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Coincidence of topologies on tensor products of Köthe echelon spaces

by

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Abstract. We investigate conditions under which the projective and the injective topologies coincide on the tensor product of two Köthe echelon or coechelon spaces. A major tool in the proof is the characterization of the $\varepsilon\pi$-continuity of the tensor product of two diagonal operators from l_p to l_q. Several sharp forms of this result are also included.

The aim of this paper is to study the coincidence of the projective topology π and the injective topology ε on the tensor product of Köthe echelon spaces or coechelon spaces.

The motivation for the above study is the well-known characterization due to Grothendieck [12] that a locally convex space (l.c.s.) E is nuclear if and only if for each l.c.s. F, $E \otimes_\varepsilon F = E \otimes_\pi F$, and Grothendieck's conjecture that if E and F are l.c.s. such that the ε and π topologies on their tensor product coincide then E or F must be nuclear. The related quadratic problem whether or not the equality $E \otimes_\varepsilon E = E \otimes_\pi E$ implies the nuclearity of E, was explicitly raised by Pietsch [20]. In 1983, Pisier [23] constructed an infinite-dimensional Banach space P such that $P \otimes_\varepsilon P = P \otimes_\pi P$, thus answering both Grothendieck's conjecture and Pietsch's question in the negative. John [16] gave several different examples exhibiting the same phenomenon; his examples were Köthe echelon spaces $\lambda_p(A)$ and $\lambda_q(B)$, with $p, q \in [1, \infty) \cup \{0\}$, and the spaces were Fréchet–Schwartz spaces, neither being nuclear. However, Pietsch's quadratic problem has an affirmative answer for certain classes of l.c.s. In fact, if E is a projective limit of a reduced spectrum of Banach spaces which are $\mathcal{L}_p$-spaces ($1 < p < \infty$) in the sense of Lindenstrauss and Pełczyński and

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if $E \otimes_\varepsilon E = E \otimes_\pi E$ then E must be nuclear (see, for example, Jarchow and John [14]). Also, Jarchow and John [15] have constructed, using Pisier's space P, a non-nuclear Fréchet–Schwartz space E such that $E \otimes_\varepsilon E = E \otimes_\pi E$.

Our main result (in §2) gives a complete characterization of the coincidence of the ε and π topologies on the tensor product $\lambda_p(A) \otimes \lambda_q(B)$ of two echelon spaces, $p, q \in [1, \infty) \cup \{0\}$. This characterization (see Theorem 5) is provided completely in terms of the matrices A and B. A major tool in the proof is the characterization of the $\varepsilon\pi$-continuity of the tensor product of two diagonal operators $A : l_p \to l_p$ and $B : l_q \to l_q$ on $l_p \otimes l_q$. This result (Theorem 1) may be of independent interest. Several sharp forms of the result are also provided. Our approach is different from John's [14] where s-numbers are used.

We also give a characterization of the coincidence of the two topologies for tensor products of Köthe coechelon (DFS)-spaces. Finally, an alternative construction of a Fréchet–Schwartz space E without the approximation property and such that $E \otimes_\varepsilon E = E \otimes_\pi E$ is provided and this construction is simpler than the one in [15].

Our notation for Banach spaces and tensor products is standard and we refer the reader to [8]; for l.c.s. we follow [13 and 18], and we follow the notation of [4 and 18] for Köthe echelon and coechelon spaces. l_0 is defined to be c_0.

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1. $\varepsilon\pi$-Continuity of the tensor product of two diagonal operators. In this section we present in Theorem 1 a characterization of the $\varepsilon\pi$-continuity of the tensor product $A \otimes B$ of two diagonal operators $A : l_p \to l_p$ and $B : l_q \to l_q$. The result is presented in the form in which it is used in proving our main result in §2. However, because of the possible use and interest in this result from the purely Banach space point of view, we also present sharper forms of the result.

Before proving the first theorem we make the following

Observation. (a) If $a, b \in c_0$ are strictly positive sequences, then the condition $\sum a_i b_{\sigma(i)} < \infty$ for each bijection $\sigma : \mathbb{N} \to \mathbb{N}$ is equivalent to $\sum a_{r(i)} b_{s(i)} < \infty$ for suitable (or for all) bijections r and s such that $(a_{r(i)})$ and $(b_{s(i)})$ are decreasing; this is easily seen by noticing that for decreasing sequences (x_i), (y_i) with $\sum x_i y_i < \infty$ and for an arbitrary bijection $\sigma : \mathbb{N} \to \mathbb{N}$, $x_i y_{\sigma(i)} \le x_i y_i + x_{\sigma(i)} y_{\sigma(i)}$ for all $i \in \mathbb{N}$.

(b) If $a, b \in c_0$ are non-negative sequences, then the condition $\sum a_i b_{\sigma(i)} < \infty$ for each bijection $\sigma : \mathbb{N} \to \mathbb{N}$ is equivalent to $\sum a_{r(i)} b_{s(i)} < \infty$ for arbitrary injections r and s from $\mathbb{N}$ into $\mathbb{N}$.

THEOREM 1. *Given $p, q \in [1, \infty) \cup \{0\}$ and bounded sequences $a = (a_i), b = (b_i)$ of non-negative numbers, let $A \in L(l_p, l_p)$ and $B \in L(l_q, l_q)$ be the diagonal operators $A[(x_i)] = (a_i x_i)$ and $B[(y_j)] = (b_j y_j)$. Then*

(1) *if $\sum a_i b_{\sigma(i)} < \infty$ for each bijection $\sigma : \mathbb{N} \to \mathbb{N}$, then $A \otimes B : l_p \otimes_\varepsilon l_q \to l_p \otimes_\pi l_q$ is continuous;*

(2) *if $A \otimes B : l_q \otimes_\varepsilon l_q \to l_p \otimes_\pi l_q$ is continuous, then there exists $r, 1 \le r < \infty$, depending only on p and q such that for each bijection $\sigma : \mathbb{N} \to \mathbb{N}$, $(a_i b_{\sigma(i)}) \in l_r$.*

Proof. (1) First consider the case where both (a_i) and (b_i) are decreasing null sequences. Consider an arbitrary C in the unit ball of $l_p \otimes_\varepsilon l_q$; we may represent it by $C = (c_{ij}) \in L(l_{p'}, l_q)$, p' being the index conjugate to p (in case $p = 1$ we embed $l_1 \otimes_\varepsilon X$ into $L(c_0, X)$). Let $M = \sum a_i b_i < \infty$ and for fixed i, let $J_i = \{j \in \mathbb{N} : j \ge i\}$ and $\widehat{J}_i = \{j \in \mathbb{N} : j > i\}$; let χ_{J_i} and $\chi_{\widehat{J}_i}$ be the corresponding characteristic functions.

Define

$$R_i = \Big(\chi_{J_i}(j) \frac{a_j}{a_i} c_{ij}\Big)_j \quad \text{and} \quad T_i = \Big(\chi_{\widehat{J}_i}(j) \frac{b_j}{b_i} c_{ji}\Big)_j .$$

The coordinates are defined to be zero if the corresponding denominator is null. Since (a_i) and (b_i) are decreasing and C is in the unit ball of $L(l_{p'}, l_q)$ we get $\|R_i\|_p \le 1$ and $\|T_i\|_q \le 1$ for each i. We verify that $(A \otimes B)C = \sum_i a_i b_i (R_i \otimes e_i + e_i \otimes T_i)$. This follows from

$$\begin{aligned}\langle (A \otimes B)C, e_j \otimes e_l \rangle &= \langle (B \circ C \circ A^t)(e_j), e_l \rangle = \langle (B \circ C)(a_j e_j), e_l \rangle \\ &= \langle B(a_j (c_{ij})_i), e_l \rangle = \langle (b_i a_j c_{ij})_i, e_l \rangle = a_j b_l c_{lj}\end{aligned}$$

and

$$\begin{aligned}\Big\langle \sum_i a_i b_i (R_i \otimes e_i + e_i \otimes T_i), e_j \otimes e_l \Big\rangle &= \Big\langle \Big[\sum_{\substack{i : i \le j \\ a_i \ne 0}} b_i a_j c_{ij} e_i \Big] + a_j b_j T_j, e_l \Big\rangle \\ &= a_j b_l c_{lj} \quad \text{for all } l, j.\end{aligned}$$

If one of a and b is not in c_0, then $(a_i b_{\sigma(i)}) \in l_1$ for each σ implies that a or b is in l_1, hence the continuity of $A \otimes B$ follows: see, for example, [13, 17.3.8]. We can suppose, hence, that $a, b \in c_0 \setminus l_1$. It is possible to reduce the problem to the case $a_i, b_i > 0$, $i \in \mathbb{N}$: If, for instance, a contains null elements, we can define the inclusion $\Phi : l_p \to l_p$, $\Phi(e_i) := e_{\varphi(i)}$, where $\varphi : \mathbb{N} \to \mathbb{N}$ is the increasing injection such that $a_j > 0$ if and only if $j \in \operatorname{Im}\varphi$, and let $P : l_p \to l_p$ the corresponding projection ($P \circ \Phi = I_{l_p}$). Then, if $\widetilde{A}$ is the diagonal operator corresponding to $(a_{\varphi(i)})$, we have $A \otimes B = (\Phi \otimes I) \circ (\widetilde{A} \otimes B) \circ (P \otimes I)$ and it is enough to show that $\widetilde{A} \otimes B$ is $\varepsilon\pi$-continuous. Finally, the assumption in (1) is equivalent to $\sum a_{\varphi(i)} b_{\sigma(i)} < \infty$ for arbitrary injections $\varphi, \sigma : \mathbb{N} \to \mathbb{N}$ by the observation above.

When $a, b \in c_0$ are strictly positive, we may also assume that (a_i) is decreasing; consider then the bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\widetilde{b} = (\widetilde{b}_i) = (b_{\sigma(i)})$ is decreasing. The map $\widetilde{\sigma} : y \to \widetilde{y}$ is an isometry and so it suffices to show that $A \otimes \widetilde{B} \in L(l_p \otimes_\varepsilon l_q, l_p \otimes_\pi l_q)$; thus the general case is reduced to the first case.

(2) Case 1: $p' \le q < \infty$. Assume that $A \otimes B$ is continuous and $\|A \otimes B\| < M$. Since for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$, the element $\sum_{i=1}^n e_i \otimes e_{\sigma(i)}$ belongs to the unit ball of $l_p \otimes_\varepsilon l_q$ for each n,

$$(A \otimes B)\Big(\sum_{i=1}^n e_i \otimes e_{\sigma(i)}\Big) = \sum_{j=1}^m \lambda_j f_j \otimes g_j$$

with $\|f_j\|_p \le 1$, $\|g_j\|_q \le 1, j = 1, \dots m$, and $\sum_{j=1}^m |\lambda_j| \le M$. Then we also have

$$\begin{aligned} a_k b_{\sigma(k)} &= \Big\langle (A \otimes B) \sum_{i=1}^n e_i \otimes e_{\sigma(i)}, e_k \otimes e_{\sigma(k)} \Big\rangle \\ &= \sum_{j=1}^m \lambda_j f_{jk} g_{j\sigma(k)}, \qquad k = 1, \dots, n. \end{aligned}$$

Since $f_j = (f_{jk})_k \in l_p$ and $g_j = (g_{j\sigma(k)})_k \in l_q$ we see, by the generalized Hölder inequality, that $f_j g_j \in l_r$ where $1/r = 1/p + 1/q$ and

$$\|f_j g_j\|_r = \Big(\sum_k |f_{jk} g_{j\sigma(k)}|^r\Big)^{1/r} \le \|f_j\|_p \|g_j\|_q \le 1$$

($r = q$ if $p = 0$); it follows that

$$(a_k b_{\sigma(k)})_k \in l_r \quad \text{and} \quad \Big(\sum_{k=1}^n a_k^r b_{\sigma(k)}^r\Big)^{1/r} \le M.$$

This is true for each n and thus the proof is complete in this case.

Case 2: $1 \le q < p' < \infty$. Let $x \in l_p$ with $\|x\|_p \le 1$ be arbitrary. Then $\sum_{i=1}^n x_i e_i \otimes e_{\sigma(i)}$ belongs to the unit ball in $l_p \otimes_\varepsilon l_q$ for each $n \in \mathbb{N}$. It now follows, as in the previous case, that $x_k a_k b_{\sigma(k)} = \sum_{j=1}^m \lambda_j f_{jk} g_{j\sigma(k)}$ for suitable λ, f and g. Since $1/p + 1/q > 1/p + 1/p' = 1$ we deduce that $(x_k a_k b_{\sigma(k)})_k$ is in the closed ball of radius M in l_1; this being true for each $x \in l_p$ with $\|x\|_p \le 1$ we obtain $(a_k b_{\sigma(k)}) \in l_{p'}, 1 < p' < \infty$.

The cases $p = 1$, $q = 0$ and $p = 1$, $q \ne 0, 1$ are also covered by the above discussion by suitably interchanging p and q.

Case 3: $p = q = 1$. We break this case into two cases.

Case 3.1. Assume both a and b are in c_0 and let $\overline{a}$, $\overline{b}$ denote the decreasing rearrangements of their non-null elements and $\overline{A}, \overline{B}$ denote the corresponding diagonal operators. Observe first that $A \otimes B$ is $\varepsilon\pi$-continuous, hence $\overline{A} \otimes \overline{B}$ is also $\varepsilon\pi$-continuous and let $\overline{K}$ denote its norm. First recall the following well-known estimate (see, for example, [8, p. 56], [11] and Proposition 3 below): for each $n \in \mathbb{N}$,

$$(*) \qquad \sqrt{n} \le \|\mathrm{id} \otimes \mathrm{id} : l_1^n \otimes_\varepsilon l_1^n \to l_1^n \otimes_\pi l_1^n\|.$$

Given n, let $\overline{A}_n = (\overline{a}_k)_{k=1}^n$ and $\overline{B}_n = (\overline{b}_k)_{k=1}^n$; we also consider $\overline{A}_n$ and $\overline{B}_n$ as diagonal operators on l_1^n (without changing notations). Then

$$\|(\overline{A}_n)^{-1} \otimes (\overline{B}_n)^{-1} : l_1^n \otimes_\pi l_1^n \to l_1^n \otimes_\pi l_1^n\| \le \|(\overline{A}_n)^{-1}\| \|(\overline{B}_n)^{-1}\| = (\overline{a}_n \overline{b}_n)^{-1}.$$

Thus, for each n, and for $z \in l_1^n \otimes l_1^n$,

$$\begin{aligned} \|z\|_\pi &= \|((\overline{A}_n)^{-1} \otimes (\overline{B}_n)^{-1}) \circ (\overline{A}_n \otimes \overline{B}_n) z\|_\pi \le (\overline{a}_n \overline{b}_n)^{-1} \|(\overline{A}_n \otimes \overline{B}_n) z\|_\pi \\ &\le (\overline{a}_n \overline{b}_n)^{-1} \cdot \overline{K} \cdot \|z\|_\varepsilon \end{aligned}$$

and it now follows from $(*)$ that $\overline{a}_n \overline{b}_n \le \overline{K}/\sqrt{n}$ and therefore $\overline{a}\overline{b} \in l_{2+\varepsilon}$ for all $\varepsilon > 0$. In view of the earlier observation this is equivalent to $(a_i b_{\sigma(i)}) \in l_{2+\varepsilon}$ for all $\varepsilon > 0$.

Case 3.2: $a \notin c_0$. We claim, in the presence of our hypothesis on the $\varepsilon\pi$-continuity of $A \otimes B$, that $b \in c_0$. If $b \notin c_0$ (and $a \notin c_0$), then there are increasing sequences $(i_k), (j_k)$ of positive integers and $\varepsilon > 0$ such that $a_{i_k} \ge \varepsilon$ and $b_{j_k} \ge \varepsilon$ for each k.

Since $A \otimes B$ is $\varepsilon\pi$-continuous, if K denotes its norm, we have as in the previous case 3.1, for each fixed n and the diagonal $n \times n$ matrices $\widehat{A}_n = (a_{i_1}, \dots, a_{i_n})$, and $\widehat{B}_n = (b_{j_1}, \dots b_{j_n})$,

$$\|(\widehat{A}_n)^{-1} \otimes (\widehat{B}_n)^{-1}\|_{\pi\text{-}\pi} \le 1/\varepsilon^2 \quad \text{and hence} \quad \sqrt{n} \le K/\varepsilon^2.$$

This being true for each n, we have the required contradiction and thus $b \in c_0$.

Now we shall show that $b \in l_{2+\varepsilon}$ for all $\varepsilon > 0$, which also implies that $ab \in l_{2+\varepsilon}$. As before, let $\overline{b}$ denote the decreasing rearrangement of the non-null elements of b. Then considering $\widehat{A}_n \otimes \overline{B}_n$ (in the notation of the previous paragraph) we conclude that $\sqrt{n} \le \varepsilon^{-1} \overline{b}_n^{-1} K$ and then $\overline{b} \in l_{2+\varepsilon}$, $\forall \varepsilon > 0$. This completes the proof in this case.

Case 4: $p = q = 0$. In this case the desired conclusion follows as in Case 3, from the observation that, for each $n \in \mathbb{N}$,

$$\sqrt{n} \le \|\mathrm{id} \otimes \mathrm{id} : l_1^n \otimes_\varepsilon l_1^n \to l_1^n \otimes_\pi l_1^n\| = \|\mathrm{id} \otimes \mathrm{id} : l_\infty^n \otimes_\varepsilon l_\infty^n \to l_\infty^n \otimes_\pi l_\infty^n\|. \ \blacksquare$$

We now improve Theorem 1 in the case of decreasing sequences a and b. Part (1) below with constant 1 and its proof were suggested to us by Kwapień. We are very thankful to him.

THEOREM 2. *Given $p, q \in [1,\infty) \cup \{0\}$ and decreasing null sequences a, b, let $A \in L(l_p, l_q)$ and $B \in L(l_q, l_q)$ be the diagonal operators associated with a and b respectively.*

(1) *If $ab \in l_1$, then $A \otimes B : l_p \otimes_\varepsilon l_q \to l_p \otimes_\pi l_q$ is continuous and $\|A \otimes B\| \le \|ab\|_1$.*

(2) *If $A \otimes B : l_p \otimes_\varepsilon l_q \to l_q \otimes_\pi l_q$ is continuous, then there is $1 \le r < \infty$ such that $ab \in l_r$ and $\|ab\|_r \le 2\|A \otimes B\|$.*

(3) *If $1/p + 1/q = 1$, then $A \otimes B : l_p \otimes_\varepsilon l_q \to l_q \otimes_\pi l_q$ is continuous if and only if $ab \in l_1$ and $\|A \otimes B\| = \|ab\|_1$.*

Proof. (1) This proof was suggested by Kwapień. We may assume $a_1 = 1 = b_1$. We denote by c_k the vector $(1, \overset{(k)}{\ldots}, 1, 0, \ldots)$, $k \in \mathbb{N}$. We may write $a = \sum \lambda_k c_k$, $b = \sum \mu_l c_l$, $0 \le \lambda_k$, $k \in \mathbb{N}$, $0 \le \mu_l$, $l \in \mathbb{N}$, and $\sum \lambda_k = 1 = \sum \mu_l$. Moreover,

$$\|ab\|_1 = \Big\|\sum_k \lambda_k c_k b\Big\|_1 = \sum_k \lambda_k \|c_k b\|_1 = \sum_{k,l} \lambda_k \mu_l \|c_k c_l\|_1.$$

On the other hand,

$$A \otimes B = \sum_k \lambda_k C_k \otimes B = \sum_{k,l} \lambda_k \mu_l C_k \otimes C_l,$$

where C_k is the diagonal operator associated with c_k. Consequently, $\|A \otimes B\| \le \sum_{k,l} \lambda_k \mu_l \|C_k \otimes C_l\|$. Accordingly, it suffices to show that $\|C_k \otimes C_l\| \le \|c_k c_l\|_1$ for all $k, l \in \mathbb{N}$.

To see this, we first assume $k \le l$ and we take an arbitrary element D in the unit ball of $l_p \otimes_\varepsilon l_q$. We may write $D = (d_{i,j}) \in L(l_{p'}, l_q)$. We have

$$(C_k \otimes C_l)D = \sum_{j=1}^{k} e_j \otimes (c_l (d_{i,j})_i),$$

hence $\|C_k \otimes C_l\| \le \min(k,l) = \|c_k c_l\|_1$, as desired.

Part (2) directly follows from an inspection of the proof of Theorem 1(2), and part (3) follows from (1) above and the proof of (2), Case 1 of Theorem 1. ■

Part (1) of the preceding theorem also follows from John [16]; his proof uses approximation numbers and gives the continuity constant 12 (instead of 1). For $p = 2$ statement (3) is due to V. Bartík, K. John and J. Korbaš [1, Lemma 1]; they use some standard techniques and the notion of singular numbers to extend it to arbitrary operators acting between Hilbert spaces. (1) and (3) for general p and q (in particular, for $p = q'$) seem to be a partial answer to a question asked there (see [1, Remark 2]).

Remark. In view of the non-symmetric nature of the hypothesis on ab_σ in (1) and the conclusion on ab_σ in (2) in Theorem 1, the following preliminary observations are in order.

(i) Consider $p = q = 4$ and $r = 2$; then from a result of Defant and Mascioni [9] it follows that there exists an $a \in l_{16/7}, a \notin l_2$ (or $a^2 \in l_{8/7}, a^2 \notin l_1$) such that $A \otimes A : l_4 \otimes_\varepsilon l_4 \to l_4 \otimes_\pi l_4$ is not continuous and thus in the theorem the hypothesis $(ab_\sigma) \in l_1$ cannot in general be weakened.

(ii) Consider $a \in l_{8/3}, a \notin l_2$, so that $a^2 \notin l_1$; again by [9], $A \otimes A : l_4 \otimes_\varepsilon l_4 \to l_4 \otimes_\pi l_4$ is continuous; thus the conclusion in (2) that $ab_\sigma \in l_r$ for some r cannot in general be improved to $ab_\sigma \in l_1$.

These observations prompt the determination of the optimal l_r, replacing l_1 in the hypothesis of (1) and in the conclusion of (2); the following results address this question.

Let us start with a quantitative version of part (2) in Theorems 1 and 2.

PROPOSITION 3. *For $1 \le p, q \le \infty$ denote by $\mu(p,q)$ the infimum of all $1 \le r \le \infty$ such that for some constant $c \ge 0$,*

$$\|ab\|_r \le c\|A \otimes B : l_p \otimes_\varepsilon l_q \to l_p \otimes_\pi l_q\|$$

holds for all decreasing null sequences a and b. Then

$$1/\mu(p,q) = \begin{cases} 1/2 + 1/q & \text{if } 2 \le p \le q; \\ 1/2 + 1/p & \text{if } 2 \le q \le p; \\ 1/p + 1/q & \text{if } p \le 2 \text{ and } 1/p + 1/q \le 1; \\ 1/p + 1/q & \text{if } q \le 2 \text{ and } 1/p + 1/q \le 1; \\ 2 - 1/p - 1/q & \text{if } p \ge 2 \text{ and } 1/p + 1/q \ge 1; \\ 2 - 1/p - 1/q & \text{if } q \ge 2 \text{ and } 1/p + 1/q \ge 1; \\ 3/2 - 1/q & \text{if } q \le p \le 2; \\ 3/2 - 1/p & \text{if } p \le q \le 2. \end{cases}$$

Our proof needs the following lemma. One implication of the lemma follows as in the proof of Case 3.1 of Theorem 1. The converse follows upon taking $a = b = (1, \overset{(n)}{\ldots}, 1, 0, 0, \ldots)$, $n \in \mathbb{N}$.

LEMMA. *For $1 \le r \le \infty$ the following are equivalent:*

(1) $\forall \varepsilon > 0\ \exists c > 0\ \forall n \in \mathbb{N} : n^{1/(r+\varepsilon)} \le c\|\mathrm{id}\otimes\mathrm{id} : l_p^n \otimes_\varepsilon l_q^n \to l_p^n \otimes_\pi l_q^n\|$;
(2) $\forall \varepsilon > 0\ \exists c > 0$ *such that for all decreasing null sequences* a, b,

$$\|ab\|_{r+\varepsilon} \le c\|A\otimes B : l_p \otimes_\varepsilon l_q \to l_p \otimes_\pi l_q\|.$$

For the proof of Proposition 3 it is then enough to give the precise estimates of (1) above. This is obtained in the remark below. We also refer to [11]. For two sequences (a_n) and (b_n) of strictly positive numbers we write $a_n \prec b_n$ if there is a constant $c > 0$ such that $a_n \le cb_n$ for all $n \in \mathbb{N}$, and we write $a_n \asymp b_n$ if $a_n \prec b_n$ and $b_n \prec a_n$.

Remark. *For* $1 \le p, q \le \infty$,

$$\|\mathrm{id}\otimes\mathrm{id} : l_p^n \otimes_\varepsilon l_q^n \to l_p^n \otimes_\pi l_q^n\| \asymp n^{1/\mu(p,q)}.$$

Proof. Denote the left sequence by $t_n(p,q)$. Since, by symmetry, $t_n(p,q) = t_n(q,p)$ and, by duality, $t_n(p,q) = t_n(p',q')$, it is enough to establish the following estimates:

(1) $t_n(p,q) \asymp n^{3/2-1/p}$ for $1 \le p \le q \le 2$,
(2) $t_n(p,q) \asymp n^{1/p+1/q}$ for $1 \le p \le 2 \le p' \le q \le \infty$.

Case 1. We denote by D_q the q-dominated norm (see e.g. [21, p. 236] or [8, p. 210]). By [8, p. 382],

$$t_n(p,q) \le \|l_q \otimes_\varepsilon l_p^n \to l_q \otimes_\pi l_p^n\| \le D_q(l_p^n \to l_p^n),$$

and, by [21, p. 316], $D_q(l_p^n \to l_p^n) \asymp n^{3/2-1/p}$ (the infimum computed in [21] is, in fact, a minimum). For the lower estimate, note first that by [9],

$$n^{3/2-1/p} \prec \|l_p^n \otimes_\varepsilon l_p^n \to l_q^n \otimes_\pi l_q^n\|,$$

hence the factorization

$$\begin{array}{ccc} l_p^n \otimes_\varepsilon l_p^n & \longrightarrow & l_q^n \otimes_\pi l_q^n \\ \big\downarrow & & \big\uparrow \\ l_p^n \otimes_\varepsilon l_q^n & \longrightarrow & l_p^n \otimes_\pi l_q^n \end{array}$$

yields the result.

Case 2. Assume $1 \le p \le 2 \le p' \le q \le \infty$. The upper estimate

$$t_n(p,q) \le D_p(l_q^n \to l_q^n) \prec n^{1/p+1/q}$$

follows similarly to Case 1. For the lower estimate first observe that $n \le t_n(p,p')$ since

$$\Big\|\sum_{k=1}^n e_k \otimes e_k\Big\|_\varepsilon = 1 \quad\text{and}\quad \Big\|\sum_{k=1}^n e_k \otimes e_k\Big\|_\pi = n$$

(the latter estimate can be seen in [8, p. 35 or 120]).

Now the factorization

$$\begin{array}{ccc} l_p^n \otimes_\varepsilon l_{p'}^n & \longrightarrow & l_p^n \otimes_\pi l_{p'}^n \\ \big\downarrow & & \big\uparrow \\ l_p^n \otimes_\varepsilon l_q^n & \longrightarrow & l_p^n \otimes_\pi l_q^n \end{array}$$

yields $n \le t_n(p,q)n^{1/p'-1/q}$, which gives the desired conclusion. ■

Note that the proof of the proposition is independent of the (simpler) proof of (the non-quantitative version of) Theorem 1(2).

Unfortunately, we only have partial results on a quantitative version of Theorem 1(1).

Examples. *For* $1 \le p, q \le \infty$ *denote by* $\gamma(p,q)$ *the supremum of all* $1 \le r \le \infty$ *such that for some constant* $c \ge 0$,

$$\|A\otimes B : l_p \otimes_\varepsilon l_q \to l_p \otimes_\pi l_q\| \le c\|ab\|_r$$

holds for all decreasing null sequences a, b. *Then*

(1) $1 \le \gamma(p,q) \le \mu(p,q) \le 2$;
(2) $\gamma(p,2) = \gamma(2,p) = 1$ *for* $1 < p < \infty$;
(3) $\gamma(p,q) = 1$ *whenever* $1/p + 1/q = 1$;
(4) $2 = \gamma(p,\infty) = \gamma(p',1)$ *for* $2 \le p \le \infty$.

For the extreme (and most important) cases $p, q \in \{1, 2, \infty\}$ this means that $\mu(p,q) = \gamma(p,q)$ is either 1 or 2. But in general $\mu \ne \gamma$; e.g. for $2 < q < \infty$ we have $\mu(2,q) = (1/2 + 1/q)^{-1} > 1 = \gamma(2,q)$. Moreover, in contrast to $\mu(p,q)$, the function $\gamma(p,q)$ is not continuous in p and q.

Proof. (1) The left inequality is Theorem 1(1) (or Theorem 2(1)) and the right inequality follows from Proposition 3.

(2) It has to be shown that every admissible r as above is 1. It is a well-known fact that l_p, $1 < p < \infty$, contains all l_2^n's uniformly complemented, i.e., there is some constant c' such that for all $n \in \mathbb{N}$ there is a factorization of the identity of l_2^n through l_p, $J_n : l_2^n \to l_p$ and $P_n : l_p \to l_2^n$ with $\|J_n\|\|P_n\| \le c'$. Using the factorization

$$\begin{array}{ccc} l_2^n \otimes_\varepsilon l_2^n & \xrightarrow{\ \mathrm{id}\ } & l_2^n \otimes_\pi l_2^n \\ {\scriptstyle J_n\otimes\mathrm{id}}\big\downarrow & & \big\uparrow{\scriptstyle P_n\otimes\mathrm{id}} \\ l_p \otimes_\varepsilon l_2^n & \xrightarrow{\ \mathrm{id}\ } & l_p \otimes_\pi l_2^n \end{array}$$

it follows from the remark in the proof of Proposition 3 that $n \le c'n^{1/r}$, which implies $r = 1$.

By the proposition, $\mu(p,q) = 1$, hence (3) is a consequence of (1).

(4) By (1) it is enough to prove $2 \le \gamma(p,q)$. We follow some ideas of [16] and give only a sketch.

(a) For every rank n operator $T: l_p \to l_\infty$, $N(T) \le cn^{1/2}\|T\|$ where $c>0$ is some universal constant and N stands for the nuclear norm. Indeed, we have

$$N(T)=I(T)=P_1(T)\le cP_2(T)\le cn^{1/2}\|T\|$$

(for the notations and necessary results see e.g. [8, 16.3, 11.3, 31.7 and 11.9]).

(b) If $S^a_{2,1}$ stands for the ideal norm coming from the approximation numbers and the Lorentz sequence ideal $l_{2,1}$, then we have $N(T)\le c' S^a_{2,1}(T)$ for all $T: l_{p'} \to l_\infty$. This follows exactly as in [22, p. 86] together with (a).

(c) Using ideas of [16, 2.2] we obtain

$$\|A\otimes B\| \le c'\Big(\sum_{k=1}^\infty (a_k(A)a_k(B))^{2-\varepsilon}\Big)^{1/(2-\varepsilon)} = c'\Big(\sum_{k=1}^\infty (a_kb_k)^{2-\varepsilon}\Big)^{1/(2-\varepsilon)}.$$

2. Coincidence of ε and π topologies. In this section we first consider the coincidence of the projective and injective topologies on the tensor product of two echelon spaces and obtain a characterization of it in terms of the defining matrices. The basic tool used is Theorem 1.

We let the following general lemma precede our discussion.

LEMMA 4. *Let $E=\mathrm{proj}_n(E_n, \varrho_{n,m}: E_m\to E_n)$ and $F=\mathrm{proj}_n(F_n, \sigma_{n,m}: F_m \to F_n)$ be Fréchet spaces which are reduced projective limits of sequences of Banach spaces. The following conditions are equivalent.*

(1) *$E\otimes_\varepsilon F = E\otimes_\pi F$ holds topologically;*
(2) *$E\widehat{\otimes}_\varepsilon F = E\widehat{\otimes}_\pi F$ holds algebraically;*
(2)′ *$E\widehat{\otimes}_\varepsilon F = E\widehat{\otimes}_\pi F$ holds algebraically and topologically;*
(3) *$\forall n\,\exists m\ge n: \varrho_{n,m}\otimes\sigma_{n,m}: E_m\otimes_\varepsilon F_m \to E_n\otimes_\pi F_n$ is continuous.*

Proof. Clearly (1) and (2)′ are equivalent and the equivalence of (2) and (2)′ follows from the open mapping theorem for Fréchet spaces.

To show that (2)′ and (3) are equivalent, we first observe that $E\widehat{\otimes}_\pi F = \mathrm{proj}_n E_n\widehat{\otimes}_\pi F_n$ and $E\widehat{\otimes}_\varepsilon F=\mathrm{proj}_n E_n\widehat{\otimes}_\varepsilon F_n$ where the projective limits are reduced (see [13, 15.4.2 and 16.3.2]). By well-known properties of projective limits, (2)′ is equivalent to the following condition:

$$\forall n\,\exists m\ge n:\quad \varrho_{n,m}\widehat{\otimes}\sigma_{n,m}: E_m\widehat{\otimes}_\varepsilon F_m\to E_n\widehat{\otimes}_\pi F_n \text{ is continuous.}$$

This is clearly equivalent to (3). The proof is complete. ∎

THEOREM 5. *Given Köthe matrices $A=(a^n)$ and $B=(b^n)$ with $a_i^n>0$ and $b_i^n>0$ for all $n,i\in\mathbb{N}$ and $p,q\in[1,\infty)\cup\{0\}$, the following are equivalent:*

(i) *$\lambda_p(A)\otimes_\varepsilon\lambda_q(B)=\lambda_p(A)\otimes_\pi\lambda_q(B)$ topologically;*
(ii) *for each n there exists m such that for each bijection $\sigma:\mathbb{N}\to\mathbb{N}$,*

$$\sum_i \frac{a_i^n b^n_{\sigma(i)}}{a_i^m b^m_{\sigma(i)}} < \infty.$$

Proof. In view of Lemma 4, (i) is equivalent to

$\forall n\,\exists m$ such that the injection

$$\varrho_{n,m}\otimes\sigma_{n,m}: E_m\otimes_\varepsilon F_m\to E_n\otimes_\pi F_n \text{ is continuous,}$$

where $E=\lambda_p(A)=\mathrm{proj}_n(l_p(a^n))$ and $F=\lambda_q(B)=\mathrm{proj}_n(l_q(b^n))$; thus (i) is equivalent to

(iii) $\forall n\,\exists m$ such that $\varrho_{n,m}\otimes\sigma_{n,m}: l_p(a^m)\otimes_\varepsilon l_q(b^m)\to l_p(a^n)\otimes_\pi l_q(b^n)$ is continuous,

which, in turn, is equivalent to

(iv) $\forall n\,\exists m$ such that $R\otimes T: l_p\otimes_\varepsilon l_q\to l_p\otimes_\pi l_q$ is continuous, where R is the diagonal map $(a^n/a^m): l_p\to l_p$ and T is the diagonal map $(b^n/b^m): l_q\to l_q$.

Now, from Theorem 1, it follows that (ii)⇒(iv)⇔(i). Also, (i)⇔(iv)⇒ (by Theorem 1) $\forall n\,\exists m$ such that $(a_i^n b^n_{\sigma(i)}/(a_i^m b^m_{\sigma(i)}))\in l_r$ for a suitable $r\ge 1$ and all bijections σ. Repeating this process a sufficient number of times so that $l_r\cdot l_r\cdot\ldots\cdot l_r$ is contained in l_1 we see that (iv)⇒(ii). ∎

The implication (ii)⇒(i) in the theorem above, for matrices with a^n/a^{n+1} and b^n/b^{n+1} decreasing, already follows from John [16], where he obtains some results for matrices obtained as powers of a given strictly decreasing null sequence. This has also to be compared with [10]. The first examples of non-nuclear Fréchet–Schwartz spaces E and F such that $E\otimes_\varepsilon F=E\otimes_\pi F$ holds topologically were obtained by John [16]. In fact, John showed that there are decreasing sequences $a=(a_i)$ and $b=(b_i)$ of strictly positive numbers such that $a,b\in c_0$ and $ab\in l_1$ but $a\notin l_p$ and $b\notin l_p$ for all $p>0$. John's examples are obtained by taking $E=\lambda_2(A)$, $F=\lambda_2(B)$, $A=(a^n):=((a_i)^{-n})$, and $B=(b^n):=((b_i)^{-n})$.

A Köthe matrix $A=(a^n)$ is called *regular* if a^n/a^{n+1} is decreasing for every $n\in\mathbb{N}$.

COROLLARY 6. *Let $A=(a^n)$ and $B=(b^n)$ be regular Köthe matrices. The following conditions are equivalent:*

(1) *$\lambda_p(A)\otimes_\varepsilon\lambda_q(B)=\lambda_p(A)\otimes_\pi\lambda_q(B)$ holds topologically;*
(2) *$\forall n\,\exists m>n: (a^nb^n)/(a^mb^m)\in l_1$.*

This result is no longer true if we drop the regularity assumption. Indeed, apply [16, Lemma 3.1] to find decreasing sequences $a,b\in c_0$ with $ab\in l_1$

but $a \notin l_p$ and $b \notin l_p$ for all $p > 0$. Define new sequences as follows: $c_{2i} := b_i^{-1}$, $c_{2i-1} := a_i^{-1}$, $d_{2i} := a_i^{-1}$, $d_{2i-1} := b_i^{-1}$ for each $i \in \mathbb{N}$, and take $C := ((c_i)^n)$ and $D := ((d_i)^n)$. The corresponding spaces $\lambda_2(C)$ and $\lambda_2(D)$ are Fréchet–Schwartz, the quotient $(c^n d^n)/(c^{n+1}d^{n+1}) = (a_j b_j)$ belongs to l_1, but $\lambda_2(C) \otimes_\varepsilon \lambda_2(D)$ and $\lambda_2(C) \otimes_\pi \lambda_2(D)$ do not coincide topologically. Indeed, if we define $\sigma : \mathbb{N} \to \mathbb{N}$ by $\sigma(2i) = 2i-1$ and $\sigma(2i-1) = 2i$ for every $i \in \mathbb{N}$, then

$$\sum_i \frac{c_i d_{\sigma(i)}}{c_i^n d_{\sigma(i)}^n} = \sum_i a_i^{n-1} + \sum_i b_i^{n-1} = \infty$$

for every $n \in \mathbb{N}$, and the conclusion follows from Theorem 5.

Next we show that the topological identity of the projective and the injective topologies on the tensor products of two Köthe echelon spaces has consequences on the structure of the spaces involved. The case $p = q = 1$ was already given in [6].

PROPOSITION 7. *If $\lambda_p(A) \otimes_\varepsilon \lambda_q(B) = \lambda_p(A) \otimes_\pi \lambda_q(B)$ holds topologically, then $\lambda_p(A)$ or $\lambda_q(B)$ is nuclear or both spaces are Fréchet–Schwartz.*

Proof. We will show that if the topological identity holds and one of the spaces is not Schwartz, the other must be nuclear. Assume that $\lambda_q(B)$ is not Schwartz. There is $n' \in \mathbb{N}$ such that $b_{n'}/b_m \notin c_0$ for all $m > n'$. Accordingly, for every $m > n$, there are $\varepsilon_m > 0$ and an injection $\delta : \mathbb{N} \to \mathbb{N}$ (which depends on m) such that $b_{n',\delta(i)}/b_{m,\delta(i)} \ge \varepsilon_m$ for every $i \in \mathbb{N}$. In order to prove that $\lambda_p(A)$ is nuclear, we fix $n > n'$. By Theorem 5, we select $m > n$ with

$$\sum_i \frac{a_{n,i} b_{n,\sigma(i)}}{a_{m,i} b_{m,\sigma(i)}} < \infty$$

for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$. We want to show that $a_n/a_m \in l_1$. If this is not the case, we may suppose without loss of generality that $(a_{n,2i}/a_{m,2i})_{i\in\mathbb{N}} \notin l_1$. We take any injection $\varrho : \mathbb{N} \to \mathbb{N}$ whose image is $\mathbb{N} \setminus \{\delta(2i) : i \in \mathbb{N}\}$ and we define the bijection $\sigma : \mathbb{N} \to \mathbb{N}$ by $\sigma(2i) := \delta(2i)$ and $\sigma(2i-1) := \varrho(i)$, $i \in \mathbb{N}$. We have

$$\infty > \sum_i \frac{a_{n,i} b_{n,\sigma(i)}}{a_{m,i} b_{m,\sigma(i)}} \ge \sum_i \frac{a_{n,2i} b_{n,\delta(2i)}}{a_{m,2i} b_{m,\delta(2i)}} \ge \varepsilon_m \sum_i \frac{a_{n,2i}}{a_{m,2i}} = \infty,$$

a contradiction. ∎

For comparison, we recall here that Pisier [23] constructed an infinite-dimensional Banach space P such that $P \otimes_\varepsilon P = P \otimes_\pi P$ holds topologically.

There are more conditions which are equivalent to the ones in Theorem 5. For the definition of vector-valued echelon spaces we refer to [18] and [13]. We refer to [8, pp. 78 ff.] for the norm Δ_p.

PROPOSITION 8. *For Köthe matrices $A = (a^n)$ and $B = (b^n)$ with $a_i^n, b_i^n > 0$ for all $i, n \in \mathbb{N}$, the following conditions are also equivalent to conditions* (i) *and* (ii) *in Theorem* 5:

(iii) *If $p \ne 0$, then $\lambda_p(A) \widehat{\otimes}_\varepsilon \lambda_q(B) = \lambda_p(A, \lambda_q(B))$ holds topologically;*
(iv) *If $p \ne 1$, then $\lambda_p(A) \widehat{\otimes}_\pi \lambda_q(B) = \lambda_p(A, \lambda_q(B))$ holds topologically;*
(v) *If $p \ne q$, then $\lambda_p(A, \lambda_q(B)) = \lambda_q(B, \lambda_p(A))$ holds topologically.*

Before we start the proof, we make the following remarks. If $p = 0$, then $\lambda_0(A) \widehat{\otimes}_\varepsilon \lambda_q(B) = \lambda_0(A, \lambda_q(B))$ holds topologically for arbitrary B. If $p = 1$, then $\lambda_1(A) \widehat{\otimes}_\pi \lambda_q(B) = \lambda_1(A, \lambda_q(B))$ holds topologically for all B. Moreover, if $p = q$, then $\lambda_p(A, \lambda_p(B)) = \lambda_p(B, \lambda_p(A)) = \lambda_p(A \otimes B)$ for all matrices A and B, with $A \otimes B = (a_i^n b_j^n)_{(i,j)\in\mathbb{N}\times\mathbb{N}}$.

Proof of Proposition 8. It is easy to see that condition (i) in Theorem 5 implies conditions (iii)–(v).

We prove that (iii) implies (ii). To do this assume first $p \ne 0$, $q \ne 0$ and $\lambda_p(A) \widehat{\otimes}_\varepsilon \lambda_q(B) = \lambda_p(A, \lambda_q(B))$. This implies

$$\forall n\, \exists m > n : \quad \|R \otimes S : l_p \otimes_\varepsilon l_q \to l_p \otimes_{\Delta_p} l_q\| < \infty,$$

where R and S are the diagonal operators associated with a^n/a^m and b^n/b^m respectively.

By a result on so-called Bennet matrices (see [5]) we have

$$\|\mathrm{id} : l_p^n \otimes_\varepsilon l_q^n \to l_p^n \otimes_{\Delta_p} l_q^n\| \ge c(p,q) n^{d(p,q)}$$

for each $n \in \mathbb{N}$, where

$$d(p,q) = \begin{cases} 1/q & \text{if } 2 \le p \le q \le \infty; \\ 1/p & \text{if } 2 \le q \le p \le \infty; \\ 1/q & \text{if } 1 \le p \le 2 \le q \le \infty; \\ 1/p & \text{if } 1 \le q \le 2 \le p \le \infty; \\ 1/2 & \text{if } 1 \le p, q \le 2. \end{cases}$$

Proceeding as in Case 3 of the proof of Theorem 1, we obtain

$$\sum_i \left(\frac{a_i^n b_{\sigma(i)}^n}{a_i^m b_{\sigma(i)}^m}\right)^{(1+\varepsilon)/\lambda} < \infty$$

for some $\lambda > 0$. This implies condition (ii) of Theorem 5.

If $q = 0$, $p \ne 0$ and $\lambda_p(A) \widehat{\otimes}_\varepsilon \lambda_0(B) = \lambda_p(A, \lambda_0(B))$, then $\lambda_p(A, \lambda_0(B)) = \lambda_0(B, \lambda_p(A))$, and the conclusion follows from the equivalence of (v) and (ii) that we are going to check now. The proof of (iv)⇒(ii) follows by duality.

To prove (v)⇒(ii), we assume $p < q$. By [8, p. 79] we have

$$\|\mathrm{id} : l_p \otimes_{\Delta_p} l_q \to l_p \otimes_{\Delta_q^t} l_q\| \le 1,$$

hence $\lambda_p(A, \lambda_q(B))$ is continuously embedded in $\lambda_q(B, \lambda_p(A))$. If they coincide topologically we have

$$\forall n\, \exists m > n: \quad \|R \otimes S : l_p \otimes_{\Delta_q^t} l_q \to l_p \otimes_{\Delta_p} l_q\| < \infty,$$

where R and S are diagonal operators as above.

Since $\|\mathrm{id} : l_p^n \otimes_{\Delta_q^t} l_q^n \to l_p^n \otimes_{\Delta_p} l_q^n\| \ge n^{1/p-1/q}$ for each $n \in \mathbb{N}$, we can proceed as above to conclude (ii). ■

We now present examples of non-nuclear (DFS)-spaces G and H such that $G \otimes_\varepsilon H = G \otimes_\pi H$ holds topologically. We recall that G is a *(DFS)-space* if G is the inductive limit of a sequence of Banach spaces G_n such that the canonical inclusions $i_{n,n+1} : G_n \to G_{n+1}$ are compact. We first need the following (more or less well-known) lemma.

LEMMA 9. *Let $G = \mathrm{ind}_n G_n$ and $H = \mathrm{ind}_n H_n$ be (DFS)-spaces. If the Banach spaces G_n and H_n have the approximation property, then $G \widehat{\otimes}_\pi H$ and $G \widehat{\otimes}_\varepsilon H$ are (DFS)-spaces and we have the following representations as injective inductive limits:*

$$G \widehat{\otimes}_\pi H = \mathrm{ind}_n (G_n \widehat{\otimes}_\pi H_n) \quad \text{and} \quad G \widehat{\otimes}_\varepsilon H = \mathrm{ind}_n (G_n \widehat{\otimes}_\varepsilon H_n).$$

Moreover, the canonical map $\widehat{\psi} : G \widehat{\otimes}_\pi H \to G \widehat{\otimes}_\varepsilon H$ is injective.

Proof. The result for injective tensor products is well known and follows from [2]. Concerning the projective tensor product, we first apply [13, 15.5.4] to deduce that $G \otimes_\pi H = \mathrm{ind}_n (G_n \otimes_\pi H_n)$ holds topologically. Moreover, the connecting maps

$$i_{n,n+1} \widehat{\otimes} j_{n,n+1} : G_n \widehat{\otimes}_\pi H_n \to G_{n+1} \widehat{\otimes}_\pi H_{n+1}$$

are compact for every $n \in \mathbb{N}$ and injective, by the approximation property. Analogously, the canonical maps

$$i_n \widehat{\otimes} j_n : G_n \widehat{\otimes}_\pi H_n \to G \widehat{\otimes}_\pi H$$

are also injective and continuous. Consequently, we have a continuous injection

$$j : \mathrm{ind}_n (G_n \widehat{\otimes}_\pi H_n) \to G \widehat{\otimes}_\pi H.$$

Since both spaces induce the same topology on $G \otimes H$, we can apply [3, 2.1] to conclude that j is also open. Since the inductive limit $\mathrm{ind}_n (G_n \widehat{\otimes}_\pi H_n)$ is complete (it is a (DFS)-space) and $G \otimes H$ is dense in it, we see that j is also surjective and the proof of this case is complete. The last statement now follows easily. One could apply [7, Theorem]. ■

If $E = \mathrm{proj}_n (E_n, \varrho_{nm} : E_m \to E_n)$ is a Fréchet–Schwartz space which is the reduced projective limit of a sequence of Banach spaces (E_n) with compact linking maps $\varrho_{n,m}$ $(n \le m)$, then its strong dual E'_b is a (DFS)-space. In fact, $E'_b = \mathrm{ind}_n E'_n$, and we will identify E'_n with a subspace of E'. The linking maps $i_{n,n+1} = \varrho^t_{n,n+1} : E'_n \to E'_{n+1}$ are injective and compact.

PROPOSITION 10. *Let $E = \mathrm{proj}_n (E_n, \varrho_{nm} : E_m \to E_n)$ and $F = \mathrm{proj}_n (F_n, \sigma_{nm} : F_m \to F_n)$ be Fréchet–Schwartz spaces which are reduced projective limits of sequences of reflexive Banach spaces with the approximation property. The following conditions are equivalent:*

(1) *$E \otimes_\varepsilon F = E \otimes_\pi F$ holds topologically;*
(2) *$E'_b \otimes_\varepsilon F'_b = E'_b \otimes_\pi F'_b$ holds topologically;*
(3) *$E'_b \widehat{\otimes}_\varepsilon F'_b = E'_b \widehat{\otimes}_\pi F'_b$ holds algebraically;*
(3)′ *$E'_b \widehat{\otimes}_\varepsilon F'_b = E'_b \widehat{\otimes}_\pi F'_b$ holds algebraically and topologically;*
(4) *$\forall n\, \exists m > n : E'_n \widehat{\otimes}_\varepsilon F'_n \subset E'_m \widehat{\otimes}_\pi F'_m$;*
(4)′ *$\forall n\, \exists m > n : E'_n \widehat{\otimes}_\varepsilon F'_n \subset E'_m \widehat{\otimes}_\pi F'_m$, with continuous inclusion;*
(5) *$\forall n\, \exists m > n : K(E_n, F'_n) \subset N(E_m, F'_m)$.*

Proof. Since $E'_b = \mathrm{ind}_n E'_n$ and $F'_b = \mathrm{ind}_n F'_n$, the proof of the equivalence of (2), (3) and (3)′ follows the same lines as the proof of Lemma 4, using the open mapping theorem for (LB)-spaces.

By Lemma 9, $E'_b \widehat{\otimes}_\varepsilon F'_b = \mathrm{ind}_n (E'_n \widehat{\otimes}_\varepsilon F'_n)$ and $E'_b \widehat{\otimes}_\pi F'_b = \mathrm{ind}_n (E'_n \widehat{\otimes}_\pi F'_n)$ with continuous injections. We then apply Grothendieck's factorization theorem to conclude that (3)′ is equivalent to (4)′. Clearly (4)′ implies (4) and (4) implies (3).

The equivalence of (4) and (5) follows from [13, 17.1.9, 17.3.3 and 18.3.4], since $K(E_n, F'_n) = E'_n \widehat{\otimes}_\varepsilon F'_n$ and $N(E_m, F'_m) = E'_m \widehat{\otimes}_\pi F'_m$.

It remains to show the equivalence of (1) and all the other conditions. Since $E \otimes_\varepsilon F$ and $E \otimes_\pi F$ are metrizable, condition (1) holds if and only if $(E \otimes_\varepsilon F)' = (E \otimes_\pi F)'$ holds algebraically, or, equivalently, if and only if $(E \widehat{\otimes}_\varepsilon F)' = (E \widehat{\otimes}_\pi F)'$ holds algebraically. Now, all the spaces involved are Fréchet–Schwartz spaces, and we can apply Buchwalter's duality (e.g. [18, §45.3]) to conclude that $(E \widehat{\otimes}_\varepsilon F)' = (E \widehat{\otimes}_\pi F)'$ holds algebraically if and only if $E'_b \widehat{\otimes}_\pi F'_b = L(E, F'_b) = E'_b \widehat{\otimes}_\varepsilon F'_b$ holds algebraically. This is exactly condition (3). ■

Remark. (1) Proceeding as in [19, 11.3.19 and 11.3.21] one can show that the conditions in Proposition 10 are equivalent to

(a) $E \otimes_\varepsilon F$ is ultrabornological;
(a)′ $E'_b \otimes_\varepsilon F'_b$ is ultrabornological.

If we also assume that E or F has the bounded approximation property, we can apply [19, 11.5.8] to conclude that all those conditions are also equivalent to

(b) $E \otimes_\varepsilon F$ is barrelled;
(b)′ $E'_b \otimes_\varepsilon F'_b$ is barrelled.

(2) If $G = \mathrm{ind}_n G_n$ and $H = \mathrm{ind}_n H_n$ are (DFS)-spaces such that G_n and H_n have the a.p. for every $n \in \mathbb{N}$, then $G \otimes_\varepsilon H$ is bornological and $G \otimes_\varepsilon H = \mathrm{ind}_n(G_n \otimes_\varepsilon H_n)$. But, in general, it seems to be unknown if $G \otimes_\varepsilon H$ is bornological for every pair of (DFS)-spaces G and H, although $G \widehat{\otimes}_\varepsilon H$ is indeed a (DFS)-space by [2].

COROLLARY 11. *Let $1 < p, q < \infty$. Let $A = (a^n)$ and $B = (b^n)$ be regular Köthe matrices such that $\lambda_p(A)$ and $\lambda_q(B)$ are Fréchet–Schwartz spaces. The following conditions are equivalent*:

(1) $\lambda_p(A)'_{\mathrm{b}} \otimes_\varepsilon \lambda_q(B)'_{\mathrm{b}} = \lambda_p(A)'_{\mathrm{b}} \otimes_\pi \lambda_q(A)'_{\mathrm{b}}$ *holds topologically*;

(2) $\forall n\, \exists m > n : (a^n b^n)/(a^m b^m) \in l_1$.

From our characterizations above it follows that, if A is regular, then $\lambda_p(A) \otimes_\varepsilon \lambda_p(A) = \lambda_p(A) \otimes_\pi \lambda_p(A)$ holds topologically, for $1 < p < \infty$, if and only if for each $n \in \mathbb{N}$ there is $m > n$ such that $a^n/a^m \in l_2$, i.e. if and only if $\lambda_p(A)$ is nuclear. This is a particular case of much deeper results due to Jarchow and John [14] (see also [1] for further references). Very recently Jarchow and John [15] have used Pisier's example [23] to construct a non-nuclear Fréchet–Schwartz space E such that $E \otimes_\varepsilon E = E \otimes_\pi E$ holds topologically. We now present a more direct construction of such a Fréchet–Schwartz space.

We recall that Schwartz's *ε-product* of two locally convex spaces E and F is defined by $E \,\varepsilon\, F = L_{\mathrm{e}}(E'_{\mathrm{co}}, F)$ (see e.g. [18]). The canonical map $\psi : E \widehat{\otimes}_\pi F \to E \,\varepsilon\, F$ is defined by $\psi(x \otimes y)(u) := \langle x, u \rangle y$ for $x \in E$, $y \in F$ and $u \in E'$.

Pisier [23, 24] has constructed an infinite-dimensional Banach space P such that both P and P' are of cotype 2 and $P \otimes_\varepsilon P = P \otimes_\pi P$ holds topologically. John [17] has proved that $K(P, P') = N(P, P')$. We can apply [13, 17.1.9 and 17.3.3] to conclude that the canonical map $\phi : P' \widehat{\otimes}_\pi P' \to P' \,\varepsilon\, P'$ is surjective.

LEMMA 12. *There is a (DFS)-space E without the approximation property such that the canonical map $\psi : E \widehat{\otimes}_\pi E \to E \,\varepsilon\, E$ is surjective.*

Proof. The dual P' of the Pisier space P does not have the approximation property (see [24]). Take an absolutely convex compact subset K_1 of P' such that the identity operator on P' cannot be uniformly approximated on K_1 by finite rank operators on P'.

The set $\widetilde{K}_1 := \{f \in P' \,\varepsilon\, P' : f(K_1^\circ) \subset K_1\}$ is an absolutely convex compact subset of $P' \,\varepsilon\, P'$ by [18, §44.3(2)]. Since $\phi : P' \widehat{\otimes}_\pi P' \to P' \,\varepsilon\, P'$ is a surjective homomorphism between Banach spaces, ϕ lifts compact subsets. Accordingly, we can apply [18, §41.4(5)] to find sequences (x_k^1) and (y_k^1) converging to 0 in P' such that for each $z \in \widetilde{K}_1$ there is $(\lambda_k) \in l_1$ with $\sum |\lambda_k| \le 1$ and $z = \phi(\sum \lambda_k (x_k^1 \otimes y_k^1))$ (and the series converges in $P' \widehat{\otimes}_\pi P'$).

We select an absolutely convex compact subset K_2 of P' such that $K_1 \cup \{x_k^1\} \cup \{y_k^1\} \subset K_2$ and such that K_1 is compact in the Banach space P'_{K_2} generated by K_2.

Proceeding by induction we can select an increasing sequence (K_n) of absolutely convex compact subsets of P' such that, if we denote the Banach space P'_{K_n} by E_n, we have

(i) the injection $j_{n,n+1} : E_n \to E_{n+1}$ is compact,

(ii) for all $z \in P' \,\varepsilon\, P'$ with $z(K_n^\circ) \subset K_n$ (polar taken in P'') there are sequences $(x_k), (y_k)$ converging to 0 in E_{n+1}, contained in K_{n+1}, and there is $(\lambda_k) \subset \mathbb{K}$ with $\sum |\lambda_k| \le 1$ and $z = \phi(\sum \lambda_k (x_k \otimes y_k))$.

We define $E := \mathrm{ind}_n E_n$. Then E is a (DFS)-space without the approximation property (see e.g. [13, 18.5.8]). We will show that the canonical map $\psi : E \widehat{\otimes}_\pi E \to E \,\varepsilon\, E$ is surjective. To do this we consider the commutative diagram

$$\begin{array}{ccc} E \widehat{\otimes}_\pi E & \xrightarrow{\ \psi\ } & E \,\varepsilon\, E \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ P' \widehat{\otimes}_\pi P' & \xrightarrow{\ \phi\ } & P' \,\varepsilon\, P' \end{array}$$

where φ_1 and φ_2 are the canonical maps associated with the continuous inclusion $j : E \to P'$.

We fix $z \in E \,\varepsilon\, E = L_{\mathrm{b}}(E', E)$. There is $n \in \mathbb{N}$ such that $z(K_n^\bullet) \subset K_n$ (the polar $\bullet$ taken in E'). Then $\varphi_2(z)(K_n^\circ) \subset K_n$ (the polar $\circ$ taken in P''). By (ii) above we can find $(x_k) \subset K_{n+1}$, $(y_k) \subset K_{n+1}$ and $(\lambda_k) \subset \mathbb{K}$ with $\sum |\lambda_k| \le 1$ such that

$$\varphi_2(z) = \phi\Big(\sum \lambda_k (x_k \otimes y_k)\Big).$$

But the series $x = \sum \lambda_k (x_k \otimes y_k)$ converges in $E_{n+1} \widehat{\otimes}_\pi E_{n+1}$, hence in $E \widehat{\otimes}_\pi E$. Accordingly,

$$\varphi_1(x) = \sum \lambda_k (x_k \otimes y_k) \quad \text{in } P' \widehat{\otimes}_\pi P'.$$

Consequently, $\varphi_2(z) = \phi(\varphi_1(z)) = \varphi_2(\psi(x))$. Since φ_2 is injective, we conclude that $z = \psi(x)$ and ψ is surjective. ∎

THEOREM 13. *There is a Fréchet–Schwartz space F without the approximation property such that $F \otimes_\pi F = F \otimes_\varepsilon F$ holds topologically.*

Proof. By Lemma 12 there is a (DFS)-space E without the approximation property such that the canonical map $\psi : E \widehat{\otimes}_\pi E \to E \,\varepsilon\, E$ is surjective.

By Buchwalter's duality (e.g. [18, 45.3]) both $E\widehat{\otimes}_\pi E$ and $E\,\varepsilon\,E$ are (DFS)-spaces. Accordingly, ψ is a surjective homomorphism that lifts bounded (or compact) sets. This implies that

$$\psi^t : (E\,\varepsilon\,E)'_{\mathrm{b}} \to (E\widehat{\otimes}_\pi E)'_{\mathrm{b}}$$

is a monomorphism. Buchwalter's duality again yields that

$$\psi^t : E'_{\mathrm{b}}\widehat{\otimes}_\pi E'_{\mathrm{b}} \to E'_{\mathrm{b}}\,\varepsilon\,E'_{\mathrm{b}}$$

is a monomorphism. It is easy to see that the restriction of ψ^t to the tensor product coincides with the identity. Consequently, $E'_{\mathrm{b}} \otimes_\pi E'_{\mathrm{b}} = E'_{\mathrm{b}} \otimes_\varepsilon E'_{\mathrm{b}}$ holds topologically.

The space $F := E'_{\mathrm{b}}$ is Fréchet–Schwartz, does not have the approximation property and the topologies ε and π coincide on $F \otimes F$. ■

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