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Operators in finite distributive subspace lattices II

by

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Abstract. In a previous paper we gave an example of a finite distributive subspace lattice $\mathcal{L}$ on a Hilbert space and a rank two operator of $\operatorname{Alg}\mathcal{L}$ that cannot be written as a finite sum of rank one operators from $\operatorname{Alg}\mathcal{L}$. The lattice $\mathcal{L}$ was a specific realization of the free distributive lattice on three generators. In the present paper, which is a sequel to the aforementioned one, we study $\operatorname{Alg}\mathcal{L}$ for the general free distributive lattice with three generators (on a normed space). Necessary and sufficient conditions are given for 1) a finite rank operator of $\operatorname{Alg}\mathcal{L}$ to be written as a finite sum of rank ones from $\operatorname{Alg}\mathcal{L}$, and 2) a realization of $\mathcal{L}$ to contain a finite rank operator of $\operatorname{Alg}\mathcal{L}$ with the preceding property. These results are then used to show the curiosity that the product of two finite rank operators of $\operatorname{Alg}\mathcal{L}$ always has the above property.

1. Introduction. This paper is a continuation of [7], of which we shall assume familiarity and whose notation we follow.

Briefly, if $\mathcal{L}$ is a subspace lattice on a normed space $\mathcal{X}$, a general question is whether every finite rank operator of $\operatorname{Alg}\mathcal{L}$ has the FRP, i.e. whether it can be written as a finite sum of rank one operators from $\operatorname{Alg}\mathcal{L}$. The question is more natural in the case of completely distributive $\mathcal{L}$, as $\operatorname{Alg}\mathcal{L}$ then has a large supply of rank one operators [4]. Indeed, in the special case of a nest $\mathcal{L}$ the answer is affirmative [1, 6] and so is the case when $\mathcal{L}$ is a complete atomic Boolean subspace lattice [5, 3]. (In some of these results $\mathcal{X}$ was assumed a Hilbert space.) For general completely distributive lattices the answer was again shown to be affirmative if the underlying space was finite-dimensional [5] but the question was finally settled negatively by Hopenwasser and Moore [2] in infinite dimensions. In the same paper they give an affirmative answer if $\mathcal{L}$ is a finite width (see [2] for the definition) commutative subspace lattice. Their example of a completely distributive subspace lattice $\mathcal{L}$ for which $\operatorname{Alg}\mathcal{L}$ fails the FRP has an infinite number of elements. This then left open the case of finite distributive subspace lattices $\mathcal{L}$, which was settled negatively in [7]. There, a specific realization of the free distributive lattice $\mathcal{L}_3$ was

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given together with an example of a rank two operator in $\operatorname{Alg}\mathcal{L}_3$ which fails the FRP.

In the present paper we systematically discuss $\operatorname{Alg}\mathcal{L}_3$, for any realization of $\mathcal{L}_3$ (on a normed space). For instance, we give necessary and sufficient conditions for a finite rank operator to be in $\operatorname{Alg}\mathcal{L}_3$ (Theorem 1) and, secondly, to have the FRP (Theorem 2). By the example in [7] it follows that the two requirements are distinct and that our statements are not vacuous. Also Theorem 3 characterizes the realizations of $\mathcal{L}_3$ which do not have the FRP.

Finally, we apply the above to show that if $T, F \in \operatorname{Alg}\mathcal{L}_3$ are of finite rank, then their product TF does have the FRP.

We shall now introduce some new notation. Let $\mathcal{X}$ be a normed space. If $T \in \mathcal{B}(\mathcal{X})$, $f^* \in \mathcal{X}^*$ and $M \subseteq \mathcal{X}$, we denote by $T|_M$ and $f^*|_M$ the restrictions of T and f^* to M respectively. Also, if $T \in \mathcal{B}(\mathcal{X})$ we define $\operatorname{Ker} T = \{x \in \mathcal{X} : Tx = 0\}$. The symbol "$\subset$" will mean proper inclusion.

Let Z_i $(i = 1, \ldots, n)$ be closed subspaces of X. We say that the set $\{Z_i : i = 1, \ldots, n\}$ is *linearly independent* if for each selection $0 \neq z_i \in Z_i$ the set $\{z_i : i = 1, \ldots, n\}$ is linearly independent. If Z_i $(i = 1, \ldots, n)$ are subspaces of $\mathcal{X}$ we shall denote by $\bigvee\{Z_i : i = 1, \ldots, n\}$ the smallest (closed) subspace of $\mathcal{X}$ which contains all the Z_i.

Let Y be a subspace of $\mathcal{X}$. We say that a subspace Z of $\mathcal{X}$ is a *complement* of Y in $\mathcal{X}$ if $Z \cap Y = 0$ and $Z + Y = \mathcal{X}$. It is easy to see that if $\mathcal{X}$ is finite-dimensional then for each $Y \subseteq \mathcal{X}$ there is a complement of Y in $\mathcal{X}$.

We know that if $T \in \mathcal{B}(\mathcal{X})$ is a finite rank operator then $\mathcal{R}(T^*) = (\operatorname{Ker} T)^\perp$. Also, for $L \subseteq \mathcal{X}$ we have $\mathcal{R}(T^*) \subseteq L^\perp$ if and only if $L \subseteq \operatorname{Ker} T$ (T finite rank).

We shall also use the following lemmas (the first one is in [4]):

LEMMA 1. *Let $\mathcal{X}$ be a normed space, let $\mathcal{L}$ be a subspace lattice on $\mathcal{X}$ and let $0 \neq e^* \in \mathcal{X}^*$ and $0 \neq f \in \mathcal{X}$. Then $e^* \otimes f \in \operatorname{Alg}\mathcal{L}$ if and only if there is an $N \in \mathcal{L}$ such that $f \in N$ and $e^* \in (N_-)^\perp$.* ∎

The next lemma is essentially a corollary of Lemma 1.

LEMMA 2. *Let $\mathcal{X}$ and $\mathcal{L}$ be as in Lemma 1. Then any non-zero finite rank operator $R \in \mathcal{B}(\mathcal{X})$ for which there is an $N \in \mathcal{L}$ with $\mathcal{R}(R) \subseteq N$ and $N_- \subseteq \operatorname{Ker} R$ necessarily has the FRP. Furthermore, R can be written as a sum of* rank R *rank one operators from* $\operatorname{Alg}\mathcal{L}$.

Proof. Let $\{x_i : i = 1, \ldots, n\}$ be a basis of $\mathcal{R}(R)$. Then there are unique $y_i^* \in \mathcal{X}^*$ such that $R = \sum_{i=1}^n y_i^* \otimes x_i$. Obviously $\mathcal{R}(R) = \langle x_i : i = 1, \ldots, n\rangle$ and $\langle y_i^* : i = 1, \ldots, n\rangle = \mathcal{R}(R^*) \subseteq N_-^\perp$. By Lemma 1 each $y_i^* \otimes x_i$ $(i = 1, \ldots, n)$ belongs to $\operatorname{Alg}\mathcal{L}$, that is, R has the FRP. ∎

LEMMA 3. *Let $R \in \mathcal{B}(\mathcal{X})$ be a finite rank operator and Y_1, Y_2 be subspaces of $\mathcal{X}$ such that $\mathcal{R}(R) \subseteq Y_1 + Y_2$. Then there are finite rank operators $R_1, R_2 \in \mathcal{B}(\mathcal{X})$ such that $R = R_1 + R_2$, $\mathcal{R}(R_i) \subseteq Y_i$ and $\mathcal{R}(R_i^*) \subseteq \mathcal{R}(R^*)$ $(i = 1, 2)$.*

Proof. As in the proof of Lemma 2, the R can be written in the form $R = \sum_{i=1}^n y_i^* \otimes x_i$. So, there are $\{t_i : i = 1, \ldots, n\} \subseteq Y_1$ and $\{s_i : i = 1, \ldots, n\} \subseteq Y_2$ such that $x_i = t_i + s_i$ $(i = 1, \ldots, n)$. Then the operators $R_1 = \sum_{i=1}^n y_i^* \otimes t_i$ and $R_2 = \sum_{i=1}^n y_i^* \otimes s_i$ satisfy the conclusions of the lemma. ∎

2. The free distributive lattice $\mathcal{L}_3$.

In this section we discuss in a systematic way the finite rank operators of $\operatorname{Alg}\mathcal{L}_3$ for any realization of the free distributive subspace lattice $\mathcal{L}_3$ on 3 generators. Our main result is a characterization of the set of finite rank operators of $\operatorname{Alg}\mathcal{L}_3$ as well as its subalgebra of operators with the FRP. As already mentioned, it follows from [7] that these two algebras need not coincide.

First we shall give some lemmas valid for general finite distributive subspace lattices on a normed space.

LEMMA 4. *Let $\mathcal{L}$ be a finite distributive subspace lattice on a normed space and $\{L_i : i = 1, \ldots, n\} \subseteq \mathcal{L}$. Then*

$$\bigcap_{i=1}^n L_{i-} = \bigvee\{M \in \mathcal{L} : L_i \nsubseteq M,\ i = 1, \ldots, n\}.$$

Proof. By distributivity we have

$$\begin{aligned}\bigcap_{i=1}^n L_{i-} &= L_{1-} \cap \ldots \cap L_{n-}\\ &= \Big(\bigvee\{K_1 \in \mathcal{L} : L_1 \nsubseteq K_1\}\Big) \cap \ldots \cap \Big(\bigvee\{K_n \in \mathcal{L} : L_n \nsubseteq K_n\}\Big)\\ &= \bigvee\{K_1 \cap \ldots \cap K_n : K_i \in \mathcal{L},\ L_i \nsubseteq K_i,\ i = 1, \ldots, n\}\\ &\subseteq \bigvee\{M \in \mathcal{L} : L_i \nsubseteq M,\ i = 1, \ldots, n\}.\end{aligned}$$

But if $L_i \nsubseteq M$ then $M \subseteq L_{i-}$ so the reverse inclusion also holds, showing equality of the two sides, as required. ∎

LEMMA 5. *Let $\mathcal{L}$ be a finite subspace lattice on a normed space $\mathcal{X}$ and $W \neq 0$ a finite-dimensional subspace of $\mathcal{X}$. Then there exists an $m \in \mathbb{N}$, a subset $\mathcal{M}_0(W) = \mathcal{M}_0 = \{M_i : i = 1, \ldots, m\}$ of $\mathcal{L}$, and subspaces $0 \neq W_i \subseteq M_i \cap W$ $(i = 1, \ldots, m)$ such that*

(1) *If $L \in \mathcal{L}$ then $L \cap W = \bigvee\{W_i : M_i \subseteq L\}$.*

In particular, if $L \cap W \neq 0$ then there is an $i \in \{1, \ldots, m\}$ such that $M_i \subseteq L$. Also (applying this to $L = \mathcal{X}$), $W = \bigvee\{W_i : i = 1, \ldots, m\}$.

(2) *For each* $i \in \{1,\ldots,m\}$ *we have* $W_i \cap (\bigvee\{W \cap L : L \in \mathcal{L}$ *and* $L \subset M_i\}) = 0$.

Proof. We define $Z_1 = \bigvee\{W \cap L : L \in \mathcal{L} \text{ and } L \subset \mathcal{X}\}$. If $Z_1 = 0$ we take $m = 1$, $M_1 = \mathcal{X}$, and $W_1 = W$. In this case no further step is to be taken. If we have proper inclusion $0 \subset Z_1 \subset W$, then we take W_1 to be a complement of Z_1 in W and $M_1 = \mathcal{X}$. Then $0 \neq W_1 \subseteq W$. If finally $Z_1 = W$ we do not define any M_i on this step.

In case $Z_1 \neq 0$, consider the maximal elements (with respect to inclusion order) of the (non-empty) set $\{L \in \mathcal{L} : L \subset \mathcal{X} \text{ and } L \cap W \neq 0\}$, which we denote by $N_1,\ldots,N_k$. We define $Z_2 = \bigvee\{Z_1 \cap L : L \in \mathcal{L} \text{ and } L \subset N_1\}$. (Note that for $L \subset \mathcal{X}$ we have $W \cap L = Z_1 \cap L$.) If $Z_2 = 0$, we set $M_{i_0} = N_1$, where $i_0 = 2$ if M_1 has been defined and $i_0 = 1$ otherwise. We also define $W_{i_0} = Z_1 \cap N_1$ $(= W \cap N_1)$. In this case no further steps are to be taken as far as N_1 is concerned but we continue in a similar manner with $N_2,\ldots,N_k$. If $0 \subset Z_2 \subset Z_1 \cap N_1$ then we define $M_{i_0} = N_1$, where $i_0 = 2$ if M_1 has been defined and $i_0 = 1$ if not. Also we take $0 \neq W_{i_0} \subseteq W$ to be a complement of Z_2 in $Z_1 \cap N_1$. Then $0 \neq W_{i_0} \subseteq W \cap M_{i_0}$. If $Z_2 = Z_1 \cap N_1$ we do not define any new M_i on this step.

We next consider the maximal elements of the set $\{L \in \mathcal{L} : L \cap W \neq 0$ and $L \subset N_1\}$ and for each of these we continue in a manner similar to the above. Since $\mathcal{L}$ is a finite lattice, this process terminates after a finite number of steps. After this we continue in a similar manner with the rest of the maximal elements $N_2,\ldots,N_k$. By deleting any of the M_i's if necessary we may suppose that they are pairwise distinct. It is clear from the way the construction was made that the conclusions of the lemma are satisfied. ■

Let $\mathcal{L}$ be a finite distributive subspace lattice on a normed space $\mathcal{X}$ and let $F \in \mathcal{B}(\mathcal{X})$ be of finite rank. By Lemma 5 applied to $W = \mathcal{R}(F)$ we find $m \in \mathbb{N}$, $\mathcal{M}_0 = \{M_i : i = 1,\ldots,m\} \subseteq \mathcal{L}$ and $0 \neq W_i \subseteq M_i \cap W, i = 1,\ldots,m$, which satisfy the conclusions of the lemma.

For such an operator F and with the notation just defined, we have the following

LEMMA 6. *The following are equivalent*:

(i) $F \in \operatorname{Alg} \mathcal{L}$,

(ii) $F(\bigcap_{i\in I} M_{i-}) \subseteq \bigvee\{W_i : i \in \{1,\ldots,m\} - I\}$, $\forall I \subseteq \{1,\ldots,m\}$.

In particular, (ii) *implies that* $F(\bigcap_{i=1}^{m} M_{i-}) = 0$.

Proof. (i)⇒(ii). Assume first that $M \in \mathcal{L}$ is such that for each $i \in I$ we have $M_i \not\subseteq M$, and that $z \in M$. Then $Fz \in F(\mathcal{X}) \cap M$, so from (1) of Lemma 5 it follows that

$$Fz \in \bigvee\{W_i : M_i \subseteq M\} \subseteq \bigvee\{W_i : i \in \{1,\ldots,m\} - I\}.$$

In the general case, it follows from Lemma 4 that

$$F\Big(\bigcap_{i\in I} M_{i-}\Big) = F\Big(\bigvee\{M \in \mathcal{L} : M_i \not\subseteq M,\ i \in I\}\Big) \subseteq \bigvee\{F(M) : M \in \mathcal{L} \text{ and } M_i \not\subseteq M,\ i \in I\}.$$

Now, if $w \in \{F(M) : M \in \mathcal{L} \text{ and } M_i \not\subseteq M, i \in I\}$ then there is an $M \in \mathcal{L}$ and a $z \in M$ such that $M_i \not\subseteq M$ $(i \in I)$ and $w = Fz$. From the first part of the proof, $Fz \in \bigvee\{W_i : i \in \{1,\ldots,m\} - I\}$, which is a closed subspace. Thus also $\bigvee\{F(M) : M \in \mathcal{L} \text{ and } M_i \not\subseteq M, i \in I\}) \subseteq \bigvee\{W_i : i \in \{1,\ldots,m\} - I\}$ as required.

(ii)⇒(i). Let $M \in \mathcal{L}$ be arbitrary. Define $I = \{i \in \{1,\ldots,m\} : M_i \subseteq M\}$ (which may be empty). For $i \notin I$ we have $M_i \not\subseteq M$ and so $M \subseteq M_{i-}$ and hence $M \subseteq \bigcap_{i\notin I} M_{i-}$. Thus $F(M) \subseteq \bigvee\{W_i : i \in I\}$, which is a subspace of M as required. ■

COROLLARY. *If for some* F *as in Lemma* 6 *the set* $\{W_i : i = 1,\ldots,m\}$ *is linearly independent*, *then* F *has the FRP. In fact*, *it can be written as a sum of* rank F *rank one operators of* $\operatorname{Alg}\mathcal{L}$.

Indeed, there are $F_i \in \mathcal{B}(\mathcal{X})$ such that $\mathcal{R}(F_i) = W_i$ and $F = \sum_{i=1}^{m} F_i$. If $z \in M_{1-}$ then $Fz \in \bigvee\{W_i : i = 2,\ldots,m\}$ (Lemma 6), so $F_1(z) = 0$ and so $F_1(M_{1-}) = 0$. Thus (Lemma 2) F_1 has the FRP. In a similar way, all other summands of F, and hence F itself, have the FRP. The final statement of the corollary is now clear. ■

Let now $\mathcal{L}_3$ denote the free distributive lattice with three generators on a normed space $\mathcal{X}$. The Hasse diagram of $\mathcal{L}_3$ is given in Figure 1 of [7].

As declared in Figure 1 of [7] the three generators of $\mathcal{L}_3$ are K_1, K_2, K_3. Moreover, we have

$$L_1 = K_1 \cap K_2, \quad N_1 = L_1 \vee L_2 = (K_1 \cap K_2) \vee (K_1 \cap K_3) = K_1 \cap (K_2 \vee K_3)$$

and cyclically for L_2, L_3, N_2, N_3. Also

$$M = (K_1 \cap K_2) \vee (K_1 \cap K_3) \vee (K_2 \cap K_3) = L_1 \vee L_2 \vee L_3 = N_1 \vee N_2 \vee N_3 = K_{1-} \cap K_{2-} \cap K_{3-}.$$

Moreover, $L_{1-} = K_3$, $L_{2-} = K_2$, $L_{3-} = K_1$, and $N_{i-} = K_{i-}$, $i = 1,2,3$. Also, since $L \supseteq M \Rightarrow L_- = \mathcal{X} \Rightarrow (L_-)^\perp = 0$, for each rank one operator in $\operatorname{Alg}\mathcal{L}_3$, the N of Lemma 1 is in $\{L_i, N_i, K_i : i = 1,2,3\}$.

The aim of the rest of this section is to prove necessary and sufficient conditions for a finite rank operator 1) to belong to $\operatorname{Alg}\mathcal{L}_3$ (Theorem 1), and 2) to have the FRP (Theorem 2). Then (Theorem 3) we characterize those realizations of $\mathcal{L}_3$ which do not have the FRP. The example in [7] shows that Theorem 3 is meaningful. Moreover, we expect difficulties in the proofs, which would take into account the subtleties of each specific realization. As the proofs are cumbersome, we shall divide them in lemmas, but keep a constant notation throughout.

Let then Q be a finite rank operator. By Lemma 5 applied to $W = \mathcal{R}(Q)$ there are $\mathcal{M}_0(\mathcal{R}(Q)) = \mathcal{M}_0(Q) \subseteq \mathcal{L}_3$, and $0 \neq W_L(Q) \subseteq L \cap \mathcal{R}(Q)$, $L \in \mathcal{M}_0(\mathcal{R}(Q))$, satisfying the conclusions of the lemma. We define

$$\mathcal{M}_1 = \{L_1, L_2, L_3, N_1, N_2, N_3, K_1, K_2, K_3\}.$$

We take a basis of $\bigvee\{W_L(Q) : L \in \mathcal{M}_0(\mathcal{R}(Q)) \cap \mathcal{M}_1\}$ and we extend it (if necessary) to a basis of $\mathcal{R}(Q)$, using vectors of $\bigvee\{W_L(Q) : L \in \mathcal{M}_0(\mathcal{R}(Q)) - \mathcal{M}_1\}$.

From this it is clear that there are finite rank operators T and S such that $Q = T + S$, and

$$\mathcal{R}(T) = \bigvee\{W_L(Q) : L \in \mathcal{M}_0(\mathcal{R}(Q)) \cap \mathcal{M}_1\}, \quad \mathcal{R}(T) \cap \mathcal{R}(S) = 0.$$

We now apply Lemma 5 for $W = \mathcal{R}(T)$ and we find $\mathcal{M}_0(\mathcal{R}(T)) \subseteq \mathcal{L}_3$, and $0 \neq W_L(T) \subseteq L \cap \mathcal{R}(T)$, $L \in \mathcal{M}_0(\mathcal{R}(T))$, which satisfy the conclusions of the lemma.

With this notation we show

LEMMA 7. *The inclusion* $\mathcal{M}_0(\mathcal{R}(T)) \subseteq \mathcal{M}_1$ *holds.*

Proof. It is sufficient to prove that for each $L_0 \in \mathcal{M}_0(\mathcal{R}(T))$ we have $M \not\subseteq L_0$. Suppose on the contrary that, for example, $L_0 = M$. (The other cases, such as $L_0 = M \vee K_1$ or $L_0 = K_1 \vee K_2$ etc. are similar.) For $0 \neq z \in W_M$, from the definition of T there exist $z_L \in W_L(T)$ for $L \in \mathcal{M}_0(\mathcal{R}(Q)) \cap \mathcal{M}_1$ such that

$$z = \sum_{L \in \mathcal{M}_0(\mathcal{R}(Q)) \cap \mathcal{M}_1} z_L.$$

Without loss of generality we may suppose that $\mathcal{M}_1 \subseteq \mathcal{M}_0(\mathcal{R}(Q))$. We have

$$z_{K_1} = z - \sum_{L \in \mathcal{M}_1 - \{K_1\}} z_L \in K_1 \cap \{M \vee K_2 \vee K_3\} = N_1.$$

So $z_{K_1} \in W_{K_1}(Q) \cap (\bigvee\{\mathcal{R}(Q) \cap L : L \subset K_1\}) = 0$ (see Lemma 5(2)). That is, $z_{K_1} = 0$ and similarly $z_{K_2} = z_{K_3} = 0$. So finally we have

$$z = \sum_{L \in \{L_1, L_2, L_3, N_1, N_2, N_3\}} z_L.$$

But this contradicts (2) of Lemma 5. The contradiction establishes the claim. ■

In the following we use the shorthand $\mathcal{M}_0 = \mathcal{M}_0(\mathcal{R}(T))$ and $W_L = W_L(T)$.

LEMMA 8. *Let N be any one of N_1, N_2, N_3. Then the set $\{W_L : L \in \mathcal{M}_0 - \{N\}\}$ is linearly independent.*

Proof. Without loss of generality we may suppose that $\mathcal{M}_0 = \mathcal{M}_1$ where $\mathcal{M}_1$ is as before. To be specific let N be N_3. For $L \in \mathcal{M}_1 - \{N_3\}$ let $0 \neq z_L \in W_L$ and $\lambda_L \in \mathbb{C}$ be such that $\sum \lambda_L z_L = 0$, where the summation runs over $L \in \mathcal{M}_1 - \{N_3\}$. We shall show that $\lambda_L = 0$ for each $L \in \mathcal{M}_1 - \{N_3\}$. Since

$$\lambda_{L_1} z_{L_1} + \lambda_{L_2} z_{L_2} + \lambda_{L_3} z_{L_3} + \lambda_{N_1} z_{N_1} + \lambda_{N_2} z_{N_2} + \lambda_{K_1} z_{K_1} + \lambda_{K_2} z_{K_2} = -\lambda_{K_3} z_{K_3} \in (M \vee K_1 \vee K_2) \cap K_3 = N_3,$$

part (2) of Lemma 5 shows that $\lambda_{K_3} = 0$. Thus

$$\lambda_{L_1} z_{L_1} + \lambda_{L_2} z_{L_2} + \lambda_{L_3} z_{L_3} + \lambda_{N_1} z_{N_1} + \lambda_{N_2} z_{N_2} + \lambda_{K_1} z_{K_1} = -\lambda_{K_2} z_{K_2} \in (M \vee K_1) \cap K_2 = N_2$$

and consequently $\lambda_{K_2} = 0$ as well. Hence also

$$\lambda_{L_1} z_{L_1} + \lambda_{L_2} z_{L_2} + \lambda_{L_3} z_{L_3} + \lambda_{N_1} z_{N_1} + \lambda_{N_2} z_{N_2} = -\lambda_{K_1} z_{K_1} \in M \cap K_1 = N_1$$

so that $\lambda_{K_1} = 0$. Moreover, we have

$$\lambda_{L_1} z_{L_1} + \lambda_{L_2} z_{L_2} + \lambda_{N_1} z_{N_1} = -\lambda_{L_3} z_{L_3} - \lambda_{N_2} z_{N_2} \in N_1 \cap N_2 = L_1,$$

so $\lambda_{N_2} z_{N_2} \in (T(\mathcal{X}) \cap L_1) \vee (T(\mathcal{X}) \cap L_3)$ and so $\lambda_{N_2} = 0$. Similarly we obtain $\lambda_{N_1} = 0$. Finally,

$$\lambda_{L_1} z_{L_1} + \lambda_{L_2} z_{L_2} = -\lambda_{L_3} z_{L_3} \in (L_1 \vee L_2) \cap L_3 = 0$$

so that $\lambda_{L_3} = 0$ and hence $\lambda_{L_1} z_{L_1} = -\lambda_{L_2} z_{L_2} \in L_1 \cap L_2 = 0$. Therefore $\lambda_{L_1} = \lambda_{L_2} = 0$. ■

Let us now discuss the case when one of N_1, N_2, N_3 happens to belong to $\mathcal{M}_0$. For notational convenience, suppose $N_3 \in \mathcal{M}_0$. Then the choice of vectors in Lemma 5 also produces the subspace W_{N_3}. At this point we shall investigate a little more closely this space which we shall split in two subspaces, according to whether they are independent or not of the set of subspaces $\{W_L : L \in \mathcal{M}_0, L \neq N_3\}$. To be precise, we proceed as follows: We define $W_{N_3,2}$ to be a complement of

$$W_{N_3,1} = \Big(\bigvee\{W_L : L \in \mathcal{M}_0 - \{N_3\}\}\Big) \cap W_{N_3}$$

in W_{N_3}. Hereafter we shall use only the $W_{N_3,1}$ and the $W_{N_3,2}$, which we shall rename W_0 and (a new) W_{N_3} respectively. With this new symbolism, the set $\{W_L : L \in \mathcal{M}_0\}$ is linearly independent and

$$(*) \qquad W_0 \subseteq \bigvee\{W_L : L \in \mathcal{M}_0 - \{N_3\}\},$$

Of course the W_0, W_{N_3} inherit the properties required by Lemma 5. In a way similar to the first part of the proof of Lemma 8, it is easy to see that in the inclusion (*) we may omit the K_i.

In fact, let $z \in W_0$. There are $z_L \in W_L$, $L \in \{L_1, L_2, L_3, N_1, N_2, K_1, K_2, K_3\}$, such that

$$z = \sum_{L \in \{L_1,L_2,L_3,N_1,N_2,K_1,K_2,K_3\}} z_L.$$

We have

$$z_{K_3} = z - \sum_{L \in \{L_1,L_2,L_3,N_1,N_2,K_1,K_2\}} z_L \in K_3 \cap \{M \vee K_1 \vee K_2\} = N_3$$

and so $z_{K_3} \in W_{K_3} \cap (W \cap N_3) = 0$ from Lemma 5(2). Similarly $z_{K_1} = z_{K_2} = 0$. That is, we have

$$W_0 \subseteq \bigvee\{W_L : L \in \mathcal{M}_0 \cap \{L_1, L_2, L_3, N_1, N_2\}\}.$$

Hence, if $\{x_j : j = 1, \ldots, k\}$ is a basis of W_0 (we assume $W_0 \neq 0$) and if $L \in \mathcal{M}_0 \cap \{L_1, L_2, L_3, N_1, N_2\}$ and $j \in \{1, \ldots, k\}$, then there are vectors $x_{j,L} \in W_L$ such that

$$x_j = \sum_{L \in \mathcal{M}_0 \cap \{L_1,L_2,L_3,N_1,N_2\}} x_{j,L}.$$

If $W_0 = 0$ we take $x_{j,L} = 0$ for each L, j.

Since (using (1) of Lemma 5) $\mathcal{R}(T) = \bigvee\{W_L : L \in \mathcal{M}_0\}$, there are (unique) finite rank $T_L \in \mathcal{B}(\mathcal{X})$ such that $\mathcal{R}(T_L) = W_L$ and

$$T = \sum_{L \in \mathcal{M}_0} T_L.$$

Using this notation we are in a position to state our theorem which characterizes the finite rank operators in $\operatorname{Alg} \mathcal{L}_3$. Of course we could state the theorem for any permutation of N_1, N_2, N_3 but the one given is just as good.

THEOREM 1. *The finite rank operator* $Q = T + S = \sum_{L \in \mathcal{M}_0} T_L + S$ *belongs to* $\operatorname{Alg} \mathcal{L}_3$ *if and only if*

(1) T_L *has the FRP for* $L \in \mathcal{M}_0 \cap \{L_2, L_3, N_3, K_1, K_2, K_3\}$.
(2) *If* $N_1 \in \mathcal{M}_0$ *then* $\mathcal{R}(T^*_{N_1}) \subseteq (N_{1-} \cap N_{3-})^\perp$.
(3) *If* $N_2 \in \mathcal{M}_0$ *then* $\mathcal{R}(T^*_{N_2}) \subseteq (N_{2-} \cap N_{3-})^\perp$.
(4) *There exist* $\lambda_j^* \in \mathcal{X}^*$, $j = 1, \ldots, k$, *such that*

$$K_3 \subseteq \operatorname{Ker}\Big(T_L - \sum_{j=1}^k \lambda_j^* \otimes x_{j,L}\Big) \quad \textit{for } L \in \{L_1, N_1, N_2\} \cap \mathcal{M}_0.$$

(5) $S = 0$.

Proof. We shall prove the theorem only in the case $\mathcal{M}_0 = \mathcal{M}_1$, $0 \subset W_0 \subset W_0 \vee W_{N_3}$. The other cases are similar.

We suppose, first, that conditions (1) to (5) hold, and we shall prove that $Q = T + S = T \in \operatorname{Alg} \mathcal{L}_3$. For this it is sufficient to prove that $T(K_i) \subseteq K_i$ $(i = 1, 2, 3)$.

From (1), each of T_{L_2}, T_{L_3}, T_{N_3}, T_{K_1}, T_{K_2}, T_{K_3} belongs to $\operatorname{Alg} \mathcal{L}_3$, so we only need to prove that the operator $T_0 = T_{L_1} + T_{N_1} + T_{N_2}$ leaves K_1, K_2, K_3 invariant. By (3), we have $K_1 \subseteq N_{2-} \cap N_{3-} \subseteq \operatorname{Ker} T_{N_2}$. Therefore $T_0(K_1) \subseteq T_{L_1}(K_1) + T_{N_1}(K_1) \subseteq L_1 \vee N_1 \subseteq K_1$, showing that $T_0(K_1) \subseteq K_1$. Similarly $T_0(K_2) \subseteq K_2$. Finally, let $u \in K_3$ be arbitrary. We have (from (4))

$$\begin{aligned} T_0(u) &= \sum_{L \in \{L_1,N_1,N_2\}} T_L(u) = \sum_{L \in \{L_1,N_1,N_2\}} \sum_{j=1}^k \lambda_j^*(u) x_{j,L} \\ &= \sum_{j=1}^k \lambda_j^*(u) \sum_{L \in \{L_1,N_1,N_2\}} x_{j,L} \\ &= \sum_{j=1}^k \lambda_j^*(u) \Big(x_j - \sum_{L \in \{L_2,L_3\}} x_{j,L}\Big) \in N_3 \vee L_2 \vee L_3 \subseteq K_3, \end{aligned}$$

as required, completing the proof of the sufficiency.

In the other direction, suppose that $Q \in \operatorname{Alg} \mathcal{L}_3$. We are to prove that conditions (1) to (5) hold.

We suppose first that there is $L_0 \in \mathcal{M}_0(Q) - \mathcal{M}_1$. Then $L_{0-} = \mathcal{X}$. Let $x \in \mathcal{X}$. From Lemma 6 we have

$$\begin{aligned} Qx &\in Q\Big(\bigcap\{L_- : L \in \mathcal{M}_0(Q) - \mathcal{M}_1\}\Big) \\ &\subseteq \bigvee\{W_L : L \in \mathcal{M}_0(Q) \cap \mathcal{M}_1\} = \mathcal{R}(T). \end{aligned}$$

So $Sx = Qx - Tx \in \mathcal{R}(S) \cap \mathcal{R}(T) = 0$, that is, $S = 0$.

If $\mathcal{M}_0(Q) \subseteq \mathcal{M}_1$ then clearly $S = 0$. So finally $S = 0$ and $T = Q \in \operatorname{Alg} \mathcal{L}_3$.

We shall now prove conditions (1) to (4).

Since $K_1 = K_{2-} \cap K_{3-} \cap N_{2-} \cap N_{3-} \cap L_{3-}$, using Lemma 6 we have

$$T(K_1) \subseteq \bigvee\{W_L : L \in \{L_1, L_2, N_1, K_1\}\}.$$

But $T = \sum_{L \in \mathcal{M}_1} T_L$ and since the set $\{W_L : L \in \mathcal{M}_1\}$ is linearly independent, we conclude $K_1 \subseteq \operatorname{Ker} T_L$ for $L \in \{L_3, N_2, N_3, K_2, K_3\}$. In particular, $L_{3-} \subseteq \operatorname{Ker} T_{L_3}$ and hence (Lemma 2) T_{L_3} has the FRP. Working similarly for $K_2 = K_{1-} \cap K_{3-} \cap N_{1-} \cap N_{3-} \cap L_{2-}$ we find that $K_2 \subseteq \operatorname{Ker} T_L$ for $L \in \{L_2, N_1, N_3, K_1, K_3\}$. Hence $L_{2-} \subseteq \operatorname{Ker} T_{L_2}$ and so T_{L_2} has the FRP.

Now, as $K_3 = K_{1-} \cap K_{2-} \cap N_{1-} \cap N_{2-} \cap L_{1-}$, Lemma 6 shows that

$$\mathcal{R}(T|_{K_3}) \subseteq \bigvee\{W_L : L \in \{L_2, L_3, N_3, K_3\}\} \vee W_0 = \bigvee\{W_L : L \in \{L_2, L_3, N_3, K_3\}\} \vee \langle x_j : j = 1, \ldots, k\rangle.$$

There are, thus, linear functionals $\lambda_j^* \in \mathcal{X}^*$ $(j = 1, \ldots, k)$ and an operator $T_1 \in \mathcal{B}(\mathcal{X})$ such that $\mathcal{R}(T_1) \subseteq \bigvee\{W_L : L \in \{L_2, L_3, N_3, K_3\}\}$ and

$$T|_{K_3} = T_1|_{K_3} + \sum_{j=1}^{k} (\lambda_j^*|_{K_3}) \otimes x_j.$$

(We can prove, in a manner similar to Lemma 6, that the set

$$\{W_L : L \in \{L_2, L_3, N_3, K_3\}\} \cup \{W_0\}$$

is linearly independent and so the $\lambda_j^*|_{K_3}$ and $T_1|_{K_3}$ are uniquely defined.) Also, since

$$N_3 = K_{1-} \cap K_{2-} \cap K_{3-} \cap N_{1-} \cap N_{2-} \cap N_{3-} \cap L_{1-} \subseteq K_3,$$

we have $T(N_3) \subseteq \bigvee\{W_L : L \in \{L_2, L_3\}\}$ and from the preceding observation we conclude that $\lambda_j^* \in (N_3)^\perp$, $j = 1, \ldots, k$. Also

$$T|_{K_3} = T_1|_{K_3} + \sum_{j=1}^{k} (\lambda_j^*|_{K_3}) \otimes \Big(\sum_{L \in \{L_1, L_2, L_3, N_1, N_2\}} x_{j,L} \Big) = T_1|_{K_3} + \sum_{L \in \{L_1, L_2, L_3, N_1, N_2\}} \sum_{j=1}^{k} (\lambda_j^*|_{K_3}) \otimes x_{j,L}.$$

Thus,

$$(**) \qquad K_3 \subseteq \operatorname{Ker}\Big(T_L - \sum_{j=1}^{k} \lambda_j^* \otimes x_{j,L}\Big) \quad \text{if } L \in \{L_1, N_1, N_2\}$$

(which is (4)), and $K_3 \subseteq \operatorname{Ker} T_L$ if $L \in \{K_1, K_2\}$.

Finally, from all the preceding relations we also conclude that

$$N_{3-} = K_1 \vee K_2 \subseteq \operatorname{Ker} T_{N_3},$$
$$K_{1-} = K_2 \vee K_3 \subseteq \operatorname{Ker} T_{K_1},$$
$$K_{2-} = K_1 \vee K_3 \subseteq \operatorname{Ker} T_{K_2},$$
$$K_{3-} = K_1 \vee K_2 \subseteq \operatorname{Ker} T_{K_3}.$$

Combining that with the previously established facts, we have (1) (from Lemma 2).

Recall that $\lambda_j^* \in (N_3)^\perp$, $j = 1, \ldots, k$, so from $(**)$ we have $N_3 \subseteq \operatorname{Ker} T_{N_1}$. Thus $N_{1-} \cap N_{3-} = N_3 \vee K_2 \subseteq \operatorname{Ker} T_{N_1}$, which is equivalent to (2).

Working similarly we can prove (3), and the proof of the theorem is complete. ■

EXAMPLE. It is perhaps instructive at this point to give an example which explains why the counterexample in [7] works. Suppose that there exist (non-zero) vectors $x_1, x_2, y_1^*, y_2^* \in \mathcal{X}$ (where $\mathcal{X}$ is now a Hilbert space) such that

$$x_1 \in N_1, \quad x_2 \in N_2, \quad y_1^* \in (N_{1-})^\perp \vee (N_{3-})^\perp, \quad y_2^* \in (N_{2-})^\perp \vee (N_{3-})^\perp$$

and, moreover,

$$x_1 + x_2 \in N_3, \quad y_3^* = y_1^* - y_2^* \in (N_{1-})^\perp \vee (N_{2-})^\perp.$$

The situation in the counterexample in [7] is precisely such a case. Set now

$$T_0 = y_1^* \otimes x_1 + y_2^* \otimes x_2$$

(as in [7]). It is easy to see that for this T_0 we have

$$\mathcal{M}_0 = \{N_1, N_2, N_3\}, \quad W_{N_1} = \langle x_1\rangle, \quad W_{N_2} = \langle x_2\rangle, \quad W_0 = \langle x_1 + x_2\rangle,$$
$$T_{N_1} = y_1^* \otimes x_1, \quad T_{N_2} = y_2^* \otimes x_2.$$

Obviously, T_0 satisfies (2) and (3). For (4) we take $\lambda^* = y_1^*$ and we have

$$T_{N_1} - \lambda^* \otimes x_1 = 0 \quad \text{and} \quad T_{N_2} - \lambda^* \otimes x_2 = (y_2^* - \lambda^*) \otimes x_2 = (-y_3^*) \otimes x_2.$$

Since $y_3^* \in (N_{1-})^\perp \vee (N_{2-})^\perp = (N_{1-} \cap N_{2-})^\perp \subseteq K_3^\perp$, condition (4) of Theorem 1 holds and so T_0 is in $\operatorname{Alg}\mathcal{L}_3$.

As we have seen, it is possible that not all finite rank operators of $\operatorname{Alg}\mathcal{L}_3$ have the FRP. Theorem 2 below characterizes those $T \in \operatorname{Alg}\mathcal{L}_3$, for any given realization of $\mathcal{L}_3$, which do have the FRP. Again, to facilitate presentation, we shall resort to lemmas. The notation used is as above.

LEMMA 9. *Let $R \in \operatorname{Alg}\mathcal{L}_3$ be a finite rank operator. If $\mathcal{R}(R) \subseteq L_1 + L_2 + L_3$, then R has the FRP.*

Proof. Let $\{z_i : i = 1, \ldots, n\}$ be a basis of $\mathcal{R}(R)$. Then there is $\{y_i^* : i = 1, \ldots, n\} \subseteq \mathcal{X}^*$ such that $R = \sum_{i=1}^{n} y_i^* \otimes z_i$. Since $z_i \in \mathcal{R}(R) \subseteq L_1 + L_2 + L_3$ for each i, there are $z_{i,j} \in \mathcal{X}$ $(j = 1, 2, 3)$ such that $z_i = z_{i,1} + z_{i,2} + z_{i,3}$ and $z_{i,j} \subseteq L_j$ $(j = 1, 2, 3)$. We define $R_j = \sum_{i=1}^{n} y_i^* \otimes z_{i,j}$ for $j = 1, 2, 3$. Then $R = R_1 + R_2 + R_3$ with $\mathcal{R}(R_j) \subseteq L_j$.

If $i_0 \in \{1, 2, 3\}$ and $z \in L_{i_0-}$ then $Rz = R_1 z + R_2 z + R_3 z \in L_{i_0-}$. Since $L_i \subseteq L_{i_0-}$ for $i \neq i_0$, it follows that $R_{i_0}(z) \in L_{i_0-}$. Thus $R_{i_0}(z) \in L_{i_0-} \cap L_{i_0} = 0$ and consequently $L_{i_0-} \subseteq \operatorname{Ker} R_{i_0}$. Therefore, by Lemma 2, R_{i_0} has the FRP for $i_0 = 1, 2, 3$ and the proof is complete. ■

LEMMA 10. *Let $P \in \mathcal{B}(\mathcal{X})$ be a finite rank operator such that for some fixed $i_0 \in \{1, 2, 3\}$ we have $K_{i_0-} \subseteq \operatorname{Ker} P$ (equivalently, $\mathcal{R}(P^*) \subseteq (K_{i_0-})^\perp$). Then $\mathcal{R}(P) = P(K_{i_0})$.*

Proof. Let $z \in \mathcal{X}$. Since $K_{i_0} \vee K_{i_0-} = \mathcal{X}$, there are sequences $\{r_n\}_{n=1}^{\infty} \subseteq K_{i_0}$ and $\{t_n\}_{n=1}^{\infty} \subseteq K_{i_0-}$ such that $r_n + t_n \to z$. Then $Pr_n + Pt_n \to Pz$. But $K_{i_0-} \subseteq \operatorname{Ker} P$, so $Pt_n = 0$ and $Pr_n \to Pz$. Thus $Pz \in \overline{P(K_{i_0})}$. Since $P(K_{i_0})$ is a finite-dimensional space, it is closed and we have $Pz \in P(K_{i_0})$. Thus $\mathcal{R}(P) \subseteq P(K_{i_0})$. The other inclusion is trivial. ■

Let T be a finite rank operator in $\operatorname{Alg}\mathcal{L}_3$ and $\mathcal{M}_0$ as in Lemma 5. We also suppose that $N_1, N_2 \in \mathcal{M}_0$. We define $W_{N_1,+} = W_{N_1} \cap (L_1 + L_2)$ and let $W_{N_1,\vee}$ be a complement of $W_{N_1,+}$ in W_{N_1}. Similarly, we define $W_{N_2,+} = W_{N_2} \cap (L_1 + L_3)$, and take $W_{N_2,\vee}$ to be a complement of $W_{N_2,+}$ in W_{N_2}. Since $\mathcal{R}(T_{N_i}) = W_{N_i} = W_{N_i,+} + W_{N_i,\vee}$ for $i \in \{1,2\}$, there are (from Lemma 3) finite rank operators $T_{N_i,+}, T_{N_i,\vee}$ such that $T_{N_i} = T_{N_i,+} + T_{N_i,\vee}$, $\mathcal{R}(T_{N_i,+}) \subseteq W_{N_i,+}$, $\mathcal{R}(T_{N_i,\vee}) \subseteq W_{N_i,\vee}$ and $\mathcal{R}(T^*_{N_i,\vee}) \subseteq \mathcal{R}(T^*_{N_i})$.

We can now formulate the second main result of this paper. Again we could state it using a permutation of $\{N_1, N_2, N_3\}$, but the following is good enough.

THEOREM 2. *An operator $T \in \operatorname{Alg}\mathcal{L}_3$ has the FRP if and only if*

(1) *If $N_1 \in \mathcal{M}_0$ then $\mathcal{R}(T^*_{N_1,\vee}) \subseteq (N_{1-})^\perp + (N_{3-})^\perp$.*

(2) *If $N_2 \in \mathcal{M}_0$ then $\mathcal{R}(T^*_{N_2,\vee}) \subseteq (N_{2-})^\perp + (N_{3-})^\perp$.*

Proof. We prove the theorem in the case $\mathcal{M}_0 = \mathcal{M}_1$. As other cases are similar and simpler, we omit them. We suppose first that $T \in \operatorname{Alg}\mathcal{L}_3$ has the FRP and we show (1) and (2).

Since $T \in \operatorname{Alg}\mathcal{L}_3$, from (1) of Theorem 1 we conclude that the operator $T_{L_1} + T_{N_1} + T_{N_2}$ has the FRP, that is, it can be written as a finite sum of rank one operators from $\operatorname{Alg}\mathcal{L}_3$. From Lemma 1 for such a rank one R there is $L \in \mathcal{L}$ such that $\mathcal{R}(R) \subseteq L$ and $\mathcal{R}(R^*) \subseteq (L_-)^\perp$. We define F_L to be the sum of those R which have the same L (that is, the same N of Lemma 1). It is clear that $\mathcal{R}(F_L) \subseteq L$, $\mathcal{R}(F_L^*) \subseteq (L_-)^\perp$ and

$$T_{L_1} + T_{N_1} + T_{N_2} = \sum_{L \in \mathcal{M}_1} F_L.$$

For each $z \in \mathcal{X}$ we have

$$F_{K_3}(z) = \Big(T_{L_1} + T_{N_1} + T_{N_2} - \sum_{L \in \mathcal{M}_1 - \{K_3\}} F_L\Big)(z)$$
$$\in K_3 \cap (M \vee K_1 \vee K_2) = K_3 \cap K_{3-} = N_3.$$

That is, $\mathcal{R}(F_{K_3}) \subseteq N_3$. Since also $K_{3-} = N_{3-}$ we can without loss of generality suppose that

$$T_{L_1} + T_{N_1} + T_{N_2} = \sum_{L \in \{L_i, N_i : i=1,2,3\}} F_L.$$

For $z \in K_{2-} \cap K_{3-} = K_{2-} \cap K_{3-} \cap N_{2-} \cap N_{3-}$, using Lemma 6 we have

$$Tz \in \bigvee\{W_L : L \in \{L_1, L_2, L_3, N_1, K_1\}\}$$

and therefore $T_{N_2}(z) = 0$. Consequently, $(T_{L_1} + T_{N_1} + T_{N_2})(z) = (F_{L_1} + F_{L_2} + F_{L_3} + F_{N_1} + F_{N_2} + F_{N_3})(z)$ gives

$$T_{L_1}(z) + T_{N_1}(z) = (F_{L_1} + F_{L_2} + F_{L_3} + F_{N_1})(z)$$

and $F_{L_3}(z) \in N_1 \cap L_3 = 0$ and so

$$F_{N_1}(z) = T_{N_1}(z) + T_{L_1}(z) - F_{L_1}(z) - F_{L_2}(z).$$

Since $\mathcal{R}(F^*_{N_1}) \subseteq (N_{1-})^\perp = (K_{1-})^\perp$, from Lemma 10 we conclude that $\mathcal{R}(F_{N_1}) = F_{N_1}(K_1)$. Since also $K_1 \subseteq K_{2-} \cap K_{3-}$, we have

$$F_{N_1}(K_1) \subseteq T_{N_1}(K_1) + T_{L_1}(K_1) + F_{L_1}(K_1) + F_{L_2}(K_1)$$
$$\subseteq T_{N_1}(K_1) + L_1 + L_2 \subseteq \mathcal{R}(T_{N_1}) + L_1 + L_2,$$

and thus

$$\mathcal{R}(F_{N_1}) \subseteq W_{N_1} + L_1 + L_2.$$

So from Lemma 3 there are three finite rank operators which have sum F_{N_1}, their ranges are in W_{N_1}, L_1 and L_2 and the ranges of their adjoints are in $\mathcal{R}(F^*_{N_1}) \subseteq (N_{1-})^\perp$. But for $i \in \{1,2\}$ we have $(N_{1-})^\perp \subseteq (L_{i-})^\perp$. Thus from Lemma 2 it is clear that we can rewrite the operator $\sum_{L \in \{L_i, N_i : i=1,2,3\}} F_L$ in such a way that $\mathcal{R}(F_{N_1}) \subseteq W_{N_1}$ (and the other assumptions for the F_L are still satisfied).

Similarly (starting from a $z \in K_{1-} \cap K_{3-}$) we can suppose that $\mathcal{R}(F_{N_2}) \subseteq W_{N_2}$. Since

$$F_{N_3} = T_{L_1} + T_{N_1} + T_{N_2} - F_{L_1} - F_{L_2} - F_{L_3} - F_{N_1} - F_{N_2},$$

we have

$$\mathcal{R}(F_{N_3}) \subseteq W_{N_1} + W_{N_2} + L_1 + L_2 + L_3.$$

Hence for $L \in \{L_1, L_2, L_3, N_1, N_2\}$ there are $F_{N_3,L}$ such that

$$F_{N_3} = \sum_{L \in \{L_1, L_2, L_3, N_1, N_2\}} F_{N_3,L},$$

$\mathcal{R}(F_{N_3,N_i}) \subseteq W_{N_i}$ and $\mathcal{R}(F^*_{N_3,N_i}) \subseteq \mathcal{R}(F^*_{N_3}) \subseteq (N_{3-})^\perp$, $i = 1, 2$. Consequently, for each $z \in \mathcal{X}$,

$$(F_{N_3,N_1} + F_{N_3,L_1} + F_{N_3,L_2} - T_{L_1} - T_{N_1} + F_{L_1} + F_{L_2} + F_{N_1})(z)$$
$$= (-F_{N_3,N_2} - F_{N_3,L_3} + T_{N_2} - F_{L_3} - F_{N_2})(z) \in N_1 \cap N_2 = L_1.$$

Therefore $\mathcal{R}(F_{N_3,N_1} - T_{N_1} + F_{N_1}) \subseteq L_1 + L_2$. Also $\mathcal{R}(F_{N_3,N_1}) \subseteq W_{N_1} = W_{N_1,+} + W_{N_1,\vee}$ so there are (Lemma 3) $F_{N_3,N_1,+}$ and $F_{N_3,N_1,\vee}$ such that

$$F_{N_3,N_1} = F_{N_3,N_1,+} + F_{N_3,N_1,\vee},$$

$$\mathcal{R}(F_{N_3,N_1,+}) \subseteq W_{N_1,+}, \quad \mathcal{R}(F_{N_3,N_1,\vee}) \subseteq W_{N_1,\vee} \quad \text{and}$$
$$\mathcal{R}(F^*_{N_3,N_1,\vee}) \subseteq \mathcal{R}(F^*_{N_3,N_1}) \subseteq (N_{3-})^{\perp}.$$

Similarly we prove the existence of $T_{N_1,+}$, $F_{N_1,+}$ (with $\mathcal{R}(T_{N_1,+})$, $\mathcal{R}(F_{N_1,+}) \subseteq W_{N_1,+}$) and $T_{N_1,\vee}$, $F_{N_1,\vee}$ (with $\mathcal{R}(T_{N_1,\vee})$, $\mathcal{R}(F_{N_1,\vee}) \subseteq W_{N_1,\vee}$) such that

$$T_{N_1} = T_{N_1,+} + T_{N_1,\vee}, \quad F_{N_1} = F_{N_1,+} + F_{N_1,\vee}$$

and $\mathcal{R}(F^*_{N_1,\vee}) \subseteq \mathcal{R}(F^*_{N_1}) \subseteq (N_{1-})^{\perp}$. So

$$\begin{aligned}\mathcal{R}(F_{N_3,N_1,\vee} &- T_{N_1,\vee} + F_{N_1,\vee})\\ &\subseteq \mathcal{R}(F_{N_3,N_1} - T_{N_1} + F_{N_1}) + \mathcal{R}(F_{N_3,N_1,+} - T_{N_1,+} + F_{N_1,+})\\ &\subseteq \{(L_1+L_2) \cap W_{N_1}\} \cap W_{N_1,\vee} = W_{N_1,+} \cap W_{N_1,\vee} = 0.\end{aligned}$$

Thus $T_{N_1,\vee} = F_{N_3,N_1,\vee} + F_{N_1,\vee}$ so $T^*_{N_1,\vee} = F^*_{N_3,N_1,\vee} + F^*_{N_1,\vee}$ and so

$$\mathcal{R}(T^*_{N_1,\vee}) \subseteq \mathcal{R}(F^*_{N_3,N_1,\vee}) + \mathcal{R}(F^*_{N_1,\vee}) \subseteq (N_{3-})^{\perp} + (N_{1-})^{\perp},$$

which is (1).

In a similar manner we obtain (2), and the proof of the necessity part is complete.

We now suppose that (1) and (2) hold. We shall prove that T has the FRP. Since $T \in \operatorname{Alg}\mathcal{L}_3$, from Theorem 1 it is sufficient to prove that $T_{N_1} + T_{N_2} + T_{L_1}$ has the FRP.

Using the hypothesis we can find finite rank operators $S_{N_1,1}$, $S_{N_1,3}$, $S_{N_2,2}$, $S_{N_2,3}$ such that

$$T_{N_1,\vee} = S_{N_1,1} + S_{N_1,3}, \quad T_{N_2,\vee} = S_{N_2,2} + S_{N_2,3},$$
$$\mathcal{R}(S^*_{N_1,i}) \subseteq (N_{i-})^{\perp}, \quad \mathcal{R}(S_{N_1,i}) \subseteq \mathcal{R}(T_{N_1,\vee}), \quad i = 1,3,$$
$$\mathcal{R}(S^*_{N_2,i}) \subseteq (N_{i-})^{\perp}, \quad \mathcal{R}(S_{N_2,i}) \subseteq \mathcal{R}(T_{N_2,\vee}), \quad i = 2,3.$$

Thus

$$\begin{aligned}T_{N_1} + T_{N_2} + T_{L_1} &= T_{N_1,+} + T_{N_1,\vee} + T_{N_2,+} + T_{N_2,\vee} + T_{L_1}\\ &= T_{N_1,+} + S_{N_1,1} + S_{N_1,3} + T_{N_2,+} + S_{N_2,2} + S_{N_2,3} + T_{L_1}\\ &= S_{N_1,1} + S_{N_1,3} + S_{N_2,2} + S_{N_2,3} + S,\end{aligned}$$

where S is the obvious operator and its range is in $L_1 + L_2 + L_3$.

Since $S_{N_1,1}$ and $S_{N_2,2}$ have the FRP from Lemma 2, it is sufficient to prove that the operator $S_{N_1,3} + S_{N_2,3} + S$, which is in $\operatorname{Alg}\mathcal{L}_3$, also has the FRP.

Let $z \in K_3$. Then $S_{N_1,3}(z) + S_{N_2,3}(z) + S(z) \in K_3 \cap M \subseteq N_3$. Hence by Lemma 10, $\mathcal{R}(S_{N_1,3} + S_{N_2,3}) \subseteq L_1 + N_3$. From this, using Lemma 3, it is clear that $S_{N_1,3} + S_{N_2,3}$ can be written as a sum of two finite rank operators with their ranges in L_1 and N_3 respectively and such that the second has the FRP (from Lemmas 3 and 2). But then it is sufficient to prove that an operator which has range in $L_1 + L_2 + L_3$ has the FRP. This follows from Lemma 9 and the proof of the theorem is complete.

Remarks. 1) Let T_0 be the operator of the example just after Theorem 1. Since $\mathcal{R}(T^*_{N_i}) = \langle y^*_i \rangle$ $(i = 1,2)$ it is clear when T_0 has and when it fails the FRP. The counterexample in [7] is such that it fails the FRP.

2) The proof of Theorem 2 also shows that if $T \in \operatorname{Alg}\mathcal{L}_3$ has the FRP then it can be written as a sum of rank one operators of $\operatorname{Alg}\mathcal{L}_3$ with at most $3\operatorname{rank}T$ terms. Notice that $\operatorname{rank}T$ as the number of summands is not always possible. Indeed, in [2] Hopenwasser and Moore construct a specific realization of $\mathcal{L}_3$ and a finite rank $T \in \operatorname{Alg}\mathcal{L}_3$ which requires strictly more than $\operatorname{rank}T$ terms in its decomposition as a sum of rank one operators from $\operatorname{Alg}\mathcal{L}_3$.

In a reflexive Banach space we have the following characterization of those realizations of $\mathcal{L}_3$ which do not have the FRP.

THEOREM 3. *Let $\mathcal{L}'_3$ be a realization of $\mathcal{L}_3$. The following are equivalent*:

(i) *$\mathcal{L}'_3$ does not have the FRP.*

(ii) *$(N_1 + N_2) \cap \{N_3 - (L_2 + L_3)\} \neq \emptyset$ and*

$$\{(N_{1-} \cap N_{3-})^{\perp} + (N_{2-} \cap N_{3-})^{\perp}\} \cap \{(N_{1-} \cap N_{2-})^{\perp} - ((N_{1-})^{\perp} + (N_{2-})^{\perp})\} \neq \emptyset.$$

(iii) *There is a rank two operator in $\operatorname{Alg}\mathcal{L}'_3$ without the FRP.*

Remark. The second condition in (ii) is simply the first one but for the lattice $\{L^{\perp} : L \in \mathcal{L}'_3\}$.

Proof of Theorem 3. (i)⇒(ii). Suppose $T \in \operatorname{Alg}\mathcal{L}'_3$ fails the FRP and $(N_1 + N_2) \cap \{N_3 - (L_2 + L_3)\} = \emptyset$. We shall use the usual notation for $\mathcal{M}_0$, W_0, x_j, T_L etc. concerning T.

Since each x_j can be written as

$$x_j = \sum_{L \in \mathcal{M}_0 \cap \{L_1, L_2, L_3, N_1, N_2\}} x_{j,L}$$

it follows that $x_j \in (N_1 + N_2) \cap N_3$. But then, from the hypothesis, $x_j \in L_2 + L_3$. Let $u \in K_3$. Using the first part of the proof of Theorem 1, we find that $(T_{L_1} + T_{N_1} + T_{N_2})u \in L_2 + L_3$, so there are $t_2 \in L_2$ and $t_3 \in L_3$ such that $T_{L_1}u + T_{N_1}u + T_{N_2}u = t_2 + t_3$ and also

$$T_{L_1}u + T_{N_1}u - t_2 = -T_{N_2}u + t_3 \in N_1 \cap N_2 = L_1.$$

Thus $T_{N_1}(K_3) \subseteq L_1 + L_2$ and $T_{N_2}(K_3) \subseteq L_1 + L_3$. Moreover, $T_{N_1,\vee}(K_3) \subseteq L_1 + L_2$ and $T_{N_2,\vee}(K_3) \subseteq L_1 + L_3$. From the definition of $T_{N_1,\vee}$, $T_{N_2,\vee}$ we obtain $T_{N_1,\vee}(K_3) = T_{N_2,\vee}(K_3) = 0$. Since (Theorem 1) $T_{N_1,\vee}(N_{1-} \cap N_{3-}) = 0$ and $T_{N_2,\vee}(N_{2-} \cap N_{3-}) = 0$ we have $T_{N_1,\vee}(N_{1-}) = T_{N_1,\vee}((N_{1-} \cap N_{3-}) \vee K_3) = 0$, that is, $T_{N_1,\vee}$ and similarly $T_{N_2,\vee}$ have the FRP. From Lemma 9 we now deduce that $T_{L_1} + T_{N_1} + T_{N_2}$, and so (Theorem 1) T, has the FRP, a contradiction. This proves the first relation of (ii). Working similarly for the lattice $\{L^\perp : L \in \mathcal{L}_3'\}$ and the operator T^*, we obtain the second relation.

(ii)⇒(iii). Let T be the rank two operator described in the Example after Theorem 1 and with the further hypothesis that $x_1+x_2 \notin L_2+L_3$ and $y_3^* \notin (N_{1-})^\perp + (N_{2-})^\perp$. Then we also have $x_1 \notin L_1+L_2$ and $x_2 \notin L_1+L_3$. In fact, if for example $x_1 = t_1+t_2$ where $t_1 \in L_1$ and $t_2 \in L_2$ then $t_1+x_2 = (x_1+x_2)-t_2 \in N_2 \cap N_3 = L_3$, that is, $x_1+x_2 \in L_2+L_3$, a contradiction. Arguing similarly for y_1^*, y_2^*, from Theorem 2 we see that T fails the FRP.

(iii)⇒(i). Obvious. ■

Remark. It is clear that relations (ii) can be replaced by others cyclically generated.

3. An application. As we have seen the FRP may fail for finite rank operators in $\operatorname{Alg}\mathcal{L}_3$, for the free distributive lattice $\mathcal{L}_3$ on 3 generators. We show here, as an application of Theorem 1, the following curiosity: the product of two finite rank operators of $\operatorname{Alg}\mathcal{L}_3$ always has the FRP. Thus for example the square F^2 of a finite rank operator $F \in \operatorname{Alg}\mathcal{L}_3$ always has the FRP. Notice, however, that F^2 may be zero. For instance, this is the case for the F in the counterexample in [7].

THEOREM 4. *If $T, R \in \operatorname{Alg}\mathcal{L}_3$ then TR has the FRP.*

Proof. Let T_L for $L \in \mathcal{M}_0$ be as defined at the beginning of the proof of Theorem 1, and R_L the respective operators for R. We have

$$T = \sum_{L\in\mathcal{M}_1} T_L,$$

where we take $T_L = 0$ if $L \in \mathcal{M}_1 - \mathcal{M}_0$. Similarly

$$R = \sum_{L\in\mathcal{M}_1} R_L.$$

Since for $L \in \{L_2, L_3, N_3, K_1, K_2, K_3\}$, each R_L has the FRP, clearly the same is true for TR_L. So to complete the proof it is sufficient to prove that the operator

$$TR - TR_{L_2} - TR_{L_3} - TR_{N_3} - TR_{K_1} - TR_{K_2} - TR_{K_3} = TR_{N_1} + TR_{N_2} + TR_{L_1}$$

of $\operatorname{Alg}\mathcal{L}_3$ has the FRP. We use the conditions (1)–(3) of Theorem 1 and so we have

$$\begin{aligned} TR_{N_1} + TR_{N_2} + TR_{L_1} &= (T_{L_1}+T_{L_2})R_{N_1} + (T_{L_1}+T_{L_3})R_{N_2} + T_{L_1}R_{L_1} \\ &= T_{L_1}(R_{N_1}+R_{N_2}+R_{L_1}) + T_{L_2}R_{N_1} + T_{L_3}R_{N_2}. \end{aligned}$$

The conclusion now follows from Lemma 9. ■

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