Contents of Volume 112, Number 1

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Detailed information for authors is given on the inside back cover. Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-293997

Correspondence concerning subscription, exchange and back numbers should be addressed to

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-293997

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by

PRINTED IN POLAND

ISSN 0039-3223

STUDIA MATHEMATICA 112 (1) (1994)

Some remarks on the asymptotic behaviour of the iterates of a bounded linear operator

by

ANATOLIĬ B. ANTONEVICH (Minsk),
JÜRGEN APPELL (Würzburg), and PETR P. ZABREĬKO (Minsk)

Abstract. We discuss the problem of characterizing the possible asymptotic behaviour of the norm of the iterates of a bounded linear operator between two Banach spaces. In particular, given an increasing sequence of positive numbers tending to infinity, we construct Banach spaces such that the norm of the iterates of a suitable multiplication operator between these spaces assumes (or exceeds) the values of this sequence.

In this paper we discuss the problem of characterizing the possible asymptotic behaviour of the norm of the iterates A^n $(n = 1, 2, \ldots)$ of a bounded linear operator which acts between two Banach spaces X and Y. Here we assume that both spaces X and Y are continuously imbedded in some topological linear space S, and hence the domains of definition

$$\mathcal{D}(A) \supseteq \mathcal{D}(A^2) \supseteq \ldots \supseteq \mathcal{D}(A^n) \supseteq \ldots \tag{1}$$

form a decreasing sequence of linear subspaces of S. The Banach spaces X and Y are required to satisfy

$$\mathcal{D}(A^n) \supseteq X, \quad \mathcal{R}(A^n) \subseteq Y \quad (n = 1, 2, \ldots), \tag{2}$$

where $\mathcal{R}(B)$ denotes as usual the range of B. In particular, in the case $X = Y = S$ the operators A^n are simply the usual iterates of A, considered in one and the same space.

Classical examples of operators which will be considered in what follows are the *multiplication operator*

$$Ax(t) = a(t)x(t) \quad (t \in \Omega), \tag{3}$$

the *shift operator* (or *composition operator*)

$$Ax(t) = x(\theta(t)) \quad (t \in \Omega), \tag{4}$$

1991 *Mathematics Subject Classification*: 47B38, 47A30, 46E30.

and the *integral operator*

$$(5)\qquad Ax(t)=\int_{\Omega} k(t,s)x(s)\,ds \qquad (t\in\Omega).$$

Here a suitable choice for S is the metric linear space of all (classes of) measurable real functions on Ω. Similarly, for X and Y one may choose Lebesgue spaces, Orlicz spaces or, more generally, arbitrary ideal spaces [8, 12] of measurable functions over Ω. Using well-known mapping and boundedness criteria for the operators (3), (4) and (5) between these spaces, one can find conditions on the functions $a:\Omega\to\mathbb{R}$ in (3), $\theta:\Omega\to\Omega$ in (4), and $k:\Omega\times\Omega\to\mathbb{R}$ in (5), respectively, under which the iterates A^n are defined and satisfy (2); sometimes it is even possible to find conditions which are both necessary and sufficient [6, 7].

The problem of studying the asymptotic behaviour of A^n plays an important role in several fields of mathematical analysis. Here we recall some variants of ergodic theory (see e.g. [4]), where, by the way, the classical case $X=Y=S$ is commonly used. Moreover, the asymptotic behaviour of A^n is also of interest in approximation theory (see e.g. [11]), in various branches of fixed point theory (see e.g. [15]), and in its applications to Banach space geometry (see e.g. [3]).

The purpose of this note is two-fold. On the one hand, we shall show that the asymptotic behaviour of the norm of the iterates A^n may be in fact almost arbitrarily prescribed. On the other hand, we shall describe some situations where the knowledge of the asymptotic behaviour of the norm of A^n gives useful information on the operator itself. For example, it turns out that the choice of the space containing the multiplicator function a in (3) determines the growth of the iterates of the corresponding operator A, and vice versa.

In the classical case $X=Y=S$, the sequence of iterates A^n has a well-known algebraic property which is trivial but useful, and which we state for further reference as

LEMMA 1. *Let A be a bounded linear operator in a Banach space X. Then the sequence $a_n=\|A^n\|$ satisfies*

$$(6)\qquad a_{m+n}\le a_m a_n,$$

i.e. it is (logarithmically) sublinear.

Lemma 1 states, loosely speaking, that the sequence $\|A^n\|$ behaves like a geometric progression. The ratio of this progression is, of course, nothing else but the spectral radius

$$\varrho(A)=\lim_{n\to\infty}\|A^n\|^{1/n}=\inf_n\|A^n\|^{1/n}=\sup\{|\lambda|:\lambda\in\sigma(A)\}.$$

We also mention the interesting formula

$$\varrho(A)=\inf\{\|A\|_*:\|\cdot\|_*\approx\|\cdot\|\},$$

where the infimum is taken over all norms on X which are equivalent to the original norm.

It is natural to ask whether or not, apart from (6), the sequence $a_n=\|A^n\|$ has other general properties. The following proposition shows that the answer is negative.

PROPOSITION 1. *Let a_n $(n=0,1,2,\ldots)$ be an arbitrary increasing sequence of positive real numbers satisfying $a_0=1$ and (6). Then one can find a Banach space X and a bounded linear operator $A:X\to X$ such that*

$$(7)\qquad \|A^n\|=a_n\quad(n=0,1,2,\ldots).$$

Proof. The proof is almost obvious. Choosing $X=l_\infty$ (or any other l_p space for $1\le p<\infty$), and defining $A:X\to X$ by

$$(8)\qquad A(x_1,x_2,x_3,x_4,\ldots)=\left(0,\frac{a_1}{a_0}x_1,\frac{a_2}{a_1}x_2,\frac{a_3}{a_2}x_3,\ldots\right),$$

we have for $n=1,2,\ldots,$

$$(9)\qquad A^n(x_1,x_2,x_3,x_4,\ldots)=\Big(\underbrace{0,\ldots,0}_{n\text{ times}},\frac{a_n}{a_0}x_1,\frac{a_{n+1}}{a_1}x_2,\frac{a_{n+2}}{a_2}x_3,\ldots\Big).$$

Consequently,

$$\|A^n\|=\sup\{a_{k+n}/a_k:k=0,1,\ldots\}.$$

But (6) implies that $a_{k+n}/a_k\le a_k a_n/a_k=a_n$, and hence $\|A^n\|=a_n$ as claimed. ∎

A trivial example for Proposition 1 is $a_n=a^n$ with $a>0$ fixed; here the operator A is simply given by

$$A(x_1,x_2,x_3,x_4,\ldots)=(0,ax_1,ax_2,ax_3,\ldots).$$

Of course, if one imposes additional conditions on the operator A, more can be said about the sequence $\|A^n\|$. For example, in [13] the problem is studied under what conditions the sequence $\|A^n\|$ is weakly equivalent to the sequence $n^\alpha\varrho^n(A)$ for some exponent α. Results of this type have important applications in the theory of approximate solutions of operator equations, but they are far from being complete.

We now pass to the (more interesting) case $X\ne Y$. Here it turns out that the sequence $\|A^n\|$ may have arbitrary growth. We state this as follows:

PROPOSITION 2. *Let M_n $(n=0,1,\ldots)$ be an arbitrary increasing sequence of positive real numbers satisfying $M_0=1$ and $M_n\to\infty$ as $n\to\infty$.*

Then one can find two Banach spaces X and Y and a bounded linear operator $A : X \to Y$ such that

$$\|A^n\| \geq M_n \quad (n = 0, 1, \ldots). \tag{10}$$

Proof. We choose $X = L_\infty = L_\infty(0, \infty)$, $Y = L_1 = L_1(0, \infty)$, and

$$Ax(t) = a(t)x(t+1), \tag{11}$$

where the function a will be specified below. (In other words, the operator A is a combination of (3) and (4).) An iterated application of (11) gives

$$A^n x(t) = a_n(t)x(t+n) \quad (n = 1, 2, \ldots), \tag{12}$$

where

$$a_n(t) = a(t)a(t+1)\ldots a(t+n-1). \tag{13}$$

Consequently, if we ensure that $a_n \in L_1$ for $n = 1, 2, \ldots$, then the operator (12) will be bounded from X into Y and

$$\|A^n\| = \|a_n\|_{L_1} = \int_0^\infty |a_n(t)|\, dt. \tag{14}$$

Now let γ_k and α_k $(k = 0, 1, \ldots)$ be sequences of positive numbers such that

$$1 > \gamma_0 \geq \gamma_1 \geq \ldots \geq \gamma_k \geq \ldots, \qquad \sum_{k=0}^\infty \gamma_k < \infty \tag{15}$$

and

$$\alpha_0 = 1 - \frac{\gamma_0}{M_1}, \quad \alpha_k = \frac{\gamma_0\gamma_1\ldots\gamma_{k-1}}{M_k} - \frac{\gamma_0\gamma_1\ldots\gamma_k}{M_{k+1}}. \tag{16}$$

Observe that the conditions (15) and (16) imply that

$$\alpha_k \geq 0 \quad (k = 1, 2, \ldots), \qquad \sum_{k=0}^\infty \alpha_k = 1.$$

Now we define the weight function a in (11) by

$$a(t) = \sum_{k=0}^\infty \gamma_k \varphi(t-k)^{\alpha_k}, \tag{17}$$

where $\varphi(t) = 1/t$ for $0 < t \leq 1$ and $\varphi(t) = 0$ otherwise. By (13) we have

$$a_n(t) = \sum_{k=0}^\infty \gamma_k \ldots \gamma_{k+n-1}\varphi(t-k)^{\alpha_k + \ldots + \alpha_{k+n-1}}. \tag{18}$$

We claim that $a_n \in L_1$ $(n = 1, 2, \ldots)$. In fact, from (18) it follows that

$$\begin{aligned}\int_0^\infty |a_n(t)|\, dt &\leq \sum_{k=0}^\infty \gamma_k \ldots \gamma_{k+n-1} \int_0^\infty \varphi(t-k)^{\alpha_k + \ldots + \alpha_{k+n-1}}\, dt \\ &= \sum_{k=0}^\infty \frac{\gamma_k \ldots \gamma_{k+n-1}}{1 - [\alpha_k + \ldots + \alpha_{k+n-1}]} \\ &\leq \sum_{k=0}^\infty \frac{\gamma_k}{1 - [\alpha_k^* + \ldots + \alpha_{k+n-1}^*]} < \infty,\end{aligned}$$

where α_k^* denotes the decreasing rearrangement of the sequence α_k (see [5]).

Now we prove the estimate (10). Again from (18) it follows that

$$a_n(t) \geq \gamma_0\gamma_1 \ldots \gamma_{n-1}\varphi(t)^{\alpha_0 + \alpha_1 + \ldots + \alpha_{n-1}},$$

hence

$$\|a_n\|_{L_1} \geq \frac{\gamma_0\gamma_1\ldots\gamma_{n-1}}{1 - [\alpha_0 + \alpha_1 + \ldots + \alpha_{n-1}]}.$$

From this and the fact that

$$1 - [\alpha_0 + \alpha_1 + \ldots + \alpha_{n-1}] = \frac{\gamma_0\gamma_1\ldots\gamma_{n-1}}{M_n},$$

by (16), we conclude that $\|a_n\|_{L_1} \geq M_n$ as claimed. ■

Here is an elementary example for Proposition 2. Let $M_n = n$ and $\gamma_n = 2^{1-n}$ $(n = 0, 1, \ldots)$. Then

$$\alpha_k = \frac{2}{k\sqrt{2^{(k-2)(k-1)}}} - \frac{2}{(k+1)\sqrt{2^{(k-1)k}}}.$$

In this case the multiplicator function (17) is given by

$$a(t) = 2^{1-k}(t-k)^{-\alpha_k} \quad (k < t \leq k+1),$$

i.e. $a(t)$ consists of infinitely many decreasing hyperbola branches such that $\lim_{t\to k-} a(t) = 2^{2-k}$ and $\lim_{t\to k+} a(t) = \infty$.

We point out that one can also choose $X = L_p$ and $Y = L_q$ for $1 \leq q < p \leq \infty$ in Proposition 2. The requirement $a_n \in L_1$ has then to be replaced by $a_n \in L_{pq/(p-q)}$, and (14) holds with the L_1 norm of a_n replaced by the $L_{pq/(p-q)}$ norm; the rest of the proof remains unchanged.

In view of Proposition 2, the problem arises to find classes of operators A between two Banach spaces X and Y for which the growth of the sequence $\|A^n\|$ $(n = 1, 2, \ldots)$ can be described more precisely. The corresponding "inverse" problem is also interesting: given an operator A, find spaces X and Y such that all iterates A^n are bounded from X into Y, and the sequence $\|A^n\|$ $(n = 1, 2, \ldots)$ has certain growth properties. We now show that these two questions can in fact be answered fairly completely and lead to interesting phenomena even in the very simple case $X = L_p = L_p(0,1)$, $Y = L_q = L_q(0,1)$, and

$$Ax(t) = a(t)x(t) \quad (0 \le t \le 1). \tag{19}$$

It is evident that, for $p < q$, the operator (19) may be bounded from L_p into L_q only if $a(t) \equiv 0$; therefore only the case $p \ge q$ is interesting. Now, for $p = q$ the multiplicator function a in (19) has to belong to L_∞ in order to generate a bounded operator. Since

$$A^n x(t) = a^n(t)x(t) \quad (n = 1, 2, \ldots) \tag{20}$$

and

$$\|A^n\| = \|a^n\|_{L_\infty} = \|a\|^n_{L_\infty} \quad (n = 1, 2, \ldots),$$

the sequence $\|A^n\|$ is simply a geometric progression in this case. A certain converse of this is also true:

LEMMA 2. *Suppose that the iterates $\|A^n\|$ of the operator (19) do not grow faster than a geometric progression, i.e.*

$$\limsup_{n\to\infty} \sqrt[n]{\|A^n\|} < \infty.$$

Then the corresponding multiplicator function a belongs to L_∞.

Proof. Let $c > 0$ and $L > 1$ be such that $\|A^n\| = \|a^n\| \le cL^n$. Passing then from a to the function

$$\widehat{a}(t) = \begin{cases} 0 & \text{if } |a(t)| \le L, \\ a(t)/L & \text{if } |a(t)| > L, \end{cases}$$

we have $\|\widehat{a}^n\| \le c$ for any n. On the other hand, the sequence $\widehat{a}^n(t)$ tends monotonically to ∞ on the set $\{t : a(t) > 0\}$, contradicting the Beppo Levi theorem. ■

We now turn to the case $p > q$. As a consequence of the classical Hölder inequality, the operator (19) is bounded from L_p into L_q if, and only if, $a \in L_{pq/(p-q)}$. Here the condition $\|a^n\|_{L_{pq/(p-q)}} < \infty$ $(n = 1, 2, \ldots)$ does *not* necessarily imply that $a \in L_\infty$, but only $a \in L_p$ for every $p \in [1, \infty)$. At this point, it is reasonable to take into account not just Lebesgue spaces, but the larger class of *Orlicz spaces*; we recall the necessary definitions and results.

Given a Young function M, consider the Orlicz space $L_M = L_M(0,1)$ equipped with the Luxemburg norm

$$\|x\|_{L_M} = \inf\left\{r : r > 0,\ \int_0^1 M[x(t)/r]\,dt \le 1\right\} \tag{21}$$

(see e.g. [6, 10]). Suppose that the Young function M grows faster than any polynomial (e.g., $M(u) = e^{|u|} - |u| - 1$), and denote by $\gamma_p(M)$ the imbedding

constant of L_M into L_p for $1 \le p < \infty$, i.e.

$$\|x\|_{L_p} \le \gamma_p(M)\,\|x\|_{L_M} \quad (x \in L_M). \tag{22}$$

Since

$$\|a^n\|_{L_1} = \|a\|^n_{L_n} \le \gamma^n_n(M)\|x\|^n_{L_M} \quad (n = 1, 2, \ldots), \tag{23}$$

all multiplicator functions a from the unit ball of L_M give rise to the same asymptotic behaviour of the iterates (20). In this connection, two facts are worth mentioning. First, it is often possible to estimate (or even calculate) the value of the imbedding constant $\gamma_p(M)$ in (22); for example, one can use the estimate

$$\gamma_p(M) \le \inf\{r : r > 0,\ M(u/r) \le (1-\lambda) + \lambda u^p\ (0 \le \lambda \le 1,\ 0 < u < \infty)\}.$$

Second, the Orlicz space L_M cannot be chosen in a "minimal" way; in fact, in [9] it is shown that there is no Young function M such that

$$\bigcap_{1<p<\infty} L_p = L_M.$$

As we have seen, only (essentially) bounded multiplicator functions a generate an operator (19) with the property that $\|A^n\|$ grows like a geometric progression. In other words, if the growth of $\|A^n\|$ is faster than any geometric progression, one should expect that the corresponding multiplicator function a belongs to some ideal space Z which contains unbounded functions. We now show that Z can always be chosen as an Orlicz space L_M generated by some real-analytic Young function M. The converse is also true: from the coefficients of the Taylor expansion of a real-analytic Young function M one can recover the growth of $\|A^n\|$ for every $a \in L_M$:

PROPOSITION 3. *Let M_n $(n = 1, 2, \ldots)$ be a sequence of positive real numbers satisfying*

$$\lim_{n\to\infty} M_n^{1/n} = \infty \tag{24}$$

and

$$M_n^2 \le M_{n-1}M_{n+1}. \tag{25}$$

Define a Young function M by

$$M(u) = \sum_{n=1}^{\infty} \frac{|u|^n}{M_n}, \tag{26}$$

and suppose that

$$\|a^n\|_{L_1} \le M_n L^n \quad (n = 1, 2, \ldots) \tag{27}$$

for some $L>0$. Then a belongs to the Orlicz space L_M, and $\|a\|_{L_M}\le 2L$. Conversely, if a belongs to L_M with M given by (26), then

$$\|a^n\|_{L_1}\le M_n\|a\|_{L_M}^n \quad (n=1,2,\ldots); \tag{28}$$

in particular, $\gamma_n^n(M)\le M_n$.

Proof. If (27) holds, from Beppo Levi's theorem we get

$$\int_0^1 M[a(t)/(2L)]\,dt=\int_0^1\sum_{n=1}^\infty \frac{|a(t)|^n}{2^nM_nL^n}\,dt$$
$$=\sum_{n=1}^\infty\frac{1}{2^nM_nL^n}\int_0^1|a(t)|^n\,dt\le\sum_{n=1}^\infty\frac{1}{2^nM_nL^n}M_nL^n=1,$$

hence $\|a\|_{L_M}\le 2L$, by the definition (21) of the Luxemburg norm. Conversely, if $\|a\|_{L_M}=L<\infty$, we have

$$\int_0^1\frac{|a(t)|^n}{M_nL^n}\,dt\le\int_0^1\sum_{n=1}^\infty\frac{|a(t)|^n}{M_nL^n}\,dt=\int_0^1 M[a(t)/L]\,dt\le 1,$$

again by (21). Consequently,

$$\int_0^1|a(t)|^n\,dt\le M_nL^n,$$

which is (28). ∎

In the preceding proposition, we found upper estimates for the L_1 norm of the function a^n, i.e. for the case when the operator A^n maps L_∞ into L_1. More generally, if A^n maps L_p into L_q $(1\le q<p\le\infty)$, we have the following analogue of Proposition 3 which is proved in exactly the same way:

PROPOSITION 4. *Let M_n $(n=1,2,\ldots)$ be a sequence of positive numbers satisfying* (24) *and* (25). *Define a Young function M by*

$$M(u)=M_{p,q}(u)=\sum_{n=1}^\infty\frac{|u|^{npq/(p-q)}}{M_n}, \tag{29}$$

and suppose that

$$\|a^n\|_{L_{pq/(p-q)}}\le M_nL^n \quad (n=1,2,\ldots) \tag{30}$$

for some $L>0$. Then a belongs to the Orlicz space L_M, and $\|a\|_{L_M}\le 2L$. Conversely, if a belongs to L_M with M given by (28), *then*

$$\|a^n\|_{L_{pq/(p-q)}}\le M_n\|a\|_{L_M}^n \quad (n=1,2,\ldots); \tag{31}$$

in particular, $\gamma_{npq/(p-q)}^n(M)\le M_n$.

We remark that the preceding two propositions can also be used to prove the assertion of Proposition 2. In fact, it suffices to choose $X=L_\infty(0,1)$, $Y=L_1(0,1)$, and $Ax(t)=a(t)x(t)$, where

$$a(t)=\frac12\sum_{n=1}^\infty\frac{M_n}{M_{n-1}}\chi_{D_n}(t), \tag{32}$$

where D_n is a sequence of mutually disjoint subsets of $[0,1]$ such that

$$\operatorname{mes}(D_n)\le 2^{-n}\frac{M_{n-1}^n}{M_n^{n-1}}.$$

We claim that then $a\in L_M$ with M given by (26), and $\|a\|_{L_M}\le 1$. To see this, put

$$\widetilde M(u)=\sup\{|u|^n/M_n:n=1,2,\ldots\}$$

and observe that

$$\widetilde M(2a(t))=\sum_{n=1}^\infty\frac{M_n^{n-1}}{M_{n-1}^n}\chi_{D_n}(t).$$

Further, since the Young functions M and $\widetilde M$ are equivalent in the sense that $\widetilde M(u)\le M(u)\le\widetilde M(2u)$, we have

$$\int_0^1 M[a(t)]\,dt\le\int_0^1\widetilde M[2a(t)]\,dt=\sum_{n=1}^\infty\frac{M_n^{n-1}}{M_{n-1}^n}\operatorname{mes}(D_n)\le 1,$$

i.e. $\|a\|_{L_M}\le 1$. On the other hand,

$$\|a^n\|_{L_1}=\int_0^1|a(t)|^n\,dt\ge\int_{D_n}\left(\frac{M_n}{M_{n-1}}\right)^n dt\ge M_n,$$

i.e. (10) holds.

We point out that the construction of the analytic Young functions (26) and (28) is similar to the construction employed in the papers [1, 2] on analytic superposition operators.

All the results proved so far refer basically to the case of multiplication operators in L_p (Propositions 3 and 4), weighted multiplication operators in L_p (Proposition 2), or multiplication operators in l_p (Proposition 1). For other types of operators A, it seems much more difficult to get precise information on the asymptotic behaviour of $\|A^n\|$. We just confine ourselves to some general remarks.

First of all, since multiplication operators are closely related (via Fourier transforms) to *convolution operators*, one could expect that the preceding two propositions carry over as well to convolution operators. This is in fact true. By means of Proposition 3, for example, one can describe the asymptotic behaviour of the iterates of a convolution operator $Ax(t)=(k*x)(t)$ if the Fourier transform $\widehat k$ of the kernel k belongs to the Orlicz space L_M with M given by (26). In general, if A is an integral operator of type (5), one may try to describe the growth of $\|A^n\|$ by combining the usual formulas for the iterated kernels

$$k^{(n)}(t,s)=\int_{\Omega} k^{(n-1)}(t,\tau)k(\tau,s)\,d\tau$$

with well-known formulas or estimates for the norm of integral operators in various function spaces (see e.g. [7, 14]).

Finally, we point out that one cannot expect any monotonicity behaviour for the sequence $\|A^n\|$ as n increases. For instance, many operators which are important in mathematical analysis are periodic, i.e. $A^p = I$ for some entire p. As typical examples, we mention the *Fourier transform* on Lebesgue spaces, the *Hilbert transform* on Hölder spaces, and all *composition operators* (4) on spaces of analytic functions (e.g. Bergman or Hardy spaces) which are generated by some Möbius transform on the unit disc [16].

References

[1] J. Appell and P. P. Zabreĭko, *On analyticity conditions for the superposition operator in ideal function spaces*, Boll. Un. Mat. Ital. 4-C (1985), 279–295.

[2] —, —, *Analytic superposition operators*, Dokl. Akad. Nauk BSSR 29 (1985), 878–881 (in Russian).

[3] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.

[4] P. R. Halmos, *Lectures in Ergodic Theory*, Chelsea, New York, 1956.

[5] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1934.

[6] M. A. Krasnosel'skiĭ and Ya. B. Rutitskiĭ, *Convex Functions and Orlicz Spaces*, Fizmatgiz, Moscow, 1958 (in Russian); English transl.: Noordhoff, Groningen, 1961.

[7] M. A. Krasnosel'skiĭ and P. P. Zabreĭko, *Geometrical Methods of Nonlinear Analysis*, Nauka, Moscow, 1975 (in Russian); English transl.: Springer, Berlin, 1984.

[8] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, North-Holland, Amsterdam, 1974.

[9] M. Novak, *Unions and intersections of families of L^p spaces*, Math. Nachr. 136 (1988), 241–251.

[10] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, M. Dekker, New York, 1991.

[11] A. Schönhage, *Approximationstheorie*, de Gruyter, Berlin, 1971.

[12] P. P. Zabreĭko, *Ideal function spaces*, Yaroslavl. Gos. Univ. Vestnik 8 (1974), 12–52 (in Russian).

[13] —, *Error estimates for successive approximations and spectral properties of linear operators*, Numer. Funct. Anal. Optim. 11 (1990), 823–838.

[14] P. P. Zabreĭko, A. I. Koshelev, M. A. Krasnosel'skiĭ, S. G. Mikhlin, L. S. Rakovshchik and V. Ya. Stetsenko, *Integral Equations*, Nauka, Moscow, 1968 (in Russian); English transl.: Noordhoff, Leyden, 1975.

[15] E. Zeidler, *Nonlinear Functional Analysis and its Applications I*, Springer, Berlin, 1990.

[16] K. Zhu, *Operator Theory in Function Spaces*, M. Dekker, New York, 1990.

BELGOSUNIVERSITET
MATEMATICHESKIĬ FAKUL'TET
PL. NEZAVISIMOSTI
BR-220050 MINSK, BELARUS

UNIVERSITÄT WÜRZBURG
MATHEMATISCHES INSTITUT
AM HUBLAND
D-97074 WÜRZBURG, GERMANY

Received March 23, 1993 (3086)
Revised version July 25, 1994