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Complex interpolation functors with a family of quasi-power function parameters

by

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Abstract. For the complex interpolation functors associated with derivatives of analytic functions, the Calderón fundamental inequality is formulated in both additive and multiplicative forms; discretization, reiteration, the Calderón–Lozanovskiĭ construction for Banach lattices, and the Aronszajn–Gagliardo construction concerning minimality and maximality are presented. These more general complex interpolation functors are closely connected with the real and other interpolation functors via function parameters which are quasi-powers with a logarithmic factor.

Let $0 < \theta < 1$. The Calderón complex interpolation space $C_\theta(\overline{X})$ for a Banach couple $\overline{X}$ is determined by $f(\theta)$ for $f \in A^{\mathrm{b}}(\mathbb{S}, \overline{X})$, where $A^{\mathrm{b}}(\mathbb{S}, \overline{X})$ consists of analytic functions from the strip

$$\mathbb{S} = \{z \in \mathbb{C} \mid 0 \le \operatorname{Re} z \le 1\}$$

to $\Sigma\overline{X}$ with boundedly continuous boundary values in X_0 or X_1. The space $C^\theta(\overline{X})$ is defined similarly with some change on the boundary. For $n = \pm 1, \pm 2, \ldots$, the generalized complex interpolation spaces $C_{\theta(n)}(\overline{X})$ are described by $f^{(n)}(\theta)$ for $f \in A^{\mathrm{b}}(\mathbb{S}, \overline{X})$ when $n > 0$; and by $f(\theta)$ for $f \in A^{\mathrm{b}}(\mathbb{S}, \overline{X})$ with $f^{(k)}(\theta) = 0$ $(1 \le k \le |n|)$ when $n < 0$. The fundamental inequality, originally stated for $n = 0$ by Calderón and generalized to arbitrary n, plays an important role in complex interpolation. For instance, it produces the duality relation between the lower $\theta(n)$- and the upper $\theta(-n)$-functors (cf. [Cal] & [FK]).

The present paper is a continuation of [FK]. We formulate the fundamental inequality in both additive and multiplicative forms, as well as applications for $n \neq 0$. We carry over some classical results (partially sometimes), such as discretization, reiteration, the Calderón–Lozanovskiĭ construction for Banach lattices, and the Aronszajn–Gagliardo construction concerning minimality and maximality (cf. [Cal], [Cw], [J]), to these more general com-

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plex interpolation functors. The connection of the $\theta(n)$-complex functors with the real and other interpolation functors is also considered.

1. Introduction and preliminaries. For details and any other unexplained notations concerning interpolation, we refer to [BK], [BL], [J] and [FK].

In the present paper, we follow the custom of using the letter c to denote a positive quantity which varies from occurrence to occurrence. $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ stand for the sets of integers, real numbers, positive real numbers and complex numbers, respectively. For positive functions f and g on some set, $f \vee g = \max\{f, g\}$, $f \wedge g = \min\{f, g\}$; $f \prec g$ means that $f \le cg$, and $f \simeq g$ means that both $f \prec g$ and $g \prec f$ hold. We use the notations $\prec$, $\subseteq$, $\simeq$ and $=$ for continuous inclusion, isometric inclusion, isomorphic equivalence and isometric equivalence, respectively, between Banach spaces, Banach couples as well as interpolation functors on the category $\overline{BC}$ of Banach couples.

Here and throughout, we assume that X is a Banach space over $\mathbf{\Lambda} = \mathbb{R}$ or $\mathbb{C}$ with dual space X', and assume that $\overline{X} = (X_0, X_1)$ is a Banach couple for which $\Delta\overline{X} = X_0 \cap X_1$ and $\Sigma\overline{X} = X_0 + X_1$. Let X be an intermediate space for $\overline{X}$. The *regularization* X^0 of X in $\overline{X}$ is the closure of $\Delta\overline{X}$ in X; and the *Gagliardo completion* X^c of X for $\overline{X}$ consists of all limits in $\Sigma\overline{X}$ of bounded sequences in X with norm

$$\|x\|_{X^c} = \inf\{c \mid x_n \to x \text{ in } \Sigma\overline{X} \text{ for } (x_n) \text{ in } X \text{ with } \|x_n\|_X \le c\}.$$

The space X (resp. the couple $\overline{X}$) is called *regular* if $X^0 = X$ (resp. $\overline{X}^0 = \overline{X}$), and *Gagliardo complete* if $X^c = X$ (resp. $\overline{X}^c = \overline{X}$). For two Banach spaces X, Y (resp. Banach couples $\overline{X}, \overline{Y}$), $\mathcal{L}(X, Y)$ (resp. $\mathcal{L}(\overline{X}, \overline{Y})$) stands for the Banach space of all bounded linear operators from X to Y (resp. from $\overline{X}$ to $\overline{Y}$).

Let F be an interpolation functor on $\overline{BC}$ over $\mathbf{\Lambda}$. We define its *regularization functor* F^0 and *Gagliardo completion functor* F^c by

$$F^0 : \overline{X} \mapsto F(\overline{X})^0 \quad \text{and} \quad F^c : \overline{X} \mapsto F(\overline{X})^c \quad \text{for } \overline{X} \in \overline{BC}.$$

F is called *regular* if $F^0 = F$, and *Gagliardo complete* if $F^c = F$. If F, F_0, F_1 are three interpolation functors, their *superposition functor* $F(F_0, F_1)$ is

$$F(F_0, F_1)(\overline{X}) = F(F_0(\overline{X}), F_1(\overline{X})) \quad \text{for } \overline{X} \in \overline{BC}.$$

We use this notation when reiteration is concerned. For a positive function ϱ on $\mathbb{R}_+$, F is called a *functor of type* ϱ if

$$\|T\|_{F(\overline{X}), F(\overline{Y})} \le c\|T\|_0 \varrho(\|T\|_1/\|T\|_0)$$

for all Banach couples $\overline{X}$, $\overline{Y}$ and for all $T \in \mathcal{L}(\overline{X}, \overline{Y})$. F is *of exact type* ϱ if

the constant is equal to one. The *characteristic function* ϱ_F of F is defined by

$$F(\mathbf{\Lambda}, (1/t)\mathbf{\Lambda}) = (1/\varrho_F(t))\mathbf{\Lambda},$$

where $t\mathbf{\Lambda}$ is the 1-dimensional space $\mathbf{\Lambda}$ with norm $\|\lambda\| = t|\lambda|$ for $t > 0$.

Suppose $\overline{A}$ is a Banach couple and A is an intermediate space for $\overline{A}$. We define the *orbit functor* $\mathrm{Orb}_A(\overline{A}, -)$ by

$$\mathrm{Orb}_A(\overline{A}, \overline{X}) = \Big\{x = \sum_{n=1}^{\infty} T_n a_n \in \Sigma\overline{X} \;\Big|\; T_n \in \mathcal{L}(\overline{A}, \overline{X}) \text{ and } a_n \in A\Big\}$$

with the *orbit norm* $\|x\|_{\mathrm{Orb}} = \inf \sum_n \|T_n\|_{\overline{A}, \overline{X}} \|a_n\|_A$, and the *coorbit functor* $\mathrm{Corb}_A(-, \overline{A})$ by

$$\mathrm{Corb}_A(\overline{X}, \overline{A}) = \{x \in \Sigma\overline{X} \mid Tx \in A \text{ for all } T \in \mathcal{L}(\overline{X}, \overline{A}) \text{ with } \|T\| \le 1\}$$

with the *coorbit norm* $\|x\|_{\mathrm{Corb}} = \sup \|Tx\|_A$. Much use is made of the important fact that $\mathrm{Orb}_A(\overline{A}, -)$ is minimal among all functors F for which $A \prec F(\overline{A})$, and $\mathrm{Corb}_A(-, \overline{A})$ is maximal among all functors H for which $H(\overline{A}) \prec A$ (under the order of contractive inclusion). If in addition A is an interpolation space with respect to $\overline{A}$, then

$$\mathrm{Orb}_A(\overline{A}, \overline{A}) \simeq \mathrm{Corb}_A(\overline{A}, \overline{A}) \simeq A.$$

Here "$\simeq$" becomes "$=$" when A is an exact interpolation space.

Next let us recall some notations and terminology of [FK]. For a Lebesgue measurable subset $\mathbb{E}$ of $\mathbf{\Lambda}$ and a Banach space X, we denote by $L^p(\mathbb{E}, X)$ ($p = 1$ or ∞) the X-valued L^p-space with respect to the Lebesgue measure. We simply write $L^p(\mathbb{E}) = L^p(\mathbb{E}, \mathbf{\Lambda})$. For the strip $\mathbb{S} = \{z \in \mathbb{C} \mid 0 \le \mathrm{Re}\, z \le 1\}$, we denote by $H_\infty(\mathbb{S}, X)$ the space of all analytic functions from $\mathrm{int}\, \mathbb{S}$, the interior of $\mathbb{S}$, to X such that

$$\sup\{\|f(z)\|_{\Sigma\overline{X}} \mid z \in \mathrm{int}\, \mathbb{S}\} < \infty.$$

Functions in $H_\infty(\mathbb{S}, X)$ need not have usual boundary values in X, while each f in $H_\infty(\mathbb{S}, X)$ has boundary values in the weak topology, say $T_{f,j}$ in the space $\mathcal{L}(L^1(\mathbb{R}_j), X)$, where $\mathbb{R}_j = \{z \in \mathbb{C} \mid \mathrm{Re}\, z = j\}$ ($j = 0, 1$) (cf. [KP, p. 146]).

Now let $\overline{X}$ be a Banach couple and let $n = 1, 2, \ldots$ We consider the following Banach spaces of analytic functions on $\mathbb{S}$:

$$A^b(\mathbb{S}, \overline{X}) = \{f \in H_\infty(\mathbb{S}, \Sigma\overline{X}) \mid f \text{ is boundedly continuous on } \mathbb{S} \text{ such that } f(j + it) \in X_j \text{ for } t \in \mathbb{R}\ (j = 0, 1)\},$$

$$A_n^b(\mathbb{S}, \overline{X}) = \{f \in A^b(\mathbb{S}, \overline{X}) \mid f^{(k)}(\theta) = 0 \text{ for } k = 1, \ldots, n\}, \tag{1.1}$$

$$H_\infty(\mathbb{S}, \overline{X}) = \{f \in H_\infty(\mathbb{S}, \Sigma\overline{X}) \mid T_{f,j} \in \mathcal{L}(L^1(\mathbb{R}), X_j)\ (j = 0, 1)\},$$

$$H_{\infty,n}(\mathbb{S}, \overline{X}) = \{f \in H_\infty(\mathbb{S}, \overline{X}) \mid f^{(k)}(\theta) = 0 \text{ for } k = 1, \ldots, n\},$$

with norm

$$\|f\|_\infty = \max_{j=0,1}\{\sup_{t\in\mathbb{R}} \|f(j+it)\|_j\}$$

for $f \in A^{\mathrm{b}}(\mathbb{S}, \overline{X})$ or $A_n^{\mathrm{b}}(\mathbb{S}, \overline{X})$, and

$$\|f\|_\infty = \|T_{f,0}\|_0 \vee \|T_{f,1}\|_1$$

for $f \in H_\infty(\mathbb{S}, \overline{X})$ or $H_{\infty,n}(\mathbb{S}, \overline{X})$. Let

$$K_{\theta(n)} = K_{\theta(n)}(A) = \{f \in A \mid f^{(n)}(\theta) = 0\},$$

a closed subspace of $A = A^{\mathrm{b}}(\mathbb{S}, \overline{X})$, $A_n^{\mathrm{b}}(\mathbb{S}, \overline{X})$, $H_\infty(\mathbb{S}, \overline{X})$ or $H_{\infty,n}(\mathbb{S}, \overline{X})$, and let

$$c_{\theta,n} = \frac{1}{n!}\bigg(-\frac{2}{\pi}\sin \pi\theta\bigg)^n.$$

With this background, one can define the *complex $\theta(n)$-interpolation spaces* by

$$C_{\theta(n)}(\overline{X}) = c_{\theta,n}A^{\mathrm{b}}(\mathbb{S}, \overline{X})/K_{\theta(n)}, \quad C^{\theta(n)}(\overline{X}) = c_{\theta,n}H_\infty(\mathbb{S}, \overline{X})/K_{\theta(n)} \tag{1.2}$$

and

$$C_{\theta(-n)}(\overline{X}) = A_n^{\mathrm{b}}(\mathbb{S}, \overline{X})/K_{\theta(0)}, \quad C^{\theta(-n)}(\overline{X}) = H_{\infty,n}(\mathbb{S}, \overline{X})/K_{\theta(0)} \tag{1.3}$$

with quotient norm

$$\|x\|_{\theta(n)} = \inf\{\|f\|_\infty \mid x = c_{\theta,n}f^{(n)}(\theta)\}$$

and

$$\|x\|_{\theta(-n)} = \inf\{\|f\|_\infty \mid x = f(\theta)\}$$

respectively. To simplify matters we usually set $n \in \mathbb{Z}$ and write C_θ and C^θ for $C_{\theta(0)}$ and $C^{\theta(0)}$. The complex interpolation with derivatives was first studied by Schechter and then by Carro–Cerdà in the distribution sense (cf. [S] & [CC]).

In the definition of the Calderón space $C_\theta(\overline{X})$, the space $A^{\mathrm{b}}(\mathbb{S}, \overline{X})$ can be replaced with some other analytic function spaces (cf. [BL], [Cw] & [Pee]). Let $A^0(\mathbb{S}, \overline{X})$ (resp. $A_n^0(\mathbb{S}, \overline{X})$) be the subspace of $A^{\mathrm{b}}(\mathbb{S}, \overline{X})$ (resp. $A_n^{\mathrm{b}}(\mathbb{S}, \overline{X})$) which consists of all functions f such that $\lim_{|t|\to\infty} f(j+it) = 0$. Then we have an equivalent definition of $C_{\theta(n)}(\overline{X})$:

$$C_{\theta(n)}(\overline{X}) = \begin{cases} c_{\theta,n}A^0(\mathbb{S}, \overline{X})/K_{\theta(n)} & \text{for } n > 0, \\ A_n^0(\mathbb{S}, \overline{X})/K_{\theta(0)} & \text{for } n < 0, \end{cases} \tag{1.4}$$

and of the regularity:

$$C_{\theta(n)}(\overline{X}) = C_{\theta(n)}(\overline{X})^0 = C_{\theta(n)}(\overline{X}^0), \quad C^{\theta(n)}(\overline{X}) = C_{\theta(n)}(\overline{X}^0), \tag{1.5}$$

according to [FK, Remark 5.8]. We will investigate further identifications in the next two sections.

2. Fundamental inequality and applications. Let us start from a pair of Banach spaces, say $\{\Delta\overline{X}, \Sigma\overline{X}\}$, with the contractive inclusion $\hat{\jmath} : \Delta\overline{X} \to \Sigma\overline{X}$. Let $\eta : \mathbb{T} \times \Delta\overline{X} \to \mathbb{R}_+$ be a measurable function such that $\|\cdot\|_t = \eta(\cdot, t)$ is a norm on $\Delta\overline{X}$ for a.e. $t \in \mathbb{T}$, the unit circle in $\mathbb{C}$; and suppose every $x \in \Delta\overline{X}$ satisfies

$$\|x\|_{\Delta\overline{X}} = \operatorname{ess\,sup}\{\|x\|_t \mid t \in \mathbb{T}\},$$

$$\|\hat{\jmath}(x)\|_{\Sigma\overline{X}} = \inf\bigg\{\int_{\mathbb{T}} \|f(t)\|_t \,\frac{dt}{2\pi} \;\bigg|\; f \in L^1(\mathbb{T}, \Delta\overline{X}) \text{ with } x = \int_{\mathbb{T}} f(t)\,\frac{dt}{2\pi}\bigg\}.$$

Let $X(t)$ be the Banach completion of $\Delta\overline{X}$ with respect to the norm $\|\cdot\|_t$ for a.e. $t \in \mathbb{T}$. The corresponding regular interpolation family $\overline{X}$ over $\mathbb{T}$ is given by $\overline{X} = \{X(t) \mid t \in \mathbb{T} \text{ a.e.}\}$. The main advantage of this setting is the validity of Fourier analysis and the technique of multipliers (cf. [FK, Secs. 1 & 2]). Let $A(\mathbb{D})$ and $H^1(\mathbb{D})$ be the disc algebra and H^1-space on the unit disk $\mathbb{D}$ respectively. For $\varphi \in H^1(\mathbb{D})$, we denote by $S_n(\varphi)$ the nth partial sum of the Fourier series of φ. Let us restate [FK, Lemma 2.5] here to make the paper self-contained.

2.1. LEMMA. *Suppose $\varphi \in H^1(\mathbb{D})$ with $u = \operatorname{Re}\varphi$.*

(i) *If $\psi \in A(\mathbb{D})$, then $S_n(\varphi\psi) = S_n(S_n(\varphi)\psi) = S_n(S_n(\varphi)S_n(\psi))$. In particular, if $S_n(\varphi) = 1$, then $S_n(\varphi\psi) = S_n(\psi)$; and hence if $\varphi(0) \neq 0$, then $S_n(\varphi S_n(1/\varphi)) = 1$ so that $S_n(S_n(1/\varphi)\varphi\psi) = S_n(\psi)$.*

(ii) *Let B_n be the quotient algebra $A(\mathbb{D})/z^{n+1}A(\mathbb{D})$. Then*

$$\|S_n(\varphi)\|_\infty \le (n+1)\|\widehat{\varphi}\|_{A(\mathbb{D})} \le (n+1)\|\varphi\|_{H^1(\mathbb{D})},$$

$$\|S_n(\exp\varphi)\|_{B_n} \le \exp u(0)(1 + \|\widehat{u}\|_{A(\mathbb{D})})^n \le \exp u(0)(1 + \|u\|_{H^1(\mathbb{D})})^n.$$

Let

$$H^\infty(\mathbb{D}, \overline{X}) = \{f : \mathbb{D} \to \Sigma\overline{X} \mid f \text{ is analytic in } \operatorname{int}\mathbb{D} \text{ and } \|f(e^{it})\|_t \in L^\infty(\mathbb{T})\}$$

with norm $\|f\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{T}} \|f(e^{it})\|_t$. For $f \in H^\infty(\mathbb{D}, \overline{X})$, let φ be an analytic function on $\operatorname{int}\mathbb{D}$ with $u(e^{it}) = \operatorname{Re}\varphi(e^{it}) = \log\|f(e^{it})\|_t$ and let $g = \exp(-\varphi)S_n(\exp\varphi)$. Then Lemma 2.1 implies that $F = gf \in H^\infty(\mathbb{D}, \overline{X})$ with $\widehat{F}(k) = \widehat{f}(k)$ for $k = 0, 1, \ldots, n$, and

$$\|F\|_\infty \le \|S_n(\exp\varphi)\|_\infty \|f\exp(-\varphi)\|_\infty \le (n+1)\int_{\mathbb{T}} \|f(e^{it})\|_t \,\frac{dt}{2\pi},$$

$$\begin{aligned} \|F\|_\infty &\le \|S_n(\exp\varphi)\|_{B_n} \|f\exp(-\varphi)\|_\infty \\ &\le \bigg(1 + \int_{\mathbb{T}} |\log\|f(e^{it})\|_t| \,\frac{dt}{\pi}\bigg)^n \exp\bigg(\int_{\mathbb{T}} \log\|f(e^{it})\|_t \,\frac{dt}{2\pi}\bigg). \end{aligned} \tag{2.2}$$

These inequalities, together with the conformal mapping m_θ $(0 < \theta < 1)$

from $\mathbb{S}$ onto $\mathbb{D}$ given by

$$m_\theta(z) = \frac{\exp(i\pi(z-\theta))-1}{\exp(-i\pi\theta)-\exp(i\pi z)} \quad \text{for } z\in\mathbb{S}, \tag{2.3}$$

as well as the estimates

$$\exp\Big(\int_{-\infty}^{\infty}\log\|f(it)\|_0 P_0(\theta,t)\,\frac{dt}{1-\theta}\Big) \le \int_{-\infty}^{\infty}\|f(it)\|_0 P_0(\theta,t)\,\frac{dt}{1-\theta},$$

$$\exp\Big(\int_{-\infty}^{\infty}\log\|f(1+it)\|_1 P_1(\theta,t)\,\frac{dt}{\theta}\Big) \le \int_{-\infty}^{\infty}\|f(1+it)\|_1 P_1(\theta,t)\,\frac{dt}{\theta}$$

for $f\in A^{\mathrm b}(\mathbb{S},\overline{X})$ imply the following

2.4. THEOREM (Fundamental inequality). *Assume $f\in A^{\mathrm b}(\mathbb{S},\overline{X})$. Then there exists $h\in A^{\mathrm b}(\mathbb{S},\overline{X})$ such that $h^{(k)}(\theta)=f^{(k)}(\theta)$ $(k=0,1,\ldots,n)$ and*

(i) $$\|h\|_\infty \le c_1\sum_{j=0,1}\int_{-\infty}^{\infty}\|f(j+it)\|_j P_j(\theta,t)\,dt,$$

(ii) $$\|h\|_\infty \le c_2\Big(1+\sum_{j=0,1}\int_{-\infty}^{\infty}|\log\|f(j+it)\|_j|P_j(\theta,t)\,dt\Big)^{|n|} \times\exp\Big(\sum_{j=0,1}\int_{-\infty}^{\infty}\log\|f(j+it)\|_j P_j(\theta,t)\,dt\Big),$$

(iii) $$\|h\|_\infty \le c_2\Big(1+\sum_{j=0,1}\int_{-\infty}^{\infty}|\log\|f(j+it)\|_j|P_j(\theta,t)\,dt\Big)^{|n|} \times\Big(\int_{-\infty}^{\infty}\|f(it)\|_0 P_0(\theta,t)\,\frac{dt}{1-\theta}\Big)^{1-\theta} \times\Big(\int_{-\infty}^{\infty}\|f(1+it)\|_1 P_1(\theta,t)\,\frac{dt}{\theta}\Big)^{\theta},$$

where c_1 and c_2 are constants depending on θ and n only.

A direct consequence is the following duality relation:

$$C_{\theta(n)}(\overline{X})' \simeq C^{\theta(-n)}((\overline{X}^0)'). \tag{2.5}$$

Let us now turn to further applications of the fundamental inequality.

2.6. LEMMA. *Assume $f\in H_\infty(\mathbb{S},\overline{X})$ such that $\|f\|_\infty\le 1$ and $f(i\cdot)\in L^\infty(\mathbb{R},X_0)$. Then there is a sequence (f_ν) in $A^{\mathrm b}(\mathbb{S},\overline{X})$ such that $\|f_\nu\|_\infty\le\|f\|_\infty$ and*

$$\int_{-\infty}^{\infty}\|f_\nu(it)-f(it)\|_0 P(\theta,t)\,dt\to 0 \quad \text{as } \nu\to\infty.$$

Furthermore,

(i) if $x_\nu=c_{\theta,n}f_\nu^{(n)}(\theta)$ and $x=c_{\theta,n}f^{(n)}(\theta)$ for $n>0$, then $x_\nu\to x$ in $C_{\theta(n)}(\overline{X})$;

(ii) there is a subsequence of (f_ν) (still denoted by (f_ν)), and a sequence (g_ν) in $A^{\mathrm b}(\mathbb{S},\overline{X})$ such that $g_\nu^{(k)}(\theta)=f_\nu^{(k)}(\theta)$ $(k=0,1,\ldots,n)$, and (g_ν) converges to a function g in $A^{\mathrm b}(\mathbb{S},\overline{X})$. If in addition $f\in A_n^{\mathrm b}(\mathbb{S},\overline{X})$, then $g\in A_n^{\mathrm b}(\mathbb{S},\overline{X})$ with $f(\theta)=g(\theta)$.

Proof. For $\nu=1,2,\ldots,$ let

$$f_\nu(z)=\nu\int_0^{1/\nu}f(z+it)\,dt \quad \text{for } z\in\mathbb{S}.$$

From the proof of the theorem in [Pee, Sec. 4], we readily see that $f_\nu\in A^{\mathrm b}(\mathbb{S},\overline{X})$, $\|f_\nu\|_\infty\le\|f\|_\infty$ and

$$\int_{-\infty}^{\infty}\|f_\nu(it)-f(it)\|_0 P(\theta,t)\,dt\to 0 \quad \text{as } \nu\to\infty.$$

(i) For $\nu>k$ and $j=0,1$, we have

$$f_\nu(j+it)-f_k(j+it) = \int_{-\infty}^{\infty}f(j+is)((\nu-k)\chi_{[t,t+1/\nu)}(s)-k\chi_{[t+1/\nu,t+1/k)}(s))\,ds.$$

This gives

$$\|f_\nu(j+it)-f_k(j+it)\|_j \le \int_{-\infty}^{\infty}|(\nu-k)\chi_{[t,t+1/\nu)}(s)-k\chi_{[t+1/\nu,t+1/k)}(s)|\,ds \le\frac{\nu-k}{\nu}\le 1.$$

It follows from Theorem 2.4(ii) that

$$\|x_\nu-x_k\|_{\theta(n)} \le c_2\Big(1-\sum_{j=0,1}\int_{-\infty}^{\infty}\log\|f_\nu(j+it)-f_k(j+it)\|_j P_j(\theta,t)\,dt\Big)^n \times\exp\Big(\sum_{j=0,1}\int_{-\infty}^{\infty}\log\|f_\nu(j+it)-f_k(j+it)\|_j P_j(\theta,t)\,dt\Big)$$

$$\le c_2' \exp\Big(\frac{1}{2}\sum_{j=0,1}\int_{-\infty}^{\infty}\log\|f_\nu(j+it)-f_k(j+it)\|_j P_j(\theta,t)\,dt\Big)$$

$$\le c_2' \exp\Big(\frac{1}{2}\int_{-\infty}^{\infty}\log\|f_\nu(it)-f_k(it)\|_0 P_0(\theta,t)\,dt\Big)$$

$$\le c_2' \Big(\int_{-\infty}^{\infty}\|f_\nu(it)-f_k(it)\|_0 P_0(\theta,t)\,\frac{dt}{1-\theta}\Big)^{(1-\theta)/2}.$$

Therefore (x_ν) is a Cauchy sequence in $C_{\theta(n)}(\overline{X})$, and so $x_\nu \to x$ in $C_{\theta(n)}(\overline{X})$ since $x_\nu \to x$ in $\Sigma\overline{X}$.

(ii) By passing to a subsequence if necessary, one can always assume that

$$\exp\Big(\frac{1}{2}\sum_{j=0,1}\int_{-\infty}^{\infty}\log\|f_\nu(j+it)-f_k(j+it)\|_j P_j(\theta,t)\,dt\Big) < 1/(c_2'\nu^2).$$

Thus by using Theorem 2.4(ii) again, one can find $h_\nu \in A^{\mathrm{b}}(\mathbb{S},\overline{X})$ such that $h_1^{(k)}(\theta) = f_1^{(k)}(\theta)$, $h_\nu^{(k)}(\theta) = f_\nu^{(k)}(\theta) - f_{\nu-1}^{(k)}(\theta)$ ($\nu \ge 2$, $k = 0,1,\ldots,n$) and $\|h_\nu\|_\infty < 1/\nu^2$. Set $g_\nu = \sum_{l=1}^{\nu} h_l$. Then (g_ν) is a Cauchy sequence, and hence a convergent sequence in $A^{\mathrm{b}}(\mathbb{S},\overline{X})$. The function $g = \lim_\nu g_\nu$ is as required. ∎

The following proposition extends some identification results for the C_θ functors obtained by Peetre (cf. [Pee]) to the case of the $C_{\theta(n)}$ functors, and its proof is directly obtained from Lemma 2.6. A Banach space X is said to satisfy the *Radon–Nikodym Property* (RNP in brief) if $\mathcal{L}(L^1(\mathbb{R}),X) = L^\infty(\mathbb{R},X)$.

2.7. PROPOSITION. (i) *Let $H^\infty(\mathbb{S},\overline{X})$ (resp. $H_n^\infty(\mathbb{S},\overline{X})$) be the subspace of $H_\infty(\mathbb{S},\overline{X})$ (resp. $H_{\infty,n}(\mathbb{S},\overline{X})$) which consists of all functions f in $H_\infty(\mathbb{S},\overline{X})$ (resp. $H_{\infty,n}(\mathbb{S},\overline{X})$) such that $f(j+it)\in X_j$ $(j=0,1)$. Then*

$$C_{\theta(n)}(\overline{X}) = \begin{cases} c_{\theta,n}H^\infty(\mathbb{S},\overline{X})/K_{\theta(n)} & \text{for } n>0,\\ H_n^\infty(\mathbb{S},\overline{X})/K_{\theta(0)} & \text{for } n<0.\end{cases}$$

(ii) *If either X_0 or X_1 has the RNP, then $C_{\theta(n)}(\overline{X}) = C^{\theta(n)}(\overline{X})$ for $n = \pm1,\pm2,\ldots$*

It is worth pointing out that if either X_0 or X_1 is reflexive, then so is the interpolation space $C_{\theta(n)}(\overline{X})$. This result was first stated by Schechter in [S, Th. 2.14].

3. Discretization. Now we employ Cwikel's idea of "periodic" complex interpolation (cf. [Cw]) to offer equivalent discrete norms for the complex $\theta(n)$-functors defined above. Assume $n>0$ for the moment. For $\gamma>0$, we consider the following subspaces of $A^{\mathrm{b}}(\mathbb{S},\overline{X})$, $A_n(\mathbb{S},\overline{X})$, $H_\infty(\mathbb{S},\overline{X})$ and

$H_{\infty,n}(\mathbb{S},\overline{X})$, respectively, which consist of analytic functions with period γ in the corresponding spaces. Specifically,

$$\begin{aligned} A^{\mathrm{b}}(\mathbb{S}_\gamma,\overline{X}) &:= \{f\in A^{\mathrm{b}}(\mathbb{S},\overline{X}) \mid f(z+i\gamma)=f(z) \text{ for } z\in \operatorname{int}\mathbb{S}\},\\ A_n^{\mathrm{b}}(\mathbb{S}_\gamma,\overline{X}) &:= \{f\in A_n^{\mathrm{b}}(\mathbb{S},\overline{X}) \mid f(z+i\gamma)=f(z) \text{ for } z\in \operatorname{int}\mathbb{S}\},\\ H_\infty(\mathbb{S}_\gamma,\overline{X}) &:= \{f\in H_\infty(\mathbb{S},\overline{X}) \mid f(z+i\gamma)=f(z) \text{ for } z\in \operatorname{int}\mathbb{S}\},\\ H_{\infty,n}(\mathbb{S}_\gamma,\overline{X}) &:= \{f\in H_{\infty,n}(\mathbb{S},\overline{X}) \mid f(z+i\gamma)=f(z) \text{ for } z\in \operatorname{int}\mathbb{S}\}.\end{aligned}\tag{3.1}$$

Observe that any f in each of these spaces has a Fourier series representation

$$f(z) \sim \sum_{\nu=-\infty}^{\infty} x_\nu e^{2\pi\nu z/\gamma},\tag{3.2}$$

where

$$x_\nu = \int_{-\gamma/2}^{\gamma/2} f(s+it)ze^{2\pi\nu(s+it)/\gamma}\,\frac{dt}{\gamma} \in \Delta\overline{X},$$

which is independent of $s\in[0,1]$. By identifying f with its Fourier series, f can also be considered as an analytic function defined on the annulus

$$\mathbb{S}_\gamma = \{z\in\mathbb{C} \mid 1\le |z| \le \exp(2\pi/\gamma)\}.$$

Replacing each of the Banach spaces $A^{\mathrm{b}}(\mathbb{S},\overline{X})$, $A_n^{\mathrm{b}}(\mathbb{S},\overline{X})$, $H_\infty(\mathbb{S},\overline{X})$ and $H_{\infty,n}(\mathbb{S},\overline{X})$ with its periodic subspace $A^{\mathrm{b}}(\mathbb{S}_\gamma,\overline{X})$, $A_n^{\mathrm{b}}(\mathbb{S}_\gamma,\overline{X})$, $H_\infty(\mathbb{S}_\gamma,\overline{X})$ and $H_{\infty,n}(\mathbb{S}_\gamma,\overline{X})$ respectively in (1.2) or (1.3) gives the periodic interpolation spaces $C_{\theta(\pm n);\gamma}(\overline{X})$ and $C^{\theta(\pm n);\gamma}(\overline{X})$. It is clear that

$$C_{\theta(\pm n);\gamma}(\overline{X}) \subseteq C_{\theta(\pm n)}(\overline{X}) \quad\text{and}\quad C^{\theta(\pm n);\gamma}(\overline{X}) \subseteq C^{\theta(\pm n)}(\overline{X}).$$

To establish the isomorphism between $C_{\theta(\pm n)}(\overline{X})$ and $C_{\theta(\pm n),\gamma}(\overline{X})$ (resp. $C^{\theta(\pm n)}(\overline{X})$ and $C^{\theta(\pm n),\gamma}(\overline{X})$), we introduce an auxiliary analytic function defined on $\mathbb{C}$ by

$$w(z) = \varphi(z)S_n(1/\varphi(z)),\tag{3.3}$$

where $\varphi(z) = \big(\frac{e^z-1}{z}\big)^{n+1}\exp(2z^2)$. It follows from Lemma 2.1(i) that $w(0) = 1$ and $w^{(k)}(0) = 0$ for $k = 1,\ldots,n$. Moreover, $w^{(k)}(2\pi i\nu) = 0$ for $k = 0,1,\ldots,n$ and $\nu = \pm1,\pm2,\ldots$; also

$$|w(s+it)| \le \begin{cases} e^{2(s^2-t^2)}\Big(\dfrac{\sqrt{2(1-\cos s)}}{|s|}\Big)^{n+1}\displaystyle\sum_{k=0}^{n}\frac{|\varphi^{(k)}(0)|}{k!}(s^2+t^2)^{k/2} & \text{if } s\ne0,\\ e^{-2t^2}\displaystyle\sum_{k=0}^{n}\frac{|\varphi^{(k)}(0)|}{k!}|t|^k & \text{if } s=0.\end{cases}$$

For $\gamma > 0$, let

$$w_{\theta,\gamma}(z) = w\left(\frac{z-\theta}{2\pi\gamma}\right). \tag{3.4}$$

Then

$$w_{\theta,\gamma}(\theta) = 1, \quad w_{\theta,\gamma}^{(k+1)}(\theta) = 0 \quad \text{and} \quad w_{\theta,\gamma}^{(k)}(\theta + i\nu\gamma) = 0 \tag{3.5}$$

for $k = 0, 1, \ldots, n$ and $\nu = \pm 1, \pm 2, \ldots$ Moreover,

$$|w_{\theta,\gamma}(j+it)| \le c_1 \exp(-c_2 t^2) \quad (j = 0, 1),$$

where c_1 and c_2 are positive numbers depending on θ and γ only.

Let $A = A^{\mathrm{b}}(\mathbb{S}, \overline{X})$, $A_n^{\mathrm{b}}(\mathbb{S}, \overline{X})$, $H_\infty(\mathbb{S}, \overline{X})$ or $H_{\infty,n}(\mathbb{S}, \overline{X})$, and let B be the corresponding periodic space. For $f \in A$, we define $F \in A$ and $G \in B$ by

$$F(z) = w_{\theta,\gamma}(z) f(z) \quad \text{and} \quad G(z) = \sum_{\nu=-\infty}^{\infty} F(z + i\nu\gamma)$$

on $\operatorname{int}\mathbb{S}_\gamma$. On the boundary of $\mathbb{S}_\gamma$, F and G are defined in the usual way if $f \in A^{\mathrm{b}}(\mathbb{S}, \overline{X})$ or $A_n^{\mathrm{b}}(\mathbb{S}, \overline{X})$, and in the sense of operators if $f \in H_\infty(\mathbb{S}, \overline{X})$ or $H_{\infty,n}(\mathbb{S}, \overline{X})$. Then $G^{(k)}(\theta) = F^{(k)}(\theta) = f^{(k)}(\theta)$ for $k = 0, 1, \ldots, n$ by (3.5). We can then formulate

3.6. THEOREM. *$C_{\theta(n)} \simeq C_{\theta(n),\gamma}$ and $C^{\theta(n)} \simeq C^{\theta(n),\gamma}$, where $n \in \mathbb{Z}$.*

It is sometimes convenient for us to use these periodic complex $\theta(n)$-functors rather than the ordinary ones. An important application comes in the final section.

4. Connection with real interpolation and other functors. Let us first look at the following function parameters, which are quasi-powers with a logarithmic factor, and at the corresponding real and other interpolation functors, closely related to the complex $\theta(n)$-functors. Let

$$\varrho_{\theta,\eta}(t) = t^\theta \left(1 + \frac{\theta(1-\theta)}{|\eta|}|\log t|\right)^\eta \quad \text{for } t \in \mathbb{R}_+ \tag{4.1}$$

with $0 < \theta < 1$, $\eta \in \mathbb{R} \setminus \{0\}$. A positive function ϱ on $\mathbb{R}_+$ is called *quasi-concave* if $\varrho(st) \le (1 \vee t)\varrho(s)$ for $s, t \in \mathbb{R}_+$. It generates three other quasi-concave functions on $\mathbb{R}_+$: ϱ^* (involution), ϱ^τ (transposition) and $\overline{\varrho}$, given by

$$\varrho^*(t) = 1/\varrho(1/t), \quad \varrho^\tau(t) = t\varrho(1/t), \quad \overline{\varrho}(t) = \sup_{s>0} \frac{\varrho(st)}{\varrho(s)}.$$

The induced homogeneous function of two variables, still denoted by ϱ, is $\varrho(t_0, t_1) = t_0 \varrho(t_1/t_0)$ with involution

$$\varrho^*(t_0, t_1) = 1/\varrho(1/t_0, 1/t_1) = t_0/\varrho(t_0/t_1).$$

A quasi-concave function ϱ is called a *quasi-power* if $\overline{\varrho}(t) \prec t^\varepsilon \vee t^{1-\varepsilon}$ on $\mathbb{R}_+$ for some $\varepsilon \in (0,1)$. It is clear that $\varrho_{\theta,\eta}$ is a quasi-power such that

$$t \wedge 1 \le \varrho_{\theta,\eta}(t) \le t \vee 1, \quad \varrho_{\theta,\eta}^* = \varrho_{\theta,-\eta}, \quad \varrho_{\theta,\eta}^\tau = \varrho_{1-\theta,\eta}, \quad \overline{\varrho}_{\theta,\eta} = \varrho_{\theta,|\eta|},$$

$$\varrho_{\theta,\eta}(t_0, t_1) = t_0^{1-\theta} t_1^\theta \left(1 + \frac{\theta(1-\theta)}{|\eta|}\left|\log \frac{t_1}{t_0}\right|\right)^\eta = \lambda^{-\theta} \varrho_{\theta,\eta}(\lambda t_0, \lambda t_1).$$

We write $\varrho_{\theta,0}(t) = \varrho_\theta(t) = t^\theta$ for simplification.

For a positive function ϱ (called a *weight*) on $\mathbb{R}_+$, we denote by l_ϱ^p the real bilateral l^p-space with weight $(1/\varrho(2^\nu))_{\nu\in\mathbb{Z}}$ when $1 \le p \le \infty$ and set

$$l_\varrho^0 = \{(\alpha_\nu) \in l_\varrho^p \mid \alpha_\nu/\varrho(2^\nu) \to 0 \text{ as } t \to \pm\infty\}.$$

Let $l_{(\theta,\eta)}^p = l_{\varrho_{\theta,\eta}}^p$. For $p = 0, 1$ or ∞ in particular, we write $l_j^p = l_{\varrho_j}^p$ with $\varrho_j(t) = t^j$ $(j = 0, 1)$ and define the Banach couples $\overline{l^p}$ by $\overline{l^p} = (l_0^p, l_1^p)$. According to [J, Sec. 4], the real and some other interpolation functors currently in use in the sense of Kalugina, Peetre, Peetre–Gustavsson and Ovchinnikov related to the function parameters $\varrho_{\theta,\eta}$ can be interpreted as the following orbit and coorbit functors:

$$\begin{aligned} &J_{(\theta,\eta)}^p = \mathrm{Orb}_{l_{(\theta,\eta)}^p}(\overline{l^1}, -), \qquad K_{(\theta,\eta)}^p = \mathrm{Corb}_{l_{(\theta,\eta)}^p}(-, \overline{l^\infty}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad (p = 0 \text{ or } 1 \le p \le \infty); \\ &G_{(\theta,\eta)}^0 = \mathrm{Orb}_{l_{(\theta,\eta)}^0}(\overline{l^0}, -), \qquad G_{(\theta,\eta)} = \mathrm{Orb}_{l_{(\theta,\eta)}^\infty}(\overline{l^0}, -); \\ &G_{(\theta,\eta)}^\infty = \mathrm{Orb}_{l_{(\theta,\eta)}^\infty}(\overline{l^\infty}, -), \qquad H^{(\theta,\eta)} = \mathrm{Corb}_{l_{(\theta,n)}^1}(-, \overline{l^1}). \end{aligned} \tag{4.2}$$

For the real functors $J_{(\theta,\eta)}^p$ and $K_{(\theta,\eta)}^p$, it is known

$$J_{(\theta,\eta)}^p \simeq K_{(\theta,\eta)}^p \quad (p = 0 \text{ or } 1 \le p \le \infty), \tag{4.3}$$

and

$$J_{(\theta,\eta)}^p(\overline{X})' = K_{(\theta,-\eta)}^{p'}(\overline{X}'), \quad K_{(\theta,\eta)}^p(\overline{X})' \simeq J_{(\theta,-\eta)}^{p'}(\overline{X}') \quad (p = 0 \text{ or } 1 \le p < \infty)$$

for any regular Banach couple $\overline{X}$. If $\eta_1 < \eta_2$, then $J_{(\theta,\eta_1)}^p \prec J_{(\theta,\eta_2)}^p$ and $K_{(\theta,\eta_1)}^\infty \prec J_{(\theta,\eta_2)}^1$ since

$$\frac{\varrho_{\theta,\eta_1}(t)}{\varrho_{\theta,\eta_2}(t)} \simeq \left(1 + \frac{\theta(1-\theta)}{\eta_2 - \eta_1}|\log t|\right)^{\eta_1-\eta_2}$$

and so $\varrho_{\theta,\eta_1}/\varrho_{\theta,\eta_2} \in L_0^\infty \cap L_0^1$. Furthermore, if $0 < \theta_0, \theta_1, \theta < 1$ with $\theta_0 \ne \theta_1$, $\alpha = (1-\theta)\theta_0 + \theta\theta_1$ and $\zeta = (1-\theta)\eta_0 + \theta\eta_1$, then

$$\varrho_{\alpha,\eta+\zeta} \simeq \varrho_{\theta,\eta}(\varrho_{\theta_0,\eta_0}, \varrho_{\theta_1,\eta_1}). \tag{4.4}$$

This, together with [J, Th. 19], implies the following reiteration result:

$$K_{(\alpha,\eta+\zeta)}^p(K_{(\theta_0,\eta_0)}^{p_0}, K_{(\theta_1,\eta_1)}^{p_1}) \simeq K_{(\theta,\eta)}^p \quad (1 \le p_0, p_1, p \le \infty). \tag{4.5}$$

Consequently,

$$K^{p_0}_{(\theta_0,\eta_0)}(\overline{X}) \cap K^{p_1}_{(\theta_1,\eta_1)}(\overline{X}) \prec J^p_{(\alpha,\eta+\zeta)}(\overline{X}) \quad \text{for } \overline{X} \in \overline{BC}. \tag{4.6}$$

Since the characteristic functions of all functors described in (4.2) are equivalent to $\varrho_{\theta,\eta}$ [J, Sec. 6], and since $C_{\theta(n)}(\overline{l^0}) \simeq l^0_{(\theta,n)}$ and $C^{\theta(n)}(\overline{l^1}) \simeq l^1_{(\theta,n)}$ [FK, Ex. 6.7], we obtain, by using the minimality/maximality of the orbit/coorbit functors, the following inclusion diagram for $1 < p < \infty$:

$$\begin{array}{ccccccccc}
J^1_{(\theta,n)} & \longrightarrow & G^0_{(\theta,n)} & \longrightarrow & G^\infty_{(\theta,n)} & \longrightarrow & G_{(\theta,n)} & \longrightarrow & (G^0_{(\theta,n)})^c \\
\Big| & & \downarrow & & & & \downarrow & & \downarrow \\
\Big| & & C_{\theta(n)} & & \longrightarrow & & C^{\theta(n)} & \longrightarrow & C^c_{\theta(n)} \\
\Big| & & \downarrow & & & & & & \downarrow \\
\Big| & & (H^{(\theta,n)})^0 & & & \longrightarrow & & & H^{(\theta,n)} \\
\downarrow & & \downarrow & & & & & & \downarrow \\
J^p_{(\theta,n)} & \longrightarrow & K^0_{(\theta,n)} & & & \longrightarrow & & & K^\infty_{(\theta,n)}
\end{array} \tag{4.7}$$

where the arrows denote continuous inclusions. This diagram generalizes the last result of [J], enriches [CCS, Prop. 5] and will be used in this and the next section.

There are several equivalent forms of $\varrho_{\theta,n}$ appearing in the literature. For instance, Carro, Cerdà and Sueiro used the function parameter $t^\theta(n + |\log t|)^n$ to study the relationship of the complex interpolation with derivatives and the corresponding real interpolation. One of the motivations for our choice of $\varrho_{\theta,n}$ is that the complex $\theta(n)$-functor is of interpolation type

$$t^\theta\left(1 + \frac{2\sin\pi\theta}{\pi}|\log t|\right)^n,$$

while $\frac{2\sin\pi\theta}{\pi} \simeq \theta(1-\theta)$ [FK, Th. 5.6]. Hence $\varrho_{\theta,n}$ preserves all information about the parameter θ and includes the classical case when $n = 0$. In fact, all functors in (4.2) are of interpolation type $\varrho_{\theta,|\eta|}$. See [Per, Th. 2.2] for the real functors $J^p_{(\theta,\eta)}$ and $K^p_{(\theta,\eta)}$, and [GP, Prop. 6.1] for the functors $G^0_{(\theta,\eta)}$ and $G_{(\theta,\eta)}$. The type of the functors $G^\infty_{(\theta,\eta)}$ and $H^{(\theta,\eta)}$ is deduced from the following proposition.

4.8. PROPOSITION. *Assume ϱ is a quasi-power function. Let $G^\infty_\varrho = \mathrm{Orb}_{l^\infty_\varrho}(\overline{l^\infty}, -)$ and $H^\varrho = \mathrm{Corb}_{l^1_\varrho}(-, \overline{l^1})$. Then the functors G^∞_ϱ and H^ϱ are of interpolation type $\overline{\varrho}$.*

Proof. Let $\overline{A} = (l^\infty_{1/\varrho}, l^\infty_{\varrho_1/\varrho})$, let A be the one-dimensional subspace of l^∞_0 generated by $e = (1)$ and let $F^\infty_\varrho = \mathrm{Orb}_A(\overline{A}, -)$. According to [J, Th. 3], we have $G^\infty_\varrho \simeq F^\infty_\varrho$. For all $\overline{X}, \overline{Y} \in \overline{BC}$, and for all $T \in \mathcal{L}(\overline{X}, \overline{Y})$, we intend

to show that

$$\|T\|_{F^\infty_\varrho(\overline{X}), F^\infty_\varrho(\overline{Y})} \le 2\overline{\varrho}(\|T\|_0, \|T\|_1), \qquad \|T\|_{H^\varrho(\overline{X}), H^\varrho(\overline{Y})} \le 2\overline{\varrho}(\|T\|_0, \|T\|_1).$$

Suppose $e_\nu = (e_{\nu k})^\infty_{k=-\infty}$ with $e_{\nu k} = 1$ if $\nu \ne k$ and $e_{\nu k} = 0$ if $\nu = k$, $\nu = 0, \pm1, \pm2, \ldots$ For any $S \in \mathcal{L}(\overline{A}, \overline{X})$, we have $Se = (Se_\nu)$ and

$$\|S\|_j = \sup_{|\lambda_\nu| \le 1} \Big\| \sum_\nu \frac{2^{\nu j}\lambda_\nu}{\varrho(2^\nu)} Se_\nu \Big\|_j \qquad (j = 0, 1).$$

We define $U \in \mathcal{L}(\overline{A})$ by $Ue_\nu = e_{\nu+\kappa}$ with

$$\kappa = \left[\frac{\log \|T\|_1/\|T\|_0}{\log 2}\right],$$

so $2^\kappa \le \|T\|_1/\|T\|_0 \le 2^{\kappa+1}$, $\|U\| = 1$ and $Ue = e$. For any $x \in F^\infty_\varrho(\overline{X})$ with $\|x\|_{F^\infty_\varrho} < 1$, choose $S \in \mathcal{L}(\overline{A}, \overline{X})$ with $x = Se$ and $\|S\| < 1$. Then $Tx = TSUe$. For $\lambda = (\lambda_\nu) \in l^\infty_0$, put $\zeta_\nu = \lambda_{\nu-\kappa}\varrho(2^\nu)/\varrho(2^{\nu-\kappa})$. Then $|\zeta_\nu| \le \overline{\varrho}(2^\kappa)|\lambda_\nu|$. Further, we have

$$\begin{aligned}
\Big\| \sum_\nu \frac{2^{\nu j}\lambda_\nu}{\varrho(2^\nu)} TSUe_\nu \Big\|_j &\le \|T\|_j \Big\| \sum_\nu \frac{2^{\nu j}\zeta_{\nu+\kappa}}{\varrho(2^{\nu+\kappa})} Se_{\nu+\kappa} \Big\|_j \\
&\le 2^{-j\kappa}\|T\|_j \Big\| \sum_\nu \frac{2^{\nu j}\zeta_\nu}{\varrho(2^\nu)} Se_\nu \Big\|_j \\
&\le 2\|T\|_0 \overline{\varrho}(2^\kappa) \le 2\overline{\varrho}(\|T\|_0, \|T\|_1)
\end{aligned}$$

and hence $Tx \in F^\infty_\varrho(\overline{Y})$ with $\|T\|_{F^\infty_\varrho} \le \|TSU\|_{\overline{l^\infty(\varrho)}, \overline{Y}} \le 2\overline{\varrho}(\|T\|_0, \|T\|_1)$.

For the coorbit functor H^ϱ, we have $S \in \mathcal{L}(\overline{X}, \overline{l^1})$ iff $Sx = (\langle x'_\nu, x\rangle)^\infty_{\nu=-\infty}$, where $x'_\nu \in \overline{X}'$ with

$$\|S\|_j = \sup\Big\{ \sum_\nu \frac{|\langle x'_\nu, x\rangle|}{2^{\nu j}} \;\Big|\; x \in X_j \text{ with } \|x\|_j \le 1 \Big\} \qquad (j = 0, 1).$$

For any $V \in \mathcal{L}(\overline{Y}, \overline{l^1})$ with $\|V\| \le 1$, let $Vy = (\langle y'_\nu, y\rangle)_\nu$ with $y'_\nu \in \overline{Y}'$. We define $S \in \mathcal{L}(\overline{X}, \overline{l^1})$ by setting $Sx = (\langle x'_\nu, x\rangle)_\nu$, where $\langle x'_\nu, x\rangle = \langle y'_{\nu-\kappa}, Tx\rangle \varrho(2^\nu)/\varrho(2^{\nu-\kappa})$ with κ as before. Observe that

$$\begin{aligned}
\|Sx\|_j &= \sum_\nu \frac{|\langle x'_\nu, x\rangle|}{2^{\nu j}} = \sum_\nu \frac{|\langle y'_{\nu-\kappa}, Tx\rangle| \varrho(2^\nu)}{2^{\nu j}\varrho(2^{\nu-\kappa})} \\
&\le 2^{-\kappa j}\overline{\varrho}(2^\kappa)\|VTx\|_j \le 2\overline{\varrho}(\|T\|_0, \|T\|_1)\|x\|_j \qquad (j = 0, 1).
\end{aligned}$$

Now if $x \in H^\varrho(\overline{X})$, then

$$\begin{aligned}
\|VTx\|_{l^1_\varrho} &= \sum_\nu \frac{|\langle y'_\nu, Tx\rangle|}{\varrho(2^\nu)} = \sum_\nu \frac{|\langle x'_{\nu+\kappa}, x\rangle|}{\varrho(2^{\nu+\kappa})} \\
&= \|Sx\|_{l^1_\varrho} \le \|S\|\|x\|_{H^\varrho} \le 2\overline{\varrho}(\|T\|_0, \|T\|_1)\|x\|_{H^\varrho}
\end{aligned}$$

and hence $Tx \in H^{\varrho}(\overline{Y})$ with $\|Tx\|_{H^\varrho} \le 2\overline{\varrho}(\|T\|_0, \|T\|_1)$ by the construction of the coorbit functor. ■

5. Reiteration and Calderón–Lozanovskiĭ construction. Our first result concerning reiteration for the complex $\theta(n)$-functors is as follows.

5.1. THEOREM (Partial reiteration). *Suppose* $0 \le \theta_0, \theta_1, \theta \le 1$, $\theta_0 \ne \theta_1$, $\alpha = (1-\theta)\theta_0 + \theta\theta_1$ *and* $kn \ge 0$. *Then*

$$C_{\alpha(n+k)} \prec C_{\theta(n)}(C_{\theta_0(k)}, C_{\theta_1(k)}) \quad \textit{for } k \ge 0,$$
$$C_{\alpha(n+k)} \succ C_{\theta(n)}(C_{\theta_0(k)}, C_{\theta_1(k)}) \quad \textit{for } k \le 0.$$

Proof. Because of the regularity in (1.5), we may assume $\overline{X}$ to be a regular Banach couple. It is known from [FK, Remark 5.8] that both inclusions hold for $k = 0$, so we may assume $k \ne 0$. Set $Y_j = C_{\theta_j(k)}(\overline{X})$ for $j = 0, 1$ and set $\overline{Y} = (Y_0, Y_1)$.

(i) If $k > 0$ and so $n \ge 0$, we show $C_{\alpha(n+k)}(\overline{X}) \prec C_{\theta(n)}(\overline{Y})$. In fact, for all $x \in C_{\alpha(n+k)}(\overline{X})$ and for all $\varepsilon > 0$, there exists $f \in A^{\mathrm{b}}(\mathbb{S}, \overline{X})$ such that $x = c_{\alpha,n+k} f^{(n+k)}(\alpha)$ and $\|f\|_\infty \le (1+\varepsilon)\|x\|_{\alpha(n+k)}$. Let

$$g(z) = f^{(k)}((1-z)\theta_0 + z\theta_1).$$

Then $g \in A^{\mathrm{b}}(\mathbb{S}, \overline{Y})$ since

$$g(j+it) = f^{(k)}((1-j)\theta_0 + j\theta_1 + it(\theta_1 - \theta_0)) \in C_{\theta_j(k)}(\overline{X}) = Y_j,$$

and $g^{(n)}(\theta) = (\theta_1 - \theta_0)^n f^{(n+k)}(\alpha)$. This implies that

$$x = \frac{c_{n+k,\alpha}}{(\theta_1 - \theta_0)^n} g^{(n)}(\theta) \in C_{\theta(n)}(\overline{Y})$$

and hence $\|x\|_{\theta(n)} \le c(1+\varepsilon)\|x\|_{\alpha(n+k)}$, where $c = |c_{\alpha,n+k}/(c_{\theta,n}(\theta_1 - \theta_0)^n)|$. This gives $\|x\|_{\theta(n)} \le c\|x\|_{\alpha(n+k)}$ and so

$$C_{\alpha(n+k)}(\overline{X}) \prec C_{\theta(n)}(C_{\theta_0(k)}(\overline{X}), C_{\theta_1(k)}(\overline{X})).$$

The "superscript" inclusion $C^{\alpha(n+k)}(\overline{X}) \prec C^{\theta(n)}(C^{\theta_0(k)}(\overline{X}), C^{\theta_1(k)}(\overline{X}))$ can be obtained with trivial modifications.

(ii) If $k < 0$ and so $n \le 0$, we invoke (i) and the duality argument to show that $C_{\alpha(n+k)}(\overline{X}) \succ C_{\theta(n)}(\overline{Y})$. By using (4.6), (4.7) and [FK, Sec. 3], we obtain the following inclusion chain:

$$\begin{aligned}\Delta\overline{X} &\prec C_{\theta_0(k)}(\overline{X}) \cap C_{\theta_1(k)}(\overline{X}) \prec K^{\infty}_{(\theta_0,k)}(\overline{X}) \cap K^{\infty}_{(\theta_1,k)}(\overline{X}) \\ &\prec J^1_{(\alpha,k)}(\overline{X}) \prec C_{\alpha(k)}(\overline{X}) \prec C_{\alpha(n+k)}(\overline{X}),\end{aligned}$$

and hence $\Delta\overline{Y} = C_{\theta_0(k)}(\overline{X}) \cap C_{\theta_1(k)}(\overline{X})$ is dense in both $C_{\theta(n)}(\overline{Y})$ and $C_{\alpha(n+k)}(\overline{X})$. For every $y \in \Delta\overline{Y} \prec C_{\alpha(n+k)}(\overline{X})$, there exists $x' \in C_{\alpha(n+k)}(\overline{X})'$ $\simeq C^{\alpha(-(n+k))}(\overline{X}')$ with $\langle x', y\rangle = \|y\|_{\alpha(n+k)}$ and $\|x'\|_{\alpha(-(n+k))} \le c$ by the Hahn–Banach theorem. Since

$$C^{\alpha(-(n+k))}(\overline{X}') \prec C^{\theta(-n)}(C^{\theta_0(-k)}(\overline{X}'), C^{\theta_1(-k)}(\overline{X}')) \simeq C^{\theta(-n)}(\overline{Y}')$$

from (i), this gives $\|y\|_{\alpha(n+k)} = \langle x', y\rangle \le \|x'\|_{\theta(-n)}\|y\|_{\theta(n)} \le c\|y\|_{\theta(n)}$ as required. ■

Assume $\overline{X} = (X_0, X_1)$ is a couple of function Banach lattices on a complete σ-finite measure space $(\Omega, \mathcal{A}, \mu)$. Let us recall that the Calderón–Lozanovskiĭ construction $\varrho_{\theta,n}(\overline{X})$ for the function parameter $\varrho_{\theta,n}$ is the space of all $\mathcal{A}$-measurable functions x such that $|x| \le c\varrho_{\theta,n}(|x_0|, |x_1|)$ μ-a.e. for some $x_j \in X_j$ with $\|x_j\|_j \le 1$ $(j = 0, 1)$, and put $\|x\|_{\varrho_{\theta,n}(\overline{X})} = \inf c$. In terms of [N, Th. 3.1] and [KMP, Th. 5], we have

$$G^0_{(\theta,n)}(\overline{X}) \simeq \varrho_{\theta,n}(\overline{X})^0 \simeq H^{(\theta,n)}(\overline{X})^0,$$
$$G^0_{(\theta,n)}(\overline{X})^{\mathrm{c}} \simeq \varrho_{\theta,n}(\overline{X})^{\mathrm{c}} \simeq H^{(\theta,n)}(\overline{X}^0).$$

This, together with (4.7), implies

5.2. THEOREM (Calderón–Lozanovskiĭ construction).

$$C_{\theta(n)}(\overline{X}) \simeq \varrho_{\theta,n}(\overline{X})^0 \quad \textit{and} \quad C_{\theta(n)}(\overline{X})^{\mathrm{c}} \simeq \varrho_{\theta,n}(\overline{X})^{\mathrm{c}}.$$

This construction may be regarded as a "real version" of the complex $\theta(n)$-functors and it extends the classical result of Calderón. As a consequence, the preceding partial reiteration for complex functors becomes reiteration when restricted to couples of Banach lattices in view of [N, Th. 3.5].

Now we can modify [N, Ex. 5.3] and deal with interpolation of weighted L^p-spaces, which extends [FK, Ex. 6.8], in the following way:

5.3. EXAMPLE. let $\overline{\omega} = (\omega_0, \omega_1)$ be a pair of weights on the complete σ-finite measure space $(\Omega, \mathcal{A}, \mu)$, and let $1 \le p_0, p_1 \le \infty$ with $p_0 \ne p_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = 1/p_0 - 1/p_1$. Let $L^{p_j}(\omega_j)$ be the weighted L^{p_j}-space on $(\Omega, \mathcal{A}, \mu)$ with weight ω_j $(j = 0, 1)$. By [N, Ex. 5.3], $x \in G^0_{(\theta,\eta)}(L^{p_0}(\omega_0), L^{p_1}(\omega_1))$ iff

$$\int_{\Omega} \Psi((\omega_1^{1/p_0}/\omega_0^{1/p_1})^q |x|/c)(\omega_0/\omega_1)^q \, d\mu \le 1 \quad \text{for some } c > 0,$$

where

$$\Psi^{-1}(t) = t^{1/p}\varrho_{\theta,\eta}(t^{-1/q}) = t^{1/p}\left(1 + \frac{\theta(1-\theta)}{q|\eta|}|\log t|\right)^{\eta}.$$

Let $\Phi_{p,\eta}$ be the normalized Young function given by

$$\Phi_{p,\eta}(t) = \int_0^t \tau^{p/p'}\left(1 + \frac{|\log\tau|}{p(p')^2|\eta|}\right)^{-\eta p} d\tau$$

and let

$$\widetilde{\Phi}_{p,\eta}(t) = \frac{t^p}{p}\left(1 + \frac{|\log t|}{p(p')^2|\eta|}\right)^{-\eta p}.$$

It is known that $\widetilde{\Phi}_{p,\eta}$ is strictly increasing and equivalent to $\Phi_{p,\eta}$ (cf. [FK, Sec. 6]). Observe that

$$\begin{aligned}\Psi^{-1}(\widetilde{\Phi}_{p,\eta})(t) \\ &= \frac{t}{\sqrt[p]{p}}\left(1 + \frac{|\log t|}{p(p')^2|\eta|}\right)^{-\eta} \\ &\quad \times \left(1 + \frac{p\theta(1-\theta)}{q|\eta|}\left|\frac{1}{p}\log\frac{1}{p} + \log t - \eta\log\left(1 + \frac{|\log t|}{p(p')^2|\eta|}\right)\right|\right)^{\eta} \simeq t.\end{aligned}$$

This shows $\Psi^{-1} \circ \widetilde{\Phi}_{p,\eta}(t) \simeq t$ and hence $\Psi \simeq \widetilde{\Phi}_{p,\eta} \simeq \Phi_{p,\eta}$. Therefore

$$G^0_{(\theta,\eta)}(L^{p_0}(\omega_0), L^{p_1}(\omega_1)) \simeq L_{\Phi_{p,\eta}(\omega_1^{q/p_0}/\omega_0^{q/p_1})}(\omega_0^q/\omega_1^q) \equiv \Phi,$$

where Φ is the weighted Orlicz space over $(\Omega, \mathcal{A}, \mu)$ with the Luxemburg norm

$$\|x\|_\Phi = \inf\left\{c > 0 \,\middle|\, \int_\Omega \Phi_{p,\eta}((\omega_1^{1/p_0}/\omega_0^{1/p_1})^q|x|/c)(\omega_0/\omega_1)^q\, d\mu \le \Phi_{p,\eta}(1)\right\}.$$

In particular,

$$C_{\theta(n)}(L^{p_0}(\omega_0), L^{p_1}(\omega_1)) \simeq L_{\Phi_{p,n}(\omega_1^{q/p_0}/\omega_0^{q/p_1})}(\omega_0^q/\omega_1^q).$$

Furthermore, we have

$$(1 \wedge 2^\nu)_\nu \in G^0_{(\theta,\eta)}(L^{p_0}(\omega_0), L^{p_1}(\omega_1)) \simeq \Phi$$

and hence the Banach lattice Φ determines a real interpolation functor K_Φ given by

$$K_\Phi(\overline{X}) = \{x \in \Sigma\overline{X} \mid \|x\|_{K_\Phi} = \|K(\cdot, x; \overline{X})\|_\Phi < \infty\}$$

in the sense of [BK, 2.6.4]. ■

The proof of the following mixed reiteration result combines the general results (cf. [J, Th. 19] and [BK, Th. 4.3.1]) with direct calculations on the crucial couples $\overline{l^1}$.

5.4. THEOREM (Mixed reiteration). *Let $\overline{X} \in \overline{\mathcal{B}}$ and let $0 < \theta_0, \theta_1, \theta < 1$ with $\theta_0 \ne \theta_1$.*

(i) *If $1 \le p \le \infty$ or $p = 0$, $\alpha = (1-\theta)\theta_0 + \theta\theta_1$, $\kappa = (1-\theta)n_0 + \theta n_1$ and $\zeta = (1-\theta)\eta_0 + \theta\eta_1$, then*

$$K^p_{(\alpha,\eta+\kappa)}(\overline{X}) \simeq K^p_{(\theta,\eta)}(C_{\theta_0(n_0)}(\overline{X}), C_{\theta_1(n_1)}(\overline{X})),$$
$$K^p_{(\alpha,n+\zeta)}(\overline{X}) \simeq C_{\theta(n)}(K^p_{(\theta_0,\eta_0)}(\overline{X}), K^p_{(\theta_1,\eta_1)}(\overline{X})).$$

(ii) *If $1 \le p_0, p_1 \le \infty$ with $p_0 \ne p_1$, $1/p = (1-\theta_0)/p_0 + \theta/p_1$ and $1/q = 1/p_0 - 1/p_1$, then*

$$K_\Phi(\overline{X}) \simeq C_{\theta(n)}(K^{p_0}_{(\theta_0,\eta_0)}(\overline{X}), K^{p_1}_{(\theta_1,\eta_1)}(\overline{X})),$$

with

$$\Phi = l_{\Phi_{p,n}(\varrho^{q/p_1}_{\theta_0,\eta_0}/\varrho^{q/p_0}_{\theta_1,\eta_1})}(\varrho^q_{\theta_1,\eta_1}/\varrho^q_{\theta_0,\eta_0}).$$

Proof. (i) The first identity follows from the diagram (4.7) and the real reiteration (4.5). For the other identity, observe that

$$\varrho^*_{\theta,\eta}(1/\varrho_{\theta_0,\eta_0}, 1/\varrho_{\theta_1,\eta_1}) = 1/\varrho_{\theta,\eta}(\varrho_{\theta_0,\eta_0}, \varrho_{\theta_1,\eta_1}) \simeq 1/\varrho_{\alpha,\eta+\zeta}$$

by (4.4). This gives $C_{\theta(n)}(l^p_{(\theta_0,\eta_0)}, l^p_{(\theta_1,\eta_1)}) \simeq G_{(\theta,n)}(l^p_{(\theta_0,\eta_0)}, l^p_{(\theta_1,\eta_1)}) \simeq l^p_{\alpha,\eta+\zeta}$ by Theorem 5.2 and [G, Th. 2.3]. Recalling the equivalence of the $J^p_{(\theta,\eta)}$ and $K^p_{(\theta,\eta)}$ functors (see (4.3)) and adapting an argument of [BK, Th. 4.3.1], we get the assertion.

(ii) Since $C_{\theta(n)}(K^{p_0}_{(\theta_0,\eta_0)}, K^{p_1}_{(\theta_1,\eta_1)}) \simeq K_{C_{\theta(n)}(l^{p_0}_{(\theta_0,\eta_0)}, l^{p_1}_{(\theta_1,\eta_1)})}$, again by [BK, Th. 4.3.1], and $C_{\theta(n)}(l^{p_0}_{(\theta_0,\eta_0)}, l^{p_1}_{(\theta_1,\eta_1)}) \simeq \Phi$ by Example 5.3, the assertion follows. ■

6. Minimality/maximality. In this final section, we study the minimality/maximality of the complex $\theta(n)$-functors within the framework of the Aronszajn–Gagliardo construction.

Let $L^1(\mathbb{T})$, $L^\infty(\mathbb{T})$, $K(\mathbb{T})$ and $M(\mathbb{T})$ (L^1, L^∞, K and M in brief) be Banach spaces of all integrable functions, essentially bounded functions, continuous functions and finite measures on the unit circle $\mathbb{T}$ respectively. For $X = L^1, L^\infty$, K or M and $j = 0, 1$, we define the sequence Banach spaces $\mathcal{F}X_j$ related to the Fourier transforms by

$$\mathcal{F}X_j := \left\{\lambda = (\lambda_\nu)_{\nu\in\mathbb{Z}} \,\middle|\, \exists\phi_j \in X,\ \mathcal{F}\phi_j(\nu) = \int_{\mathbb{T}} \phi_j(t)e^{-\nu i t}\frac{dt}{2\pi} = \lambda_\nu e^{-\nu j}\right\}$$

with norm $\|\lambda\|_{\mathcal{F}X_j} = \|\phi_j\|_X$; and consider the corresponding Banach couples $\overline{\mathcal{F}} = (\mathcal{F}X_0, \mathcal{F}X_1)$. For $n \in \mathbb{Z}$ and X given above, let

$$\mathcal{F}X_{(\theta,n)} := \left\{\lambda = (\lambda_\nu)_{\nu\in\mathbb{Z}} \,\middle|\, \exists\phi_{\theta,n} \in X,\ \mathcal{F}\phi_{\theta,n}(\nu) = \begin{cases}\lambda_\nu e^{-\nu\theta}/\nu^n & \text{if } \nu \ne 0\\ 0 & \text{if } \nu = 0\end{cases}\right\}$$

with norm $\|\lambda\|_{\mathcal{F}X_{(\theta,n)}} = \|\phi_{\theta,n}\|_X + |\lambda_0|$. Observe that

$$\begin{aligned}(\overline{\mathcal{F}M})^0 = \overline{\mathcal{F}L^1}, \quad (\overline{\mathcal{F}L^1})' = \overline{\mathcal{F}L^\infty}, \quad (\overline{\mathcal{F}K})' = \overline{\mathcal{F}M};\\ (\mathcal{F}L^1_{(\theta,n)})' = \mathcal{F}L^\infty_{(\theta,-n)}, \quad (\mathcal{F}K_{(\theta,n)})' = \mathcal{F}M_{(\theta,-n)}.\end{aligned} \tag{6.1}$$

The first formula goes back to an argument in [J, Th. 2.1], while the others come from the duality relations $L^1(\mathbb{T})' = L^\infty(\mathbb{T})$ and $K(\mathbb{T})' = M(\mathbb{T})$. Now we compute the $\theta(n)$-interpolation spaces for these couples.

6.2. Proposition. *If $n \neq 0$, then*

$$C_{\theta(n)}(\overline{\mathcal{F}L^1}) = C_{\theta(n)}(\overline{\mathcal{F}M}) \simeq \mathcal{F}L^1_{(\theta,n)}, \qquad C_{\theta(n)}(\overline{\mathcal{F}K}) \simeq \mathcal{F}K_{(\theta,n)},$$
$$C^{\theta(n)}(\overline{\mathcal{F}L^1}) = C^{\theta(n)}(\overline{\mathcal{F}M}) \simeq \mathcal{F}M_{(\theta,n)}, \quad C^{\theta(n)}(\overline{\mathcal{F}L^\infty}) \simeq \mathcal{F}L^\infty_{(\theta,n)}.$$

Proof. The proof proceeds in three steps:

(i) $\mathcal{F}L^1_{(\theta,n)} \prec C_{\theta(n)}(\overline{\mathcal{F}L^1})$ and $\mathcal{F}K_{(\theta,n)} \prec C_{\theta(n)}(\overline{\mathcal{F}K})$ for $n > 0$,

(ii) $\mathcal{F}L^1_{(\theta,n)} \prec C_{\theta(n)}(\overline{\mathcal{F}L^1})$ and $\mathcal{F}K_{(\theta,n)} \prec C_{\theta(n)}(\overline{\mathcal{F}K})$ for $n < 0$,

(iii) $\mathcal{F}L^\infty_{(\theta,n)} \prec C^{\theta(n)}(\overline{\mathcal{F}L^\infty})$ and $\mathcal{F}M_{(\theta,n)} \prec C^{\theta(n)}(\overline{\mathcal{F}M})$ for $n \neq 0$;

and the required identities are obtained by (6.1) together with the standard duality argument.

Let $n > 0$ and let $X = L^1(\mathbb{T})$ or $K(\mathbb{T})$ for the moment. By density, it suffices to estimate the norms for finite scalar sequences in $\mathcal{F}X_{(\theta,n)}$. For such a $\lambda = (\lambda_\nu)$, set

$$\text{(iv)} \qquad \phi_{\theta,n}(t) = \sum_{\nu\neq 0} \lambda_\nu e^{\nu(it-\theta)}/\nu^n.$$

Then $\phi_{\theta,n} \in X$ with $\|\lambda\|_{\mathcal{F}X_{(\theta,n)}} = \|\phi_{\theta,n}\|_X + |\lambda_0|$.

For the inclusion $\mathcal{F}X_{(\theta,n)} \prec C_{\theta(n)}(\overline{\mathcal{F}X})$, we consider the analytic function $f(z) = (f_\nu(z))_{\nu\in\mathbb{Z}}$, where

$$f_\nu(z) = \begin{cases} \dfrac{\lambda_\nu}{c_{\theta,n}\nu^n} e^{\nu(z-\theta)} & \text{if } \nu \neq 0, \\ \lambda_0 m_\theta(z)^n e^{z-\theta} & \text{if } \nu = 0. \end{cases}$$

Then $\lambda = c_{\theta,n} f^{(n)}(\theta)$ and

$$\|f(j+is)\|_{\mathcal{F}X_j} = \Big\|\sum_{\nu\neq 0} f_\nu(j+is)e^{\nu(it-j)}/\nu^n\Big\|_X + |f_0(j+is)|$$
$$\leq \frac{1}{|c_{\theta,n}|}\|\phi_{\theta,n}\|_X + |\lambda_0|e^{j-\theta} \leq c\|\lambda\|_{\mathcal{F}X_{\theta,n}}$$

for $j = 0, 1$, where the constant c only depends on θ and n. This gives (i).

We verify (ii) by using induction and partial reiteration. First recall $C_\theta(\overline{\mathcal{F}X}) \simeq \mathcal{F}X_\theta$ [J, Sec. 7], and assume $\mathcal{F}X_{(\theta,-n)} \prec C_{\theta(-n)}(\overline{\mathcal{F}X})$. Then we choose $0 < \theta_0, \theta_1, \alpha < 1$ with $\theta_0 < \theta < \theta_1$ and $\theta = (1-\alpha)\theta_0 + \alpha\theta_1$, and we claim

$$\text{(v)} \qquad \mathcal{F}X_{(\theta,-(n+1))} \prec C_{\alpha(-1)}(\mathcal{F}X_{(\theta_0,-n)}, \mathcal{F}X_{(\theta_1,-n)}).$$

To prove the claim, set $\overline{Y} = (\mathcal{F}X_{(\theta_0,-n)}, \mathcal{F}X_{(\theta_1,-n)})$. For a finite scalar sequence $\lambda = (\lambda_\nu)$, we need an analytic function $g \in A^{\mathrm{b}}(\mathbb{S}, \overline{Y})$ such that $\lambda = g(\alpha)$, and $g'(\alpha) = 0$. In fact, we define $g(z) = (g_\nu(z))_{\nu\in\mathbb{Z}}$ by

$$g_\nu(z) = \lambda_\nu(1 - c_{\alpha,1}(\theta_1-\theta_0)\nu m_\alpha(z))e^{\nu(\theta_1-\theta_0)(z-\alpha)},$$

where m_α is the standard conformal mapping from the strip $\mathbb{S}$ to the unit disc $\mathbb{D}$ in the form (2.3), and hence $c_{\alpha,1} = 1/m'_\alpha(\alpha)$. Let $\psi(t) = i(\pi - t)$ for $t \in (0, 2\pi)$. Then $\psi \in L^1(\mathbb{T})$ with $\|\psi\|_{L^1} = \pi/2$, $\mathcal{F}\psi(\nu) = 1/\nu$ if $\nu \neq 0$ and $\mathcal{F}\psi(\nu) = 0$ if $\nu = 0$. Now we have

$$\|g(j+is)\|_{\mathcal{F}X_{(\theta_j,-n)}} = \Big\|\sum_{\nu\neq 0} \nu^n e^{-\nu\theta_j} g_\nu(j+is)e^{i\nu t}\Big\|_{X_j} + |g_0(j+is)|$$
$$= \Big\|\sum_{\nu\neq 0} \nu^n e^{-\nu\theta}\lambda_\nu(1 - c_{\alpha,1}(\theta_1-\theta_0)\nu m_\alpha(j+is))e^{i\nu t'}\Big\|_X + |\lambda_0|$$
$$\text{(where } t' = t + (\theta_1-\theta_0)s)$$
$$\leq \Big\|\sum_{\nu\neq 0} \frac{1}{\nu}\nu^{n+1}e^{-\nu\theta}\lambda_\nu e^{i\nu t'}\Big\|_X$$
$$+ (\theta_1-\theta_0)|c_{\alpha,1}|\Big\|\sum_{\nu\neq 0}\nu^{n+1}e^{-\nu\theta}\lambda_\nu e^{i\nu t'}\Big\|_X + |\lambda_0|$$
$$= \|\psi * \phi_{\theta,-(n+1)}\|_X + (\theta_1-\theta_0)|c_{\alpha,1}|\|\phi_{\theta,-(n+1)}\|_X + |\lambda_0|$$
$$\leq (\|\psi\|_{L^1} + (\theta_1-\theta_0)|c_{\alpha,1}|)\|\phi_{\theta,-(n+1)}\|_X + |\lambda_0| < \pi\|\lambda\|_{\mathcal{F}X_{(\theta,-(n+1))}}$$

and so $\|\lambda\|_{\alpha(-1)} \leq \|g\|_\infty < \pi\|\lambda\|_{\mathcal{F}X_{(\theta,-(n+1))}}$. Therefore,

$$\begin{aligned} \mathcal{F}X_{(\theta,-(n+1))} &\prec C_{\alpha(-1)}(\mathcal{F}X_{(\theta_0,-n)}, \mathcal{F}X_{(\theta_1,-n)}) && \text{(by (v))} \\ &\prec C_{\alpha(-1)}(C_{\theta_0(-n)}(\overline{\mathcal{F}X}), C_{\theta_1(-n)}(\overline{\mathcal{F}X})) && \text{(by induction)} \\ &\prec C_{\theta(-(n+1))}(\overline{\mathcal{F}X}) && \text{(by Theorem 5.1).} \end{aligned}$$

That is, the inclusions in (ii) also hold true.

Now we come to the proof of the inclusion $\mathcal{F}L^\infty_{(\theta,n)} \prec (\mathcal{F}K_{(\theta,n)})^{\mathrm{c}}$ for $n \in \mathbb{Z}$. For any $\lambda = (\lambda_\nu) \in \mathcal{F}L^\infty_{(\theta,n)}$, let $\phi \in L^\infty(\mathbb{T})$ with

$$\mathcal{F}\phi(\nu) = \begin{cases} \lambda_\nu e^{-\nu\theta}/\nu^n & \text{if } \nu \neq 0, \\ 0 & \text{if } \nu = 0. \end{cases}$$

Assume K_N is the Nth Fejér kernel and set $\phi_N = K_N * \phi$ for $N = 1, 2, \ldots$ Then $\phi_N \in K(\mathbb{T})$ with $\|\phi_N\|_\infty \leq \|\phi\|_\infty \leq \|\phi\|_\infty + |\lambda_0| = \|\lambda\|_{\mathcal{F}L^\infty_{(\theta,n)}}$, and $\langle\psi, \phi_N\rangle \to \langle\psi, \phi\rangle$ for any $\psi \in L^1(\mathbb{T})$. This yields $\lambda \in (\mathcal{F}K_{(\theta,n)})^{\mathrm{c}}$ with

$$\|\lambda\|_{(\mathcal{F}K_{(\theta,n)})^{\mathrm{c}}} \leq \sup_N \|\phi_N\|_\infty + |\lambda_0| \leq \|\lambda\|_{\mathcal{F}L^\infty_{(\theta,n)}}.$$

The proof for the inclusion $\mathcal{F}M_{(\theta,n)} \prec (\mathcal{F}L^1_{(\theta,n)})^{\mathrm{c}}$ is essentially the same. Combining this with the inclusions

$$\mathcal{F}K_{(\theta,n)} \prec C_{\theta(n)}(\overline{\mathcal{F}K}) \prec C^{\theta(n)}(\overline{\mathcal{F}L^\infty}) = C^{\theta(n)}(\overline{\mathcal{F}L^\infty})^{\mathrm{c}},$$
$$\mathcal{F}L^1_{(\theta,n)} \prec C_{\theta(n)}(\overline{\mathcal{F}L^1}) \prec C^{\theta(n)}(\overline{\mathcal{F}M}) = C^{\theta(n)}(\overline{\mathcal{F}M})^{\mathrm{c}},$$

we obtain the inclusions in (iii) and hence complete the proof. ■

The following minimality/maximality theorem generalizes results of Janson as well as Nilsson for $n = 0$ (cf. [J, Th. 22] & [N, Th. 4.1]). The first part can be reduced to the case of regular Banach couples and can be proved via the discretization stated in Section 3 and a fairly straightforward adaptation of arguments used in [J, Th. 22].

6.3. THEOREM. *For* $n > 0$, *we have*

(i) $$C_{\theta(n)} \simeq \mathrm{Orb}_{\mathcal{F}L^1_{(\theta,n)}}(\overline{\mathcal{F}L^1}, -) \simeq \mathrm{Orb}_{\mathcal{F}L^1_{(\theta,n)}}(\overline{\mathcal{F}M}, -),$$

(ii) $$C^{\theta(n)} \simeq \mathrm{Orb}_{\mathcal{F}M_{(\theta,n)}}(\overline{\mathcal{F}L^1}, -) \simeq \mathrm{Orb}_{\mathcal{F}M_{(\theta,n)}}(\overline{\mathcal{F}M}, -),$$

(iii) $$C^{\mathrm{c}}_{\theta(-n)} \simeq \mathrm{Corb}_{\mathcal{F}L^\infty_{(\theta,-n)}}(-, \overline{\mathcal{F}L^\infty}).$$

Proof. Let $F_1 = \mathrm{Orb}_{\mathcal{F}L^1_{(\theta,n)}}(\overline{\mathcal{F}L^1}, -)$, $F_2 = \mathrm{Orb}_{\mathcal{F}M_{(\theta,n)}}(\overline{\mathcal{F}L^1}, -)$ and $H = \mathrm{Corb}_{\mathcal{F}L^1_{(\theta,-n)}}(-, \overline{\mathcal{F}L^\infty})$.

(i) By the minimality of the orbit functors and by Proposition 6.2, we only need to examine the inclusions

$$C_{\theta(n)}(\overline{X}) \prec F_1(\overline{X}) \quad \text{and} \quad C^{\theta(n)}(\overline{X}) \prec F_2(\overline{X})$$

for any regular Banach couple $\overline{X}$.

In fact, let $x \in C_{\theta(n)}(\overline{X})$ with $x = c_{\theta,n} f^{(n)}(\theta)$ for some $f \in A^{\mathrm{b}}(\mathbb{S}_{2\pi}, \overline{X})$ with $\|f\|_\infty \le c\|x\|_{\theta(n)}$. For all scalar sequences $\lambda = (\lambda_\nu)$ with finite support we define $T\lambda$ by

$$T\lambda = \int_{s-i\pi}^{s+i\pi} \sum_\nu \lambda_\nu e^{-\nu z} f(z)\,\frac{dz}{2\pi i} = \int_{-\pi}^{\pi} \sum_\nu \lambda_\nu e^{-\nu(s+it)} f(s+it)\,\frac{dt}{2\pi},$$

where the value of the integral is independent of $s \in [0,1]$. Observe that

$$\|T\lambda\|_{X_j} \le \Big\| \int_{-\pi}^{\pi} \sum_\nu \lambda_\nu e^{-\nu(j+it)} f(j+it)\,\frac{dt}{2\pi} \Big\|_{X_j} \le \|f\|_\infty \|\lambda\|_{\mathcal{F}L^1_j}$$

for $j = 0, 1$, hence T can be uniquely extended to an operator in $\mathcal{L}(\overline{\mathcal{F}L^1}, \overline{X})$, still denoted by T, with $\|T\| \le \|f\|_\infty$.

For $N = 1, 2, \ldots$, set $\lambda_N = (\lambda_{N\nu})$ with

$$\lambda_{N\nu} = \begin{cases} c_{\theta,n}\Big(1 - \dfrac{|\nu|}{N+1}\Big)\nu^n e^{\nu\theta} & \text{if } 1 \le |\nu| \le N, \\ 0 & \text{otherwise.} \end{cases}$$

and further put

$$x_N = T\lambda_N = c_{\theta,n} \int_{-\pi}^{\pi} \sum_{1 \le |\nu| \le N} \nu^n \Big(1 - \frac{|\nu|}{N+1}\Big) e^{\nu(\theta - it)} f(it)\,\frac{dt}{2\pi}.$$

Since

$$\int_{-\pi}^{\pi} f^{(n)}(\theta - it)\,dt = i(f^{(n-1)}(\theta - i\pi) - f^{(n-1)}(\theta + i\pi)) = 0,$$

we use integration by parts to obtain

$$x_N = c_{\theta,n} \int_{-\pi}^{\pi} \sum_{\nu=-N}^{N} \Big(1 - \frac{|\nu|}{N+1}\Big) e^{i\nu t} f^{(n)}(\theta - it)\,\frac{dt}{2\pi}$$
$$= c_{\theta,n} \int_{-\pi}^{\pi} K_N(t) f^{(n)}(\theta - it)\,\frac{dt}{2\pi}.$$

Thus $x_N \in F_1(\overline{X})$ with $\|x_N\|_{F_1} \le \|T\| \|\lambda_N\|_{\mathcal{F}L^1_{(\theta,n)}} \le c\|x_N\|_{\theta(n)}$. Replace x_N with $x_N - x_{N'}$ and notice that $x_N \to x$ in $C_{\theta(n)}(\overline{X})$ as $N \to \infty$. We readily see that (x_N) is a Cauchy sequence in $F_1(\overline{X})$ and so $x \in F_1(\overline{X})$ with $\|x\|_{F_1} \le c\|x\|_{\theta(n)}$ by passing to the limit.

(ii) If $x \in C^{\theta(n)}(\overline{X})$, then $x = c_{\theta,n} f^{(n)}(\theta)$ for some $f \in H_\infty(\mathbb{S}_{2\pi}, \overline{X})$ with $\|f\|_\infty \le c\,\|x\|_{\theta(n)}$. For $\lambda = (\lambda_\nu) \in \Sigma\overline{\mathcal{F}L^1}$, define T as before. If $\lambda \in \mathcal{F}L^1_j$, assume $\phi_j \in L^1(\mathbb{T})$ with $\mathcal{F}\phi_j(\nu) = \lambda_\nu e^{-\nu j}$. Then

$$T\lambda = \int_{-\pi}^{\pi} \phi_j(-t) f(j+it)\,\frac{dt}{2\pi} \in X_j$$

since $f(j+it) \in \mathcal{L}(L^1(\mathbb{T}), X_j)$ and hence $\|T\lambda\|_{X_j} \le \|f\|_\infty \|\lambda\|_{\mathcal{F}L^1_j}$. Thus $T \in \mathcal{L}(\overline{\mathcal{F}L^1}, \overline{X})$ with $\|T\| \le \|f\|_\infty$. Take now

$$\lambda_\nu = \begin{cases} c_{\theta,n}\nu^n e^{\nu\theta} & \text{if } \nu \ne 0, \\ 0 & \text{if } \nu = 0. \end{cases}$$

Then $\lambda = (\lambda_\nu) \in \mathcal{F}M_{(\theta,n)}$ with

$$T\lambda = c_{\theta,n} \int_{-\pi}^{\pi} f^{(n)}(\theta - it)\,d\delta_0 = c_{\theta,n} f^{(n)}(\theta) = x,$$

where δ_0 is the Dirac measure on $\mathbb{T}$ concentrated at $t = 0$. Therefore, the inclusion $C^{\theta(n)}(\overline{X}) \prec F_2(\overline{X})$ follows.

(iii) It is easy to see that $\overline{\mathcal{F}L^1}$ is a regular Banach couple with $\|e_0\|_{\Delta\overline{\mathcal{F}L^1}} = \|e_0\|_{\Sigma\overline{\mathcal{F}L^1}} = 1$; $\Delta\overline{\mathcal{F}L^1}$ is dense in $\mathcal{F}L^1_{(\theta,n)}$ with $(e^{\nu\theta}) \in \mathcal{F}L^1_{(\theta,n)} \setminus (\mathcal{F}L^1_0 \cup \mathcal{F}L^1_1)$; $\overline{\mathcal{F}L^1}$ satisfies the metric approximation property in the sense that for every pair $\{C_0, C_1\}$ of compact subsets of $\{X_0, X_1\}$ and every $\varepsilon > 0$, there exists $P \in \mathcal{L}(\overline{\mathcal{F}L^1})$ of finite rank with $\|P\| \le 1$ such that $\|P_j x - x\|_j < \varepsilon$ for all $x \in C_j$ $(j = 0, 1)$; and $\mathcal{F}L^\infty_j$ $(j = 0, 1)$ are projective with respect to quotient maps. According to [KP, VII.5.6, VII.5.8 & VIII.2.5], we have

$H(\overline{X}) = H(\overline{X}^0)$, $F_1(\overline{X})' = H((\overline{X}^0)')$ and hence H is the maximal funct with this property. Combining the preceding statement with [BK, 2.4.31 2.4.12], we obtain

$$H(\overline{X}) = H(\overline{X}^0) = \Sigma\overline{X}^0 \cap F((\overline{X}^0)')' \simeq \Sigma\overline{X}^0 \cap C_{\theta(n)}((\overline{X}^0)')' \simeq \Sigma\overline{X}^0 \cap C^{\theta(-n)}((((\overline{X}^0)')^0)') \simeq C^{c}_{\theta(-n)}(\overline{X}). \blacksquare$$

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