Some Sawyer type inequalities for martingales

by

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Abstract. Some martingale analogues of Sawyer's two-weight norm inequality for the Hardy-Littlewood maximal function $Mf$ are shown for the Doob maximal function of martingales.

1. Introduction. Throughout this paper, we will only consider closed discrete martingales $f = (f_n)$ with respect to a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_n\}_{n \geq 0}$ with $\mathcal{F} = \bigvee_{n \geq 0} \mathcal{F}_n$. That is, $f_n = E(f \mid \mathcal{F}_n)$ for all $n$ and $f \in L^1(\Omega, \mathcal{F}, P)$. We will also follow the convention of Zygmund to denote by $c_p$ a constant which only depends on $p$. However, it may be different in different lines or different theorems. Recall the fundamental work of B. Muckenhoupt [4], who showed in 1972 the weighted Hardy-Littlewood inequality.

Theorem (Muckenhoupt). If $p > 1$, then the weighted norm inequality

$$\int_{\mathbb{R}^n} (Mf)^p w \, dx \leq c_p \int_{\mathbb{R}^n} |f|^p w \, dx$$

holds for every $f \in L^p(w)$ if and only if the weight $w$ satisfies Muckenhoupt's $A_p$-condition

$$(A_p) \quad \sup_Q \left[ \left( \frac{1}{|Q|} \int_Q w \, dx \right)^{1/(p-1)} \left( \frac{1}{|Q|} \int_Q \left( \frac{1}{w} \right)^{1/(p-1)} \, dx \right)^{p-1} \right] < \infty,$$

where $Mf = \sup_Q \frac{1}{|Q|} \int_Q |f| \, dy$, the Hardy-Littlewood maximal function, and $Q$ denotes an arbitrary cube in $\mathbb{R}^n$.

On the other hand, the martingale version of the Hardy-Littlewood inequality is the famous Doob inequality [1]. So it is natural to consider the martingale analogue of inequality (1). To this end, we need the counterpart of the $A_p$-condition for martingales. Say a martingale $w = (w_n)$ satisfies the

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\( A_p \)-condition if
\[
(A_p) \quad \sup_n \| E(w | F_n) E(1/w)^{1/(p-1)} | F_n \|_{p-1} < \infty.
\]

In 1977, Izumisawa and Kazamaki [3] proved the following martingale analogue of Muckenhoupt’s weighted inequality for the Doob maximal function \( f^* = \sup_n | f_n | \) of a martingale \( f = (f_n) \).

**Theorem (Izumisawa–Kazamaki).** For all \( p > 1 \), and every martingale \( f = (f_n) \), we have

(a) The \( A_p \)-condition is a necessary condition for the weighted norm inequality
\[
(2) \quad \int \Omega f^* w \, dP \leq c_p \int \Omega |f|^p w \, dP.
\]
to hold.

(b) If \( w = (w_n) \) also satisfies the step regular condition: \( w_n \leq c w_{n+1} \) for all \( n \geq 0 \), then the \( A_p \)-condition is sufficient for (2) to hold.

For a given weight \( w \), we denote by \( \sigma(w) = (1/w)^{1/(p-1)} = w^{1-p'} \) (or simply \( \sigma \)) its conjugate weight, where \( p' \) is the conjugate exponent of \( p \). Thus the \( A_p \)-condition can be expressed as \( \sup_n \| w_n \sigma_n \|_{p-1} < \infty \), where \( w_n = E(w | F_n) \), and \( \sigma_n = E(\sigma | F_n) \). Say a pair \( (v, w) \) satisfies \( A_p \) if \( \sup_n \| v_n \sigma_n \|_{p-1} < \infty \).

Muckenhoupt not only considered the one weight problem, but also raised the question: What condition on a pair of weights \( (v, w) \) is necessary and sufficient for the two-weight norm inequality
\[
(3) \quad \int_{R^n} (Mf)^p v \, dx \leq c_p \int_{R^n} |f|^p w \, dx
\]
to hold? Suggested by the \( A_p \)-condition, the first natural candidate seems to be
\[
(4) \quad \sup_Q \left[ \left( \frac{1}{|Q|} \int Q v \, dx \right)^{1/q} \left( \frac{1}{|Q|} \int Q \left( \frac{1}{w} \right)^{1/(p-1)} \, dx \right)^{p-1} \right] < \infty.
\]
But it turns out that (4) is equivalent to the weak \( (p, p) \) inequality for the operator \( Mf \). The right condition on \( (v, w) \) which fully characterizes (3) was only found in 1982 by E. Sawyer [5].

**Definition.** Say a pair of weights \( (v, w) \) satisfies Sawyer’s \( S_p \)-condition
\[
(S_p) \quad \int_Q (M(\chi_Q \sigma))^p v \, dx \leq c_p \int_Q \sigma \, dx
\]
for every cube \( Q \) in \( R^n \), where \( \chi_Q \) is the characteristic function of \( Q \).

**Theorem (Sawyer).** If \( p > 1 \), then the Hardy–Littlewood maximal operator \( M \) is bounded from \( L^p(w \, dx) \) to \( L^p(w \, dx) \) if and only if the weights \( (v, w) \) in \( R^n \) satisfy the Sawyer \( S_p \)-condition.

We will show some Sawyer type inequalities for martingales in Section 2. We end this section with notations of the tailed maximal function for a martingale \( f = (f_n) \); \( f_n = \sup_{m \geq n} (|f_m|) \), and of the weighted conditional expectation with respect to \( w \) and \( P \); \( \hat{E}_w (\cdot | \cdot) \). Here we assume \( w \) is nonnegative and \( E(w) = 1 \). It is not hard to verify the formula
\[
(5) \quad \hat{E}_w (f | F_n) = \frac{1}{w_n} E(f w | F_n).
\]
It is also easy, from the definition of conditional expectation, to check the following chain formula:
\[
(6) \quad E(XE(Y | F')') = E(E(X | F') Y)
\]
where \( X \) and \( Y \) are random variables, and \( F' \) is a sub-\( \sigma \)-algebra of \( F \).

**2. Sawyer-type inequalities for martingales.** First of all, we need the martingale counterpart of the \( S_p \)-condition.

**Definition.** For a given pair of nonnegative martingales \( v = (v_n) \) and \( w = (w_n) \), we say \( (v, w) \) satisfies the \( S_p \)-condition if there exists a constant \( c_p \) such that for all \( n \geq 0 \),
\[
(S_p) \quad E((\sigma_n)^p v | F_n) \leq c_p \sigma_n.
\]

We say that \( w \) satisfies \( A_p \), or \( S_p \), if \( (w, w) \) has that property. In the equal weight case, Hunt–Kurtz–Neugebauer [2] have shown, in the classical setting, that \( A_p \Rightarrow S_p \). The following theorem shows that our \( S_p \) is a proper martingale analogue of the \( S_p \)-condition.

**Theorem 1.** The martingale \( w = (w_n) \) satisfies \( A_p \) if and only if it satisfies \( S_p \).

The direction \( S_p \Rightarrow A_p \) is easy. Before proving the other direction let us first prove a lemma, which is probably well known, but does not appear in the standard references.

**Lemma (conditional version of Doob’s inequality).** If \( p > 1 \), then for any \( n \geq 0 \), we have
\[
E((\sigma_n)^p | F_n) \leq c_p E((\sigma)^p | F_n)
\]
for any martingale \( g = (g_n) \).

**Proof.** In fact, for any fixed \( F \in F_n \) take \( f = \chi_{F \cap g} \). Then for any \( m \geq n, f_m = E(\chi_{F \cap g} | F_m) = \chi_{F \cap g} \). So \( f_n = \chi_{F \cap g} \). By applying
Doob’s inequality to \( f \) we get
\[
\int \chi_F \varphi_{m}^{p} \, dP = \int \chi_{F}^{+} \varphi_{m}^{p} \, dP \leq \int \chi_{F} \varphi_{m}^{p} \, dP = \int \chi_{F} \varphi_{m}^{p} \, dP,
\]
i.e., \( \int \chi_{F}^{+} \varphi_{m}^{p} \, dP \leq \int \chi_{F} \varphi_{m}^{p} \, dP \), which implies \( E(\varphi_{m}^{p} \mid F_{n}) \leq c_{p} E(\varphi_{m}^{p} \mid F_{n}) \).

**Proof of Theorem 1.** Suppose \( w \) satisfies \( A_{p} \). Then for \( m \geq n \), we have \( \sigma_{m} \leq c_{p}[1/w_{n}]^{1/(p-1)} \). Since
\[
\mathcal{E}_{w}(w^{-1} \mid F_{n}) = \frac{1}{w_{n}}E(w^{-1} \mid F_{n}) = \frac{1}{w_{n}},
\]
we get
\[
\sigma_{m} \leq c_{p} \mathcal{E}_{w}(w^{-1} \mid F_{n})^{p'/p}
\]
or
\[
(\sup_{m \geq n} \sigma_{m})^{p} \leq c_{p} \mathcal{E}_{w}(w^{-1} \mid F_{n})^{p'}. \tag{8}
\]
Taking the conditional expectation with respect to \( \mathcal{E}_{w} \), and using the conditional version of Doob’s inequality with index \( p' \), we have
\[
\mathcal{E}_{w}[((\sup_{m \geq n} \sigma_{m})^{p} \mid F_{m})] \leq c_{p} \mathcal{E}_{w}[((\sup_{m \geq n} \sigma_{m})^{p} \mid F_{m})^{p'} \mid F_{n}]
\]
\[
\leq c_{p} \mathcal{E}_{w}[w^{-p'} \mid F_{n}] = c_{p} \frac{1}{w_{n}}E[w^{-p'} \mid F_{n}] = c_{p} \sigma_{n}.
\]
Finally, \( E[(\sup_{m \geq n} \sigma_{m})^{p} w \mid F_{n}] = \mathcal{E}_{w}[((\sup_{m \geq n} \sigma_{m})^{p} \mid F_{n}) w_{n} \leq \sigma_{n} w_{n} \leq \sigma_{n}, \)
which is \( S_{p} \).

**Theorem 2.** \( S_{p} \) is a necessary condition for the two-weight norm inequality
\[
\int \chi_{F}^{p} \varphi_{m}^{p} \, dP \leq c_{p} \int \chi_{F}^{p} \varphi_{m}^{p} \, dP \]
to hold.

**Proof.** Take \( f = \chi_{F} \), for \( F \) an arbitrary set in \( F_{n} \). Then for any \( m \geq n, f_{m} = E(\chi_{F} \varphi_{m} \mid F_{m}) = \chi_{F} E(\varphi_{m} \mid F_{m}) = \chi_{F} \sigma_{m} \), and therefore \( f_{m}^{p} = \chi_{F}^{p} \sigma_{m}^{p} \). Hence from (7) we know that \( \int \chi_{F}^{p} \sigma_{m}^{p} \, dP \leq c_{p} \int \chi_{F}^{p} \sigma_{m}^{p} \, dP \). But \( \sigma = w^{-1} \Rightarrow \sigma_{m} = \sigma_{m}^{1-p} \), so \( \sigma_{m} \sigma_{m}^{1-p} = \sigma \), hence by the definition of conditional expectation the theorem is proved.

Whether \( S_{p} \) is also a sufficient condition is still unknown, at least to this author. It seems that we need some sort of “regular” condition, like the one in Izumiwasa–Kazamaki’s theorem, to guarantee the sufficiency of \( S_{p} \). Nevertheless, we present the following two theorems, which, we hope, will shed some light for the further study.

**Theorem 3.** If \( (v,w) \) satisfies the uniform regular condition \( v_{n}/\sigma_{n} \leq c_{p}/\sigma_{n} \), then \( A_{p} \) is a sufficient condition for (7) to hold.

**Proof.** For any \( \lambda > 0 \), define two stopping times
\[
\tau = \inf\{n : |f_{n}| > \lambda\}, \quad T = \inf\{n : |f_{n}| > 2\lambda\}.
\]
Clearly \( \tau \leq T \), and \( \{T < \infty\} = \{\tau < \infty, |f_{T}| > 2\lambda\} \). We now show
\[
P_{\lambda}(f^{*} > 2\lambda) \leq \frac{c_{p}}{\tau < \infty} \int Y \, dP,
\]
where \( Y = |f|/(v/\sigma)^{1/p'} \). First, by the chain formula (8), we have
\[
P_{\lambda}(f^{*} > 2\lambda) = P_{\lambda}(T < \infty) = P_{\lambda}(T < \infty, |f_{T}| > 2\lambda)
\]
\[
\leq \frac{1}{2\lambda} \int \int E(|f_{T}| \mid F_{T})\chi_{(T < \infty)} \, dP
\]
\[
= \frac{1}{2\lambda} \int \int E(\chi_{(T < \infty)} \mid F_{T}) \, dP = \frac{1}{2\lambda} \int \int \frac{|f_{T}| \mid F_{T}}{F_{T}} \, dP.
\]
Now, using the Hölder inequality, we see that
\[
P_{\lambda}(f^{*} > 2\lambda) \leq \frac{1}{2\lambda} \int \int \left| f_{T}(\frac{\sigma}{\sigma_{T}}) \right| \chi_{(T < \infty)} \, dP
\]
\[
= \frac{1}{2\lambda} \int \int \sigma_{T}^{-1} \left| f_{T}(\sigma_{T}^{-1/p} v^{1/p'} \mid F_{T}) \right| \, dP
\]
\[
\leq \frac{1}{2\lambda} \int \sigma_{T}^{-1} \left| f_{T} E(\sigma_{T}^{1/p} v^{1/p'}) \mid F_{T} \right| \, dP
\]
Finally, by the \( A_{p}^{-1} \)-condition and the uniform regular condition, we get
\[
P_{\lambda}(f^{*} > 2\lambda) \leq \frac{c_{p}}{2\lambda} \int \int \sigma_{T}^{-1} \left| f_{T} \sigma_{T}^{1/p} v^{1/p'} \right| \, dP
\]
\[
= \frac{c_{p}}{2\lambda} \int \int \left| f_{T}(v/\sigma)^{1/p'} \right| \, dP \leq \frac{c_{p}}{2\lambda} \int \int \left| f_{T}(v/\sigma)^{1/p'} \right| \, dP.
\]
Multiply both sides of (8) by \( p\lambda^{p-1} \) and integrate with respect to \( \lambda \) from 0 to \( \infty \), to obtain
\[
\int_{0}^{\infty} p\lambda^{p-1} P_{\lambda}(f^{*} > 2\lambda) \, d\lambda \leq \int_{0}^{\infty} p\lambda^{p-1} \frac{c_{p}}{\tau < \infty} \int Y \, d\lambda.
\]
By Fubini’s theorem and Hölder’s inequality we have
\[
\int_\Omega \left( (f^+/2)^{p} \right) dP \leq c_p \int_\Omega Y \int_0^Y \lambda^{p-1} d\lambda dP = c_p \int_\Omega Y^{1/p'} f^{*(p-1)} f^{\sigma-1/p'} dP \\
\leq c_p \left( \int_\Omega f^{p} dP \right)^{1/p'} \left( \int_\Omega f^{\sigma-1/p'} dP \right)^{1/p}.
\]
Since \(\sigma^{-1/p'} = \sigma^{-(p-1)} = w\), Theorem 3 is proved.

Before we give the next theorem, first we need the following definitions of reverse Hölder inequality for a pair of weights, and the strong \(S_p\)-condition. Hereafter, we assume \(\sigma dP = \sigma(w) dP\) is a probability measure.

**Definition.** We say a pair \((v, w)\) satisfies the reverse Hölder inequality of order \(p\), and write \((v, w) \in RH_p\), if there exists a constant \(c_p\) such that
\[
\sigma^{1/p} v^{1/p'} \leq c_p E(\sigma^{1/p} v^{1/p'}) |F_T|
\]
for every stopping time \(T\).

**Definition.** We say a pair of weights \((v, w)\) satisfies the strong \(S_p\)-condition, denoted by \(SS_p\), if for any stopping times \(T_1 \leq T_2\), there exists a constant \(c_p\) such that
\[
E(\sigma^{1/p} v | F_{T_2}) \leq c_p \sigma(T_2).
\]
If \(T_1 \leq T_2\) are stopping times, let \((T_1 f^{T_2})^* = \sup_{T_1 \leq n \leq T_2} |f_n|\) denote the cut maximal function.

**Theorem 4.** If \((v, w)\) satisfies the strong \(S_p\)-condition \(SS_p\) and the reverse Hölder inequality \(RH_p\), then we have (7).

**Proof.** Similar to what we did before, define two stopping times
\[T_1 = \inf \{ n : f_n > \lambda \}, \quad T_2 = \inf \{ n : f_n > 2\lambda \}.\]
Then \(T_1 \leq T_2\) and \(\{ f^* > 2\lambda \} \subset \{ T_1 < \infty, f_{T_2}^* - f_{T_1-1}^* > \lambda \}\), so that \(P_e(f^* > 2\lambda) \leq P_e(T_1 < \infty, f_{T_2}^* - f_{T_1-1}^* > \lambda)\) or
\[
\int_\Omega v dP = \int_\Omega v dP \leq \int_\Omega v dP.
\]
Noticing that \(F_{T_1} \subset F_{T_2}\), we have the estimate of the maximal function by

the weighted maximal function:
\[
(T_1 f^{T_2})^* = \sup_{T_1 \leq n \leq T_2} |E(f | F_n)| = \sup_{T_1 \leq n \leq T_2} \{ E_{\sigma} (\sigma^{-1} | F_n) \mid \sigma \}
\leq \sup_{T_1 \leq n \leq T_2} \{ E_{\sigma} (\sigma^{-1} | F_n) \mid \sigma \} \leq (T_1 (f^{T_2})^*)^*(\sigma_{T_1}),
\]
where \(\ast\) denotes the maximal functional with respect to the weighted measure \(\sigma dP\). Apply the Hölder inequality to obtain
\[
\int_\Omega v dP \leq \frac{1}{\lambda} \int_\Omega (f_{T_2}^* - f_{T_1-1}^*) v dP \leq \frac{1}{\lambda} \int_\Omega (T_1 f^{T_2})^* v dP
\]
\[
\leq \frac{1}{\lambda} \int_\Omega (T_1 (f^{T_2})^*)^*(\sigma_{T_1}) v dP
\]
\[
= \frac{1}{\lambda} \int_\Omega (T_1 (f^{T_2})^*)^* E(\sigma_{T_1} | F_{T_2}) dP
\]
\[
\leq \frac{1}{\lambda} \int_\Omega (T_1 (f^{T_2})^*)^* E(\sigma_{T_1}^p v | F_{T_2})^{1/p} E(v | F_{T_2})^{1/p'} dP.
\]
Then using the strong \(S_p\)-condition and reverse Hölder inequality, we get
\[
\int_\Omega v dP \leq \frac{C_p}{\lambda} \int_\Omega (T_1 (f^{T_2})^* \sigma_{T_2}^{1/p} v_{T_2}^{1/p'}) dP
\]
\[
\leq \frac{C_p}{\lambda} \int_\Omega (T_1 (f^{T_2})^*)^* E(\sigma_{T_2}^{1/p} v_{T_2}^{1/p'}) |F_{T_2}| dP
\]
\[
= \frac{C_p}{\lambda} \int_\Omega (T_1 (f^{T_2})^*)^* \sigma_{T_2}^{1/p} v_{T_2}^{1/p'} dP
\]
\[
\leq \frac{C_p}{\lambda} \int_\Omega (f^{T_2})^* \sigma_{T_2}^{1/p} v_{T_2}^{1/p'} dP.
\]
The \(F_{T_2}\) measurability of \((T_1 (f^{T_2})^*)^*\) is the key point in the above argument. Proceeding as in the previous proof, by the Fubini theorem, Hölder inequality, and the Doob maximal inequality for the weighted probability measure \(\sigma dP\), we have
\[
\int_\Omega f^{p} v dP \leq c_p \int_\Omega [(\sigma^{-1})^p]^{p} \sigma dP \leq c_p \int_\Omega |f|^p \sigma^{-p} \sigma dP = c_p \int_\Omega |f|^p w dP.
\]
This completes the proof of Theorem 4.

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Oscillatory kernels in certain Hardy-type spaces

by

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Abstract. We consider a convolution operator \( T f = \text{p.v.} \, \Omega * f \) with \( \Omega(x) = K(x) e^{ih(x)} \), where \( K(x) \) is an \((n, \beta)\) kernel near the origin and an \((\alpha, \beta)\), \( \alpha \geq n \), kernel away from the origin; \( h(x) \) is a real-valued \( C^\infty \) function on \( \mathbb{R}^n \setminus \{0\} \). We give a criterion for such an operator to be bounded from the space \( B^p_\beta(\mathbb{R}^n) \) into itself.

1. Introduction and notations. Let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( h(x) \) be a real-valued function. Consider the oscillatory kernel \( \Omega(x) = K(x) e^{ih(x)} \) with \( K(x) \) being an \((n, \beta)\) kernel near the origin of \( \mathbb{R}^n \) and an \((\alpha, \beta)\) kernel away from the origin. An \((\alpha, \beta)\) kernel \( K \) is a function on \( \mathbb{R}^n \setminus \{0\} \) satisfying

\[
|D^J K(x)| \leq C_J |x|^{-\alpha - |J|} \tag{1.1}
\]

with \( |J| \leq \beta \), \( x \neq 0 \). The phase function \( h(x) \) is a \( C^\infty \) function on \( \mathbb{R}^n \setminus \{0\} \) satisfying (1.2) and (1.3):

\[
|D^J h(x)| \leq C_J |x|^{\beta - |J|} \tag{1.2}
\]

for all multi-indices \( J \) with \( |J| \leq M, \ x \neq 0 \), where \( M \) and \( b \) are positive integers, and

\[
|\nabla h(x)| \geq C |x|^{b-1}, \tag{1.3}
\]

where \( \nabla = (\partial_{x_1}, \ldots, \partial_{x_n}) \) is the gradient operator.

For the above defined kernel \( \Omega(x) \), the associated oscillatory singular integral \( T \) is defined by

\[
T f(y) = \text{p.v.} \int_{\mathbb{R}^n} e^{ih(y-x)} K(y-x) f(x) \, dx, \tag{1.4}
\]

where \( K(x) \) satisfies (1.1) and in addition, there exists an \( \varepsilon > 0 \) such that

\[
\text{p.v.} \int_{0<|x| \leq \varepsilon} K(x) \, dx = 0. \tag{1.5}
\]

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