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Some Sawyer type inequalities for martingales

by

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Abstract. Some martingale analogues of Sawyer's two-weight norm inequality for the Hardy–Littlewood maximal function Mf are shown for the Doob maximal function of martingales.

1. Introduction. Throughout this paper, we will only consider closed discrete martingales $f = (f_n)$ with respect to a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_n\}_{n\geq 0}$ with $\mathcal{F} = \bigvee_{n\geq 0} \mathcal{F}_n$. That is to say, $f_n = E(f \mid \mathcal{F}_n)$ for all n and $f \in L^1(\Omega, \mathcal{F}, P)$. We will also follow the convention of Zygmund to denote by c_p a constant which only depends on p. However, it may be different in different lines or different theorems. Recall the fundamental work of B. Muckenhoupt [4], who showed in 1972 the weighted Hardy–Littlewood inequality.

THEOREM (Muckenhoupt). *If $p > 1$, then the weighted norm inequality*

$$\int_{\mathbb{R}^n} (Mf)^p w\,dx \leq c_p \int_{\mathbb{R}^n} |f|^p w\,dx \tag{1}$$

holds for every $f \in L^p(w)$ if and only if the weight w satisfies Muckenhoupt's $\mathbf{A}_p$-condition

$$\sup_Q \left[\left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q \left(\frac{1}{w}\right)^{1/(p-1)} dx\right)^{p-1}\right] < \infty, \tag{$\mathbf{A}_p$}$$

where $Mf = \sup_Q \frac{1}{|Q|}\int_Q |f|\,dy$, the Hardy–Littlewood maximal function, and Q denotes an arbitrary cube in $\mathbb{R}^n$.

On the other hand, the martingale version of the Hardy–Littlewood inequality is the famous Doob inequality [1]. So it is natural to consider the martingale analogue of inequality (1). To this end, we need the counterpart of the $\mathbf{A}_p$-condition for martingales. Say a martingale $w = (w_n)$ satisfies the

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$\mathcal{A}_p$-*condition* if

$$(\mathcal{A}_p) \qquad \sup_n \|E(w \mid \mathcal{F}_n)E((1/w)^{1/(p-1)} \mid \mathcal{F}_n)^{p-1}\|_\infty < \infty.$$

In 1977, Izumisawa and Kazamaki [3] proved the following martingale analogue of Muckenhoupt's weighted inequality for the Doob maximal function $f^* = \sup_n |f_n|$ of a martingale $f = (f_n)$.

THEOREM (Izumisawa–Kazamaki). *For all $p > 1$, and every martingale $f = (f_n)$, we have*

(a) *The $\mathcal{A}_p$-condition is a necessary condition for the weighted norm inequality*

$$(2) \qquad \int_\Omega f^{*p} w\, dP \le c_p \int_\Omega |f|^p w\, dP.$$

to hold.

(b) *If $w = (w_n)$ also satisfies the step regular condition: $w_n \le c w_{n+1}$ for all $n \ge 0$, then the $\mathcal{A}_p$-condition is sufficient for* (2) *to hold.*

For a given weight w, we denote by $\sigma(w) = (1/w)^{1/(p-1)} = w^{1-p'}$ (or simply σ) its conjugate weight, where p' is the conjugate exponent of p. Thus the $\mathcal{A}_p$-condition can be expressed as $\sup_n \|w_n \sigma_n^{p-1}\|_\infty < \infty$, where $w_n = E(w \mid \mathcal{F}_n)$, and $\sigma_n = E(\sigma \mid \mathcal{F}_n)$. Say a pair (v, w) satisfies $\mathcal{A}_p$ if $\sup_n \|v_n \sigma(w)_n^{p-1}\|_\infty < \infty$.

Muckenhoupt not only considered the one weight problem, but also raised the question: What condition on a pair of weights (v, w) is necessary and sufficient for the two-weight norm inequality

$$(3) \qquad \int_{\mathbb{R}^n} (Mf)^p v\, dx \le c_p \int_{\mathbb{R}^n} |f|^p w\, dx$$

to hold? Suggested by the $\mathbf{A}_p$-condition, the first natural candidate seems to be

$$(4) \qquad \sup_Q \left[\left(\frac{1}{|Q|}\int_Q v\, dx\right)\left(\frac{1}{|Q|}\int_Q \left(\frac{1}{w}\right)^{1/(p-1)} dx\right)^{p-1}\right] < \infty.$$

But it turns out that (4) is equivalent to the weak type (p, p) inequality for the operator Mf. The right condition on (v, w) which fully characterizes (3) was only found in 1982 by E. Sawyer [5].

DEFINITION. Say a pair of weights (v, w) satisfies Sawyer's $\mathbf{S}_p$-*condition* if

$$(\mathbf{S}_p) \qquad \int_Q [M(\chi_Q \sigma)]^p v\, dx \le c_p \int_Q \sigma\, dx$$

for every cube Q in $\mathbb{R}^n$, where χ_Q is the characteristic function of Q.

THEOREM (Sawyer). *If $p > 1$, then the Hardy–Littlewood maximal operator Mf is bounded from $L^p(w dx)$ to $L^p(v dx)$ if and only if the weights (v, w) in $\mathbb{R}^n$ satisfy the Sawyer $\mathbf{S}_p$-condition.*

We will show some Sawyer type inequalities for martingales in Section 2. We end this section with notations of the tailed maximal function for a martingale $f = (f_n)$: ${}^*f_n = \sup_{m \ge n}(|f_m|)$, and of the weighted conditional expectation with respect to $w dP$: $\widehat{E}_w(\cdot \mid \cdot)$. Here we assume w is nonnegative and $E(w) = 1$. It is not hard to verify the formula

$$(5) \qquad \widehat{E}_w(f \mid \mathcal{F}_n) = \frac{1}{w_n} E(fw \mid \mathcal{F}_n).$$

It is also easy, from the definition of conditional expectation, to check the following chain formula:

$$(6) \qquad E(XE(Y \mid \mathcal{F}')) = E(E(X \mid \mathcal{F}')Y)$$

where X and Y are random variables, and $\mathcal{F}'$ is a sub-σ-algebra of $\mathcal{F}$.

2. Sawyer type inequalities for martingales. First of all, we need the martingale counterpart of the $\mathbf{S}_p$-condition.

DEFINITION. For a given pair of nonnegative martingales $v = (v_n)$ and $w = (w_n)$, we say (v, w) satisfies the $\mathcal{S}_p$-*condition* if there exists a constant c_p such that for all $n \ge 0$,

$$(\mathcal{S}_p) \qquad E[({}^*\sigma_n)^p v \mid \mathcal{F}_n] \le c_p \sigma_n.$$

We say that w satisfies $\mathcal{A}_p$, or $\mathcal{S}_p$, if (w, w) has that property. In the equal weight case, Hunt–Kurtz–Neugebauer [2] have shown, in the classical setting, that $\mathbf{A}_p \Leftrightarrow \mathbf{S}_p$. The following theorem shows that our $\mathcal{S}_p$ is a proper martingale analogue of the $\mathbf{S}_p$-condition.

THEOREM 1. *The martingale $w = (w_n)$ satisfies $\mathcal{A}_p$ if and only if it satisfies $\mathcal{S}_p$.*

The direction $\mathcal{S}_p \Rightarrow \mathcal{A}_p$ is easy. Before proving the other direction let us first prove a lemma, which is probably well known, but does not appear in the standard references.

LEMMA (conditional version of Doob's inequality). *If $p > 1$, then for any $n \ge 0$, we have*

$$E({}^*g_n^p|\mathcal{F}_n) \le c_p E(|g|^p \mid \mathcal{F}_n)$$

for any martingale $g = (g_n)$.

Proof. In fact, for any fixed $F \in \mathcal{F}_n$ take $f = \chi_F g$. Then for any $m \ge n$, $f_m = E(\chi_F g \mid \mathcal{F}_m) = \chi_F g_m$, so that ${}^*f_n = \chi_F {}^*g_n$. By applying

Doob's inequality to f we get

$$\int_\Omega \chi_F {}^*g_n^p\,dP = \int_\Omega {}^*f_n^p\,dP \le \int_\Omega f^{*p}\,dP \le c_p \int_\Omega |f|^p\,dP = c_p \int_\Omega \chi_F |g|^p\,dP,$$

i.e., $\int_F {}^*g_n^p\,dP \le c_p \int_F |g|^p\,dP$, which implies $E({}^*g_n^p \mid \mathcal{F}_n) \le c_p E(|g|^p \mid \mathcal{F}_n)$.

Proof of Theorem 1. Suppose w satisfies $\mathcal{A}_p$. Then for $m \ge n$, we have $\sigma_m \le c_p[1/w_m]^{1/(p-1)}$. Since

$$\widehat{E}_w(w^{-1} \mid \mathcal{F}_m) = \frac{1}{w_m} E(w^{-1}w \mid \mathcal{F}_m) = \frac{1}{w_m},$$

we get

$$\sigma_m \le c_p[\widehat{E}_w(w^{-1} \mid \mathcal{F}_m)]^{p'/p}$$

or

$$(\sup_{m\ge n} \sigma_m)^p \le c_p[\sup_{m\ge n} \widehat{E}_w(w^{-1} \mid \mathcal{F}_m)]^{p'}.$$

Taking the conditional expectation with respect to $\widehat{E}_w$ and using the conditional version of Doob's inequality with index p', we have

$$\widehat{E}_w[(\sup_{m\ge n} \sigma_m)^p \mid \mathcal{F}_n] \le c_p \widehat{E}_w[(\sup_{m\ge n} \widehat{E}_w(w^{-1} \mid \mathcal{F}_m))^{p'} \mid \mathcal{F}_n]$$
$$\le c_p \widehat{E}_w[w^{-p'} \mid \mathcal{F}_n] = c_p \frac{1}{w_n} E[w^{-p'} w \mid \mathcal{F}_n] = c_p \frac{\sigma_n}{w_n}.$$

Finally, $E[(\sup_{m\ge n} \sigma_m)^p w \mid \mathcal{F}_n] = \widehat{E}_w[(\sup_{m\ge n} \sigma_m)^p \mid \mathcal{F}_n] w_n \le c_p \sigma_n$, which is $\mathcal{S}_p$.

THEOREM 2. *$\mathcal{S}_p$ is a necessary condition for the two-weight norm inequality*

$$\int_\Omega f^{*p} v\,dP \le c_p \int_\Omega |f|^p w\,dP \tag{7}$$

to hold.

Proof. Take $f = \chi_F \sigma$, for F an arbitrary set in $\mathcal{F}_n$. Then for any $m \ge n$, $f_m = E(\chi_F \sigma \mid \mathcal{F}_m) = \chi_F E(\sigma \mid \mathcal{F}_m) = \chi_F \sigma_m$, and therefore ${}^*f_n^p = \chi_F {}^*\sigma_n^p$. Hence from (7) we know that $\int_\Omega \chi_F {}^*\sigma_n^p v\,dP \le c_p \int_\Omega \chi_F \sigma^p w\,dP$. But $\sigma = w^{1-p'} \Rightarrow w = \sigma^{1-p}$, so $\sigma^p w = \sigma^p \sigma^{1-p} = \sigma$, hence by the definition of conditional expectation the theorem is proved.

Whether $\mathcal{S}_p$ is also a sufficient condition is still unknown, at least to this author. It seems that we need some sort of "regular" condition, like the one in Izumisawa–Kazamaki's theorem, to guarantee the sufficiency of $\mathcal{S}_p$. Nevertheless, we present the following two theorems, which, we hope, will shed some light for the further study.

THEOREM 3. *If (v, w) satisfies the uniform regular condition $v_n/\sigma_n \le cv/\sigma$, then $\mathcal{A}_p$ is a sufficient condition for (7) to hold.*

Proof. For any $\lambda > 0$, define two stopping times

$$\tau = \inf\{n : |f_n| > \lambda\}, \quad T = \inf\{n : |f_n| > 2\lambda\}.$$

Clearly $\tau \le T$, and $\{T < \infty\} = \{\tau < \infty,\ |f_T| > 2\lambda\}$. We now show

$$P_v\{f^* > 2\lambda\} \le \frac{c_p}{\lambda} \int_{\{\tau<\infty\}} Y\,dP, \tag{8}$$

where $Y = |f|(v/\sigma)^{1/p'}$. First, by the chain formula (6), we have

$$P_v\{f^* > 2\lambda\} = P_v\{T < \infty\} = P_v\{\tau < \infty,\ |f_T| > 2\lambda\}$$
$$\le \frac{1}{2\lambda} \int_{\{\tau<\infty\}} |f_T| v\,dP \le \frac{1}{2\lambda} \int_\Omega E(|f| \mid \mathcal{F}_T)\chi_{\{\tau<\infty\}} v\,dP$$
$$= \frac{1}{2\lambda} \int_\Omega |f| E(\chi_{\{\tau<\infty\}} v \mid \mathcal{F}_T)\,dP = \frac{1}{2\lambda} \int_{\{\tau<\infty\}} |f| v_T\,dP.$$

Now, using the Hölder inequality, we see that

$$P_v\{f^* > 2\lambda\} \le \frac{1}{2\lambda} \int_{\{\tau<\infty\}} |f| E\left(\frac{\sigma_T}{\sigma_T} v \,\middle|\, \mathcal{F}_T\right) dP$$
$$= \frac{1}{2\lambda} \int_{\{\tau<\infty\}} \sigma_T^{-1} |f| E(\sigma_T v^{1/p} v^{1/p'} \mid \mathcal{F}_T)\,dP$$
$$\le \frac{1}{2\lambda} \int_{\{\tau<\infty\}} \sigma_T^{-1} |f| E(\sigma_T^p v \mid \mathcal{F}_T)^{1/p} E(v \mid \mathcal{F}_T)^{1/p'}\,dP.$$

Finally, by the $\mathcal{A}_p$-condition and the uniform regular condition, we get

$$P_v\{f^* > 2\lambda\} \le \frac{c_p}{2\lambda} \int_{\{\tau<\infty\}} \sigma_T^{-1} |f| \sigma_T^{1/p} v_T^{1/p'}\,dP$$
$$= \frac{c_p}{2\lambda} \int_{\{\tau<\infty\}} |f| (v_T/\sigma_T)^{1/p'}\,dP \le \frac{c_p}{2\lambda} \int_{\{\tau<\infty\}} |f| (v/\sigma)^{1/p'}\,dP.$$

Multiply both sides of (8) by $p\lambda^{p-1}$ and integrate with respect to λ from 0 to ∞, to obtain

$$\int_0^\infty p\lambda^{p-1} P_v\{f^* > 2\lambda\}\,d\lambda \le \int_0^\infty p\lambda^{p-1} \frac{c_p}{\lambda} \int_{\{\tau<\infty\}} Y\,dP\,d\lambda.$$

By Fubini's theorem and Hölder's inequality we have

$$\begin{aligned}\int_\Omega (f^*/2)^p v\,dP &\le c_p \int_\Omega Y \int_0^{f^*} \lambda^{p-2}\,d\lambda\,dP\\ &= c_p \int_\Omega v^{1/p'} f^{*(p-1)}|f|\sigma^{-1/p'}\,dP\\ &\le c_p\Big(\int_\Omega f^{*p}v\,dP\Big)^{1/p'}\Big(\int_\Omega |f|^p\sigma^{-p/p'}\,dP\Big)^{1/p}.\end{aligned}$$

Since $\sigma^{-p/p'} = \sigma^{-(p-1)} = w$, Theorem 3 is proved.

Before we give the next theorem, first we need the following definitions of reverse Hölder inequality for a *pair* of weights, and the strong $\mathcal{S}_p$-condition. Hereafter, we assume $\sigma dP = \sigma(w)dP$ is a probability measure.

DEFINITION. We say a pair (v,w) satisfies the *reverse Hölder inequality* of order p, and write $(v,w)\in\mathcal{RH}_p$, if there exists a constant c_p such that

$$(\mathcal{R}_p)\qquad \sigma_T^{1/p}v_T^{1/p'} \le c_p E(\sigma^{1/p}v^{1/p'} \mid \mathcal{F}_T)$$

for every stopping time T.

DEFINITION. We say a pair of weights (v,w) satisfies the *strong $\mathcal{S}_p$-condition*, denoted by $\mathbf{S}\mathcal{S}_p$, if for any stopping times $T_1\le T_2$, there exists a constant c_p such that

$$(\mathbf{S}\mathcal{S}_p)\qquad E({}^*\sigma_{T_1}^p v \mid \mathcal{F}_{T_2}) \le c_p\sigma_{T_2}.$$

If $T_1\le T_2$ are stopping times, let $({}^{T_1}f^{T_2})^* = \sup_{T_1\le n\le T_2}|f_n|$ denote the cut maximal function.

THEOREM 4. *If (v,w) satisfies the strong $\mathcal{S}_p$-condition $\mathbf{S}\mathcal{S}_p$ and the reverse Hölder inequality $\mathcal{RH}_p$, then we have* (7).

Proof. Similar to what we did before, define two stopping times

$$T_1 = \inf\{n : f_n^* > \lambda\},\qquad T_2 = \inf\{n : f_n^* > 2\lambda\}.$$

Then $T_1\le T_2$ and $\{f^*>2\lambda\} = \{T_2<\infty\}\subset\{T_1<\infty,\ f_{T_2}^* - f_{T_1-1}^* > \lambda\}$, so that $P_v(f^*>2\lambda) \le P_v(T_1<\infty,\ f_{T_2}^* - f_{T_1-1}^* > \lambda)$ or

$$\int_{\{f^*>2\lambda\}} v\,dP = \int_{\{T_2<\infty\}} v\,dP \le \int_{\{T_1<\infty,\, f_{T_2}^*-f_{T_1-1}^*>\lambda\}} v\,dP.$$

Noticing that $\mathcal{F}_{T_1}\subset\mathcal{F}_{T_2}$, we have the estimate of the maximal function by

the weighted maximal function:

$$\begin{aligned}({}^{T_1}f^{T_2})^* &= \sup_{T_1\le m\le T_2}|E(f\mid\mathcal{F}_m)| = \sup_{T_1\le m\le T_2}\{|\widehat{E}_\sigma(f\sigma^{-1}\mid\mathcal{F}_m)|\sigma_m\}\\ &\le \sup_{T_1\le m\le T_2}\{|\widehat{E}_\sigma(f\sigma^{-1}\mid\mathcal{F}_m)|\}\sup_{T_1\le m\le T_2}\sigma_m \le ({}^{T_1}(f\sigma^{-1})^{T_2})^{\hat{*}}({}^*\sigma_{T_1}),\end{aligned}$$

where $\hat{*}$ denotes the maximal functional with respect to the weighted measure σdP. Apply the Hölder inequality to obtain

$$\begin{aligned}\int_{\{f^*>2\lambda\}} v\,dP &\le \frac{1}{\lambda}\int_{\{T_1<\infty\}}(f_{T_2}^* - f_{T_1-1}^*)v\,dP \le \frac{1}{\lambda}\int_{\{T_1<\infty\}}({}^{T_1}f^{T_2})^* v\,dP\\ &\le \frac{1}{\lambda}\int_{\{T_1<\infty\}}({}^{T_1}(f\sigma^{-1})^{T_2})^{\hat{*}}({}^*\sigma_{T_1})v\,dP\\ &= \frac{1}{\lambda}\int_{\{T_1<\infty\}}({}^{T_1}(f\sigma^{-1})^{T_2})^{\hat{*}}E[({}^*\sigma_{T_1})v\mid\mathcal{F}_{T_2}]\,dP\\ &\le \frac{1}{\lambda}\int_{\{T_1<\infty\}}({}^{T_1}(f\sigma^{-1})^{T_2})^{\hat{*}}E({}^*\sigma_{T_1}^p v\mid\mathcal{F}_{T_2})^{1/p}E(v\mid\mathcal{F}_{T_2})^{1/p'}\,dP.\end{aligned}$$

Then using the strong $\mathcal{S}_p$-condition and reverse Hölder inequality, we get

$$\begin{aligned}\int_{\{f^*>2\lambda\}} v\,dP &\le \frac{c_p}{\lambda}\int_{\{T_1<\infty\}}({}^{T_1}(f\sigma^{-1})^{T_2})^{\hat{*}}\sigma_{T_2}^{1/p}v_{T_2}^{1/p'}\,dP\\ &\le \frac{c_p}{\lambda}\int_{\{T_1<\infty\}}({}^{T_1}(f\sigma^{-1})^{T_2})^{\hat{*}}E(\sigma^{1/p}v^{1/p'}\mid\mathcal{F}_{T_2})\,dP\\ &= \frac{c_p}{\lambda}\int_{\{T_1<\infty\}}({}^{T_1}(f\sigma^{-1})^{T_2})^{\hat{*}}\sigma^{1/p}v^{1/p'}\,dP\\ &\le \frac{c_p}{\lambda}\int_{\{T_1<\infty\}}(f\sigma^{-1})^{\hat{*}}\sigma^{1/p}v^{1/p'}\,dP.\end{aligned}$$

The $\mathcal{F}_{T_2}$ measurability of $({}^{T_1}(f\sigma^{-1})^{T_2})^{\hat{*}}$ is the key point in the above argument. Proceeding as in the previous proof, by the Fubini theorem, Hölder inequality, and the Doob maximal inequality for the weighted probability measure σdP, we have

$$\int_\Omega f^{*p}v\,dP \le c_p\int_\Omega[(f\sigma^{-1})^{\hat{*}}]^p\sigma\,dP \le c_p\int_\Omega|f|^p\sigma^{-p}\sigma\,dP = c_p\int_\Omega|f|^p w\,dP.$$

This completes the proof of Theorem 4.

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References

[1] J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953.
[2] R. A. Hunt, D. S. Kurtz and C. J. Neugebauer, *A note on the equivalence of $\mathcal{A}_p$ and Sawyer's condition for equal weights*, in: Conf. Harmonic Analysis in Honor of A. Zygmund, Wadsworth, Belmont, Calif., 1981, 156–158.
[3] M. Izumisawa and N. Kazamaki, *Weighted norm inequalities for martingales*, Tôhoku Math. J. 29 (1977), 115–124.
[4] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.
[5] E. Sawyer, *A characterization of a two weight norm inequality for maximal operators*, Studia Math. 75 (1982), 1–11.

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Oscillatory kernels in certain Hardy-type spaces

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Abstract. We consider a convolution operator $Tf = \text{p.v.}\,\Omega * f$ with $\Omega(x) = K(x)e^{ih(x)}$, where $K(x)$ is an (n,β) kernel near the origin and an (α,β), $\alpha \ge n$, kernel away from the origin; $h(x)$ is a real-valued C^∞ function on $\mathbb{R}^n \setminus \{0\}$. We give a criterion for such an operator to be bounded from the space $H_0^p(\mathbb{R}^n)$ into itself.

1. Introduction and notations. Let $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and $h(x)$ be a real-valued function. Consider the oscillatory kernel $\Omega(x) = K(x)e^{ih(x)}$ with $K(x)$ being an (n,β) kernel near the origin of $\mathbb{R}^n$ and an (α,β) kernel away from the origin. An (α,β) *kernel* K is a function on $\mathbb{R}^n \setminus \{0\}$ satisfying

$$|D^J K(x)| \le C_J |x|^{-\alpha-|J|} \tag{1.1}$$

with $|J| \le \beta$, $x \ne 0$. The *phase function* $h(x)$ is a C^∞ function on $\mathbb{R}^n \setminus \{0\}$ satisfying (1.2) and (1.3):

$$|D^J h(x)| \le C_J |x|^{b-|J|} \tag{1.2}$$

for all multi-indices J with $|J| \le M$, $x \ne 0$, where M and b are positive integers, and

$$|\nabla h(x)| \ge C|x|^{b-1}, \tag{1.3}$$

where $\nabla = (\partial_{x_1},\ldots,\partial_{x_n})$ is the gradient operator.

For the above defined kernel $\Omega(x)$, the associated oscillatory singular integral T is defined by

$$Tf(y) = \text{p.v.} \int_{\mathbb{R}^n} e^{ih(y-x)} K(y-x) f(x)\,dx, \tag{1.4}$$

where $K(x)$ satisfies (1.1) and in addition, there exists an $\varepsilon > 0$ such that

$$\text{p.v.} \int_{0<|x|\le\varepsilon} K(x)\,dx = 0. \tag{1.5}$$

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