the function $\boldsymbol{\Delta}_{n,j}$ corresponding to $\delta_{n,j}$ in accordance with (5.4) is given by $\boldsymbol{\Delta}_{n,j} = \widehat{\kappa}_n\boldsymbol{\Psi}_j$. It follows that for $1 \le j \le N$, $\boldsymbol{\Delta}_{n,j} \to \boldsymbol{\Psi}_j$ pointwise on $\mathbb{R}$ as $n \to \infty$, and $\sup_n \mathrm{var}(\boldsymbol{\Delta}_{n,j}, \mathbb{R}) \le 2\|\boldsymbol{\Psi}_j\|_{\mathrm{BV}(\mathbb{R})}$. Notice that for any $n \ge 1$, (5.13) holds with $\{\delta_{n,j}\}_{j=1}^N$ in place of $\{\psi_j\}_{j=1}^N$. Hence we can apply the outcome of the preceding special case to $\{\delta_{n,j}\}_{j=1}^N$, and thus obtain for $a > 0$, and $\{x_j\}_{j=1}^N \subseteq \{E(a)-E(-a)\}X$,

$$\Big\{\int\limits_{D^N}\Big\|\sum_{j=1}^N \varepsilon_j \int\limits_{[-a,a]} \boldsymbol{\Delta}_{n,j}\,dE\,x_j\Big\|_X^p\,d\varepsilon\Big\}^{1/p} \le c^2K\Big\{\int\limits_{D^N}\Big\|\sum_{j=1}^N \varepsilon_j x_j\Big\|_X^p\,d\varepsilon\Big\}^{1/p}.$$

Letting $n \to \infty$, we can apply the limit theorem for spectral integrals [5, Proposition (2.10)] to infer (5.14) and thereby complete the proof of Theorem (5.12). ■

(5.15) COROLLARY. *Assume the hypotheses and notation of Theorem* B, *and suppose* $1 < p < \infty$. *Then for each* $\psi \in \mathrm{BV}(\mathbb{R})$, *we have, in the notation of* (5.4) *and* (5.5),

$$\|\mathfrak{T}_\psi\| \le c^2\|\psi\|_{M_{p,X}(\mathbb{R})}.$$

(5.16) COROLLARY. *Assume the hypotheses and notation of Theorem* B. *Then* $\{E(\lambda) : \lambda \in \mathbb{R}\}$ *has the R-property. In fact, if* $1 < p < \infty$, *then for* $N \in \mathbb{N}$, $\{\lambda_j\}_{j=1}^N \subseteq \mathbb{R}$, *and* $\{x_j\}_{j=1}^N \subseteq X$,

$$\Big\{\int\limits_{D^N}\Big\|\sum_{j=1}^N \varepsilon_j E(\lambda_j)x_j\Big\|_X^p\,d\varepsilon\Big\}^{1/p} \le c^2C_{p,X}\Big\{\int\limits_{D^N}\Big\|\sum_{j=1}^N \varepsilon_j x_j\Big\|_X^p\,d\varepsilon\Big\}^{1/p}.$$

Proof. For $n \in \mathbb{N}$, and $1 \le j \le N$, let $\psi_{n,j} \in \mathrm{BV}(\mathbb{R})$ be the characteristic function of the interval $[\lambda_j - n, \lambda_j + 1/n]$. Apply Lemma (3.5) and Theorem (5.12) to $\{\psi_{n,j}\}_{j=1}^N$, and let $n \to \infty$ to infer the desired inequality. ■

Since Theorem (1.3) has already been demonstrated, we shall not require the counterparts of the foregoing in the context of Theorem C. Nevertheless, we include them here in order to complete the circle of ideas. Under the hypotheses of Theorem C, we shall work with $\mathfrak{M}(\mathbb{Z})$ rather than $\mathrm{BV}(\mathbb{Z})$. In particular, for $\alpha \in \mathfrak{M}(\mathbb{Z})$, we define $\mathfrak{T}_\alpha$ in accordance with Theorem (1.3)(ii) by writing

$$\text{(5.17)}\qquad \mathfrak{T}_\alpha = \sum_{n=1}^\infty \alpha(-n)P_{-n} + \sum_{n=0}^\infty \alpha(n)P_n.$$

Recall that, by Theorem (4.5), for $1 < p < \infty$, α has a corresponding multiplier transform T_α acting on $L^p(\mathbb{T}, X)$.

(5.18) THEOREM. *Assume the hypotheses and notation of Theorem C, and suppose that* $1 < p < \infty$. *Let* $N \in \mathbb{N}$, *suppose* $\{\alpha_j\}_{j=1}^N \subseteq \mathfrak{M}(\mathbb{Z})$, *and let* K *be a constant such that*

$$\left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j T_{\alpha_j} f_j \Big\|_p^p \, d\varepsilon \right\}^{1/p} \le K \left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j f_j \Big\|_p^p \, d\varepsilon \right\}^{1/p} \tag{5.19}$$

for all $\{f_j\}_{j=1}^N \subseteq L^p(\mathbb{T}, X)$. *Then, in the notation of* (5.17), *we have*

$$\left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j \mathfrak{T}_{\alpha_j} x_j \Big\|_X^p \, d\varepsilon \right\}^{1/p} \le c^2 K \left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j x_j \Big\|_X^p \, d\varepsilon \right\}^{1/p} \tag{5.20}$$

for all $\{x_j\}_{j=1}^N \subseteq X$.

Proof. Suppose first that each $\alpha_j \in L^1(\mathbb{Z})$, and let $k_j : \mathbb{T} \to \mathbb{C}$ be defined by

$$k_j(z) = \sum_{m=-\infty}^{\infty} \alpha_j(m) z^m.$$

The hypothesis in (5.19) can now be written in the form

$$\left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j \, k_j * f_j \Big\|_p^p \, d\varepsilon \right\}^{1/p} \le K \left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j f_j \Big\|_p^p \, d\varepsilon \right\}^{1/p}$$

for all $\{f_j\}_{j=1}^N \subseteq L^p(\mathbb{T}, X)$. Hence by Theorem (2.6),

$$\left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j H_{k_j} x_j \Big\|_X^p \, d\varepsilon \right\}^{1/p} \le c^2 K \left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j x_j \Big\|_X^p \, d\varepsilon \right\}^{1/p} \tag{5.21}$$

for all $\{x_j\}_{j=1}^N \subseteq X$. It is easy to see directly that for $m \in \mathbb{Z}$, and $1 \le j \le N$, we have

$$H_{k_j} P_m = \alpha_j(m) P_m = \mathfrak{T}_{\alpha_j} P_m.$$

Hence $H_{k_j} = \mathfrak{T}_{\alpha_j}$, and, in this case, (5.21) coincides with the desired conclusion (5.20).

In the general case, let κ_n denote the Fejér kernel of order n for $\mathbb{T}$, and notice that the hypothesis (5.19) continues to hold for $\{\widehat{\kappa}_n \alpha_j\}_{j=1}^N \subseteq L^1(\mathbb{Z}) \subseteq \mathfrak{M}(\mathbb{Z})$. Hence by the preceding special case, we can infer that for all $\{x_j\}_{j=1}^N \subseteq X$,

$$\left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j \mathfrak{T}_{\widehat{\kappa}_n \alpha_j} x_j \Big\|_X^p \, d\varepsilon \right\}^{1/p} \le c^2 K \left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j x_j \Big\|_X^p \, d\varepsilon \right\}^{1/p} \tag{5.22}$$

However,

$$\mathfrak{T}_{\widehat{\kappa}_n \alpha_j} = \sum_{m=-n}^{n} \left(1 - \frac{|m|}{n+1}\right) \alpha_j(m) P_m$$

is the nth Cesàro mean of the bilateral series defining $\mathfrak{T}_{\alpha_j}$, and so for $1 \le j \le N$, $\mathfrak{T}_{\widehat{\kappa}_n \alpha_j} \to \mathfrak{T}_{\alpha_j}$ in the strong operator topology as $n \to \infty$. Using this fact in (5.22) completes the proof of Theorem (5.18). ∎

As previously, the case $N = 1$ yields the following corollary of Theorem (5.18).

(5.23) COROLLARY. *Assume the hypotheses and notation of Theorem* C. *If* $\alpha \in \mathfrak{M}(\mathbb{Z})$ *and* $1 < p < \infty$, *then, in the notation of* (5.17), *we have* $\|\mathfrak{T}_\alpha\| \le c^2 \|\alpha\|_{M_{p,X}(\mathbb{Z})}$.

(5.24) COROLLARY. *Assume the hypotheses and notation of Theorem* C. *Then* $\{\sum_{n=-\infty}^{m} P_n : m \in \mathbb{Z}\}$ *has the R-property. In fact, if* $1 < p < \infty$, *then for* $N \in \mathbb{N}$, $\{m_j\}_{j=1}^N \subseteq \mathbb{Z}$, *and* $\{x_j\}_{j=1}^N \subseteq X$, *we have*

$$\left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j E_{m_j} x_j \Big\|_X^p \, d\varepsilon \right\}^{1/p} \le c^2 C_{p,X} \left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j x_j \Big\|_X^p \, d\varepsilon \right\}^{1/p},$$

where, for $m \in \mathbb{Z}$, E_m *is the sum* $\sum_{n=-\infty}^{m} P_n$ *in the strong operator topology of* $\mathfrak{B}(X)$.

Proof. For $1 \le j \le N$, let α_j be the characteristic function, defined on $\mathbb{Z}$, of $\{n \in \mathbb{Z} : n \le m_j\}$. In particular, $\alpha_j \in \mathrm{BV}(\mathbb{Z}) \subseteq \mathfrak{M}(\mathbb{Z})$. By Lemma (3.5),

$$\left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j T_{\alpha_j} f_j \Big\|_p^p \, d\varepsilon \right\}^{1/p} \le C_{p,X} \left\{ \int_{D^N} \Big\| \sum_{j=1}^N \varepsilon_j f_j \Big\|_p^p \, d\varepsilon \right\}^{1/p}$$

for all $\{f_j\}_{j=1}^N \subseteq L^p(\mathbb{T}, X)$. Since $\mathfrak{T}_{\alpha_j} = E_{m_j}$ for $1 \le j \le N$, an application of Theorem (5.18) completes the proof of Corollary (5.24). ∎

Proof of Theorem (1.1)(i). Keeping p fixed in the range $1 < p < \infty$, let $X, U, E(\cdot)$, and c be as in Theorem A, and denote by $\mathcal{D}_\mathbb{T}$ the collection of subsets of $\mathbb{T}$ consisting of the singleton set $\{1\}$ together with the arcs Γ_n and the singleton sets $\{\omega_n\}$ for all $n \in \mathbb{Z}$. $\mathcal{E}(\cdot)$ is a projection-valued function initially defined on $\mathcal{D}_\mathbb{T}$, as described in (1.1)(i), and $\mathcal{E}(\alpha)\mathcal{E}(\beta) = 0$ for $\alpha \in \mathcal{D}_\mathbb{T}$, $\beta \in \mathcal{D}_\mathbb{T}$, $\alpha \ne \beta$. Notice that each $\sigma \in \Sigma_{\mathrm{d}}(\mathbb{T})$ can be expressed as the union of a uniquely determined subcollection $\mathcal{A}_\sigma$ of $\mathcal{D}_\mathbb{T}$. The uniqueness assertion in (1.1)(i) follows immediately from this fact, and we now turn to the existence assertion.

Suppose first that $N \in \mathbb{N}$, $j_k \in \mathbb{Z}$ and $\varepsilon_k = \pm 1$ for $k = 1, \ldots, N$, and $j_k \neq j_n$ for $k \neq n$. For $1 \leq k \leq N$, let $a_k \in (0, 2\pi)$ and $b_k \in (0, 2\pi)$ satisfy

$$t_{j_k} < a_k < b_k < t_{j_k+1},$$

and denote by χ_k the characteristic function defined on $\mathbb{T}$ of $\{e^{it} : a_k \leq t \leq b_k\}$. Put $\phi = \sum_{k=1}^{N} \varepsilon_k \chi_k$. Obviously $\phi \in \mathrm{BV}(\mathbb{T})$ and $\|\phi\|_{\mathfrak{M}(\mathbb{T})} \leq 3$. By Theorem (4.5), $\|\phi\|_{M_{p,X}(\mathbb{T})} \leq C_{p,X}$, and it follows by Corollary (5.9) that

$$\|\mathfrak{T}_\phi\| \leq c^2 C_{p,X}. \tag{5.25}$$

Direct calculations from (5.3) show that

$$\mathfrak{T}_\phi = \sum_{k=1}^{N} \varepsilon_k [2^{-1}\{E(b_k) + E(b_k^-)\} - 2^{-1}\{E(a_k) + E(a_k^-)\}].$$

Using this in (5.25), and letting $a_k \to t_{j_k}^+$ and $b_k \to t_{j_k+1}^-$ for $k = 1, \ldots, N$, we see that

$$\Big\| \sum_{k=1}^{N} \varepsilon_k \mathcal{E}(\Gamma_{j_k}) \Big\| \leq c^2 C_{p,X}. \tag{5.26}$$

If we replace the Γ_{j_k} by singleton sets $\{\omega_{j_k}\}$, $k = 1, \ldots, N$, with the ω_{j_k}'s distinct, and if, for sufficiently small $\varepsilon > 0$, we let χ_k in this situation be the characteristic function of the arc $\{e^{it} : t_{j_k} - \varepsilon \leq t \leq t_{j_k} + \varepsilon\}$, then similar considerations show that

$$\Big\| \sum_{k=1}^{N} \varepsilon_k \mathcal{E}(\{\omega_{j_k}\}) \Big\| \leq c^2 C_{p,X}. \tag{5.27}$$

Combining (5.26) and (5.27), we see that if $M \in \mathbb{N}$, $\varepsilon_n = \pm 1$ for $1 \leq n \leq M$, and $\alpha_1, \ldots, \alpha_M$ are distinct elements of $\mathcal{D}_{\mathbb{T}}$, then

$$\Big\| \sum_{n=1}^{M} \varepsilon_n \mathcal{E}(\alpha_n) \Big\| \leq c^2 C_{p,X}. \tag{5.28}$$

Notice that the span of the ranges $\mathcal{E}(\alpha)X$ for $\alpha \in \mathcal{D}_{\mathbb{T}}$ is dense in X. It follows from this observation and (5.28) that for any sequence $\{\alpha_j\}_{j\geq 1}$ of distinct elements of $\mathcal{D}_{\mathbb{T}}$, we have pointwise on X the unconditional convergence of $\sum_{j\geq 1} \mathcal{E}(\alpha_j)$ to a projection whose norm does not exceed the bound $c^2 C_{p,X}$ in (5.28). We now extend $\mathcal{E}(\cdot)$ from $\mathcal{D}_{\mathbb{T}}$ to a projection-valued function $\mathcal{E}(\cdot)$ on $\Sigma_{\mathrm{d}}(\mathbb{T})$ by writing

$$\mathcal{E}(\sigma)x = \sum_{\alpha \in \mathcal{A}_\sigma} \mathcal{E}(\alpha)x \quad \text{for } \sigma \in \Sigma_{\mathrm{d}}(\mathbb{T}),\ x \in X.$$

Obviously $\mathcal{E}(\sigma)X$ is the closed linear span of the ranges $\mathcal{E}(\alpha)X$ for $\alpha \in \mathcal{A}_\sigma$. Similar reasoning from (5.28) now shows that $\mathcal{E}(\cdot)$ is strongly countably additive on $\Sigma_{\mathrm{d}}(\mathbb{T})$. The remaining properties of a spectral measure on $\Sigma_{\mathrm{d}}(\mathbb{T})$ are also readily verified for $\mathcal{E}(\cdot)$, and the proof of Theorem (1.1)(i) is complete. ■

The proof of Theorem (1.2)(i) can be carried out in an entirely analogous fashion by using Corollary (5.15) in place of Corollary (5.9). We omit the details for this, and pass to consideration of (1.1)(ii) and (1.2)(ii).

Proof of Theorem (1.1)(ii). Assume the setting and notation of Theorem A, and let $\phi \in \mathfrak{M}(\mathbb{T})$. By Corollary (5.10), $\{E(\lambda) : 0 \leq \lambda \leq 2\pi\}$ has the R-property. The remaining hypothesis of Theorem (4.4) is also satisfied, since, in terms of the spectral measure $\mathcal{E}(\cdot)$ in Theorem (1.1)(i),

$$\Big\| \sum_{n=N}^{M} \varepsilon_n \{E(t_{n+1}) - E(t_n)\} \Big\| = \Big\| \sum_{n=N}^{M} \varepsilon_n \mathcal{E}(\sigma_n) \Big\| \leq c^2 C_X,$$

where $\sigma_n = \{e^{it} : t_n < t \leq t_{n+1}\}$. Hence $\int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda)$ exists, and we now turn our attention to obtaining an estimate of the form

$$\Big\| \int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda) \Big\| \leq c^2 C_X \|\phi\|_{\mathfrak{M}(\mathbb{T})}. \tag{5.29}$$

This will be accomplished by successive reductions.

Notice first that if $0 \leq a < b \leq 2\pi$, then $\int_{[a,b]} \phi(e^{i\lambda})\, dE(\lambda)$ exists and

$$\int_{[a,b]} \phi(e^{i\lambda})\, dE(\lambda) = \Big\{ \int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda) \Big\} \{E(b) - E(a)\}. \tag{5.30}$$

Using the symbol "χ" to denote "characteristic function" relative to $[0, 2\pi]$, we see from (5.30) that, in terms of the sequence $\{t_j\}_{j=-\infty}^{\infty}$ of dyadic points in $(0, 2\pi)$,

$$\int_{[0,2\pi]} \chi_{(t_{-N},t_N]}(\lambda)\phi(e^{i\lambda})\, dE(\lambda) = \int_{[t_{-N},t_N]} \phi(e^{i\lambda})\, dE(\lambda) \to \int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda)$$

in the strong operator topology as $N \to \infty$. So it suffices to establish (5.29) in the special case when ϕ vanishes outside the arc $\{e^{it} : t_{-N} \leq t \leq t_N\}$, for some $N \in \mathbb{N}$. We now consider this special case. In particular, $\phi \in \mathrm{BV}(\mathbb{T})$. Let $\Phi \in \mathrm{BV}([0, 2\pi])$ be the 2π-periodic function corresponding to ϕ in accordance with (5.2), and notice that $\|\Phi\|_{\mathfrak{M}(\mathbb{T})} \leq 2\|\phi\|_{\mathfrak{M}(\mathbb{T})}$ (here, and in what follows, we identify a function f defined on $\mathbb{T}$ with the function $t \mapsto f(e^{it})$ defined for $0 \leq t \leq 2\pi$). Put $\psi = \phi - \Phi \in \mathrm{BV}(\mathbb{T})$. Clearly,

$$\|\psi\|_{\mathfrak{M}(\mathbb{T})} \leq 3\|\phi\|_{\mathfrak{M}(\mathbb{T})}. \tag{5.31}$$

By Corollary (5.9) and Theorem (4.5), we can introduce $p \in (1, \infty)$ in a transitory role to infer that

$$\left\| \int_{[0,2\pi]} \Phi(\lambda)\, dE(\lambda) \right\| \le c^2 C_X \|\phi\|_{\mathfrak{M}(\mathbb{T})}.$$

Hence, in view of (5.31), it now suffices for (5.29) to show that

(5.32) $$\left\| \int_{[0,2\pi]} \psi(\lambda)\, dE(\lambda) \right\| \le c^2 C_X \|\psi\|_{\mathfrak{M}(\mathbb{T})}.$$

Since the discontinuity set of ϕ is countable, there is a sequence $\{\lambda_j\}_{j\ge1} \subseteq [t_{-N}, t_N]$ such that ψ vanishes on the set-theoretic difference $[0,2\pi] \setminus \{\lambda_j : j \ge 1\}$. Moreover, since $\psi \in \mathrm{BV}(\mathbb{T})$, $\sum_{j\ge1} |\psi(\lambda_j)| \le 2^{-1} \operatorname{var}(\psi, [0,2\pi]) < \infty$. Hence the sum $\sum_{j\ge1} \psi(\lambda_j)\chi_{\{\lambda_j\}}$ is absolutely convergent in the space $\mathrm{BV}([0,2\pi])$, and equals ψ. Consequently, we can apply the limit theorem for spectral integrals [5, Proposition (2.10)] to see that as $n \to \infty$,

(5.33) $$\int_{[0,2\pi]} \sum_{j=1}^{n} \psi(\lambda_j)\chi_{\{\lambda_j\}}\, dE \to \int_{[0,2\pi]} \psi\, dE$$

in the strong operator topology. Notice that for $n \in \mathbb{N}$,

$$\left\| \sum_{j=1}^{n} \psi(\lambda_j)\chi_{\{\lambda_j\}} \right\|_{\mathfrak{M}(\mathbb{T})} \le \|\psi\|_{\mathfrak{M}(\mathbb{T})}.$$

In view of the reduction in (5.32), it now suffices for (5.29) to obtain

(5.34) $$\left\| \int_{[0,2\pi]} F(t)\, dE(t) \right\| \le c^2 C_X \|F\|_{\mathfrak{M}(\mathbb{T})}$$

for any function F having the form $F = \sum_{j=1}^{m} \alpha_j \chi_{\{y_j\}}$, where $m \in \mathbb{N}$, $\{\alpha_j\}_{j=1}^{m} \subseteq \mathbb{C}$, and $\{y_j\}_{j=1}^{m} \subseteq (0,2\pi)$.

Observe that

$$\int_{[0,2\pi]} F(t)\, dE(t) = \sum_{j=1}^{m} \alpha_j \{E(y_j) - E(y_j^-)\}.$$

In particular, Theorem (1.1)(i) permits us to assume further, without loss of generality, that each y_j is not a dyadic point of $(0,2\pi)$. Accordingly, let $\varepsilon > 0$ be small enough so that the intervals $[y_j - \varepsilon, y_j + \varepsilon]$, $1 \le j \le m$, are disjoint, and so that for each j, $[y_j - \varepsilon, y_j + \varepsilon]$ is contained in the interior of some corresponding dyadic interval of $(0,2\pi)$.

Define $f_\varepsilon \in \mathrm{BV}([0,2\pi])$ by writing $f_\varepsilon = \sum_{j=1}^{m} \alpha_j \chi_{[y_j-\varepsilon, y_j+\varepsilon]}$. It is easy to see that $\|f_\varepsilon\|_{\mathfrak{M}(\mathbb{T})} = \|F\|_{\mathfrak{M}(\mathbb{T})}$. Applying Corollary (5.9) and Theorem (4.5) to f_ε (regarded as an element of $\mathrm{BV}(\mathbb{T})$), we now obtain

(5.35) $$\|\mathfrak{T}_{f_\varepsilon}\| \le c^2 C_X \|F\|_{\mathfrak{M}(\mathbb{T})}.$$

Direct calculations easily show that

$$\mathfrak{T}_{f_\varepsilon} = \sum_{j=1}^{m} [2^{-1}\alpha_j \{E(y_j+\varepsilon) + E((y_j+\varepsilon)^-)\} - 2^{-1}\alpha_j \{E(y_j-\varepsilon) + E((y_j-\varepsilon)^-)\}].$$

Substituting this in (5.35), and letting $\varepsilon \to 0^+$, we obtain (5.34). This gives (5.29).

In order to complete the proof of (1.1)(ii), it remains only to show that the mapping $\mathfrak{M}(\mathbb{T}) \ni \phi \mapsto \int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda)$ is multiplicative. It follows from (5.30) that

$$\Big\{ \int_{[t_{-N}, t_N]} \phi(e^{i\lambda})\, dE(\lambda) \Big\}_{N=1}^{\infty}$$

is a uniformly bounded sequence of operators which, as noted earlier, converges in the strong operator topology to $\int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda)$. If, also, $\psi \in \mathfrak{M}(\mathbb{T})$, then both ϕ and ψ belong to $\mathrm{BV}([t_{-N}, t_N])$, and so

$$\int_{[t_{-N}, t_N]} \phi(e^{i\lambda})\, dE(\lambda) \int_{[t_{-N}, t_N]} \psi(e^{i\lambda})\, dE(\lambda) = \int_{[t_{-N}, t_N]} \phi(e^{i\lambda})\psi(e^{i\lambda})\, dE(\lambda).$$

The desired multiplicativity now follows by letting $N \to \infty$, and so Theorem (1.1)(ii) is demonstrated. ∎

The proof of Theorem (1.2)(ii) can be accomplished similarly, by starting from Corollary (5.16) and Theorem (4.1), and so the main results are now established.

(5.36) A COUNTEREXAMPLE. We give an example showing that the conclusions of Theorem (1.1) can fail to hold in the Hilbert space $L^2(\mathbb{N})$ for a spectral family of projections which decomposes an invertible operator U that is not power-bounded. By the construction of a suitable conditional basis for $L^2(\mathbb{N})$ it was shown in [15] that there is a sequence $\{F_n\}_{n=1}^{\infty}$ of projection operators defined on $L^2(\mathbb{N})$ such that:

(i) $F_n F_m = 0$ for $n \ne m$;

(ii) $\sum_{n=1}^{\infty} F_n$ converges to I in the strong operator topology;

(iii) $\|\sum_{j=1}^{n} F_{2j}\| \to \infty$ as $n \to \infty$.

We now define a bounded, strictly decreasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ by putting, for each $n \in \mathbb{N}$, $\lambda_n = 2^{-n-1}\pi$. Thus for all $n \in \mathbb{N}$, λ_n is the dyadic point t_{-n} of $(0,2\pi)$. For $t \in (0,\lambda_1]$ let N be the smallest $N \in \mathbb{N}$ such that $\lambda_N \le t$, and put $E(t) = \sum_{n=N}^{\infty} F_n$, the series converging in the strong operator topology by (5.36)(ii). Also define $E(t) = 0$ for $t \le 0$, and $E(t) = I$ for $t > \lambda_1$. It is easy to see that $E(\cdot)$ is a spectral family of projections in $L^2(\mathbb{N})$ concentrated on $[0,\lambda_1] = [0,\pi/4]$. Now put $U = \int_{[0,2\pi]}^{\oplus} e^{it}\, dE(t)$. As was shown in [5, (5.10)],

$$\sup\{\|U^n\| : n \in \mathbb{Z}\} = \infty.$$

Nevertheless, it follows from general spectral-theoretic considerations (see [5, Propositions (2.11) and (2.17)]) that since the existence part of the conclusion in Theorem A is satisfied here, so is the uniqueness part. Thus, in the present context, all the hypotheses of Theorem (1.1) are fulfilled except the power-boundedness of U, and we now proceed to show that the spectral family $E(\cdot)$ just defined does not enjoy the properties described in either (1.1)(i) or (1.1)(ii).

Suppose first that $E(\cdot)$ gave rise to a countably additive spectral measure $\mathcal{E}(\cdot)$ on $\Sigma_{\rm d}(\mathbb{T})$, as formulated in (1.1)(i). Then, upon writing $\sigma_n = \{e^{it} : t_{-n-1} < t \le t_{-n}\}$ for $n \in \mathbb{N}$, we have

$$\mathcal{E}(\sigma_n) = E(t_{-n}) - E(t_{-n-1}) = E(\lambda_n) - E(\lambda_{n+1}) = F_n.$$

It follows from this equality and the strong countable additivity of $\mathcal{E}(\cdot)$ that for each sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\varepsilon_n = \pm 1$ for all $n \in \mathbb{N}$, we must have the strong convergence of the series $\sum_{n=1}^{\infty} \varepsilon_n F_n$, in contradiction to (5.36)(iii).

In order to establish the failure of (1.1)(ii) in the present context, we first observe directly from the definition of $E(\cdot)$ that if $\phi : \mathbb{T} \to \mathbb{C}$ is a bounded function such that $\int_{[0,2\pi]} \phi(e^{i\lambda})\,dE(\lambda)$ exists, then

$$\int_{[0,2\pi]} \phi(e^{i\lambda})\,dE(\lambda) = \sum_{n=1}^{\infty} \phi(e^{i\lambda_n})F_n,$$

with series convergence in the strong operator topology. Consequently, if $\int_{[0,2\pi]} \phi(e^{i\lambda})\,dE(\lambda)$ were to exist for every $\phi \in \mathfrak{M}(\mathbb{T})$, we would have the strong convergence of $\sum_{n=1}^{\infty} \varepsilon_n F_n$ for every sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\varepsilon_n = \pm 1$ for all $n \in \mathbb{N}$. As noted above, this is impossible.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
1409 WEST GREEN ST.
URBANA, ILLINOIS 61801
U.S.A.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF EDINBURGH
JAMES CLERK MAXWELL BUILDING
EDINBURGH EH9 3JZ, SCOTLAND

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