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One-parameter subgroups and the B-C-H formula

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Abstract. An algebraic scheme for Lie theory of topological groups with "large" families of one-parameter subgroups is proposed. Such groups are quotients of "ER-groups", i.e. topological groups equipped additionally with the continuous exterior binary operation of multiplication by real numbers, and generated by special ("exponential") elements. It is proved that under natural conditions on the topology of an ER-group its group multiplication is described by the B-C-H formula in terms of the associated Lie algebra.

1. Introduction. The notion of a Lie group of infinite dimensions is not well founded. The differential manifold approach which is basic for the classical (finite-dimensional) theory may be successfully applied only to Banach–Lie groups ([1], [9]). This class, however, appears to be too restrictive to incorporate most of interesting infinite-dimensional examples. Difficulties in extending the manifold approach beyond the frames of Banach case are of two kinds, which correspond to the two main limitations of the differential calculus in non-Banach spaces: lack of the *existence and uniqueness theorem* for ordinary differential equations and lack of the *inverse map theorem* for smooth mappings. In the classical theory one associates with a given group G its Lie algebra g which is usually defined to be the Lie algebra of all left (or equivalently right) invariant vector fields on G. This step presents no difficulty whatsoever, but to make it meaningful g has to be better connected with the group structure of G. Classically this is achieved by associating with each $X \in g$ its properly selected integral curve, which happens to be a one-parameter subgroup of G. Thus the validity of the existence and uniqueness theorem provides a one-to-one map i from g to $\Lambda(G)$, the set of all continuous one-parameter subgroups of G. In the absence of this theorem, e.g. for Fréchet–Lie groups, it is not known whether such a group has a single nontrivial one-parameter subgroup (cf. [9]). On the other hand, no examples disproving bijectivity of $i : g \to \Lambda(G)$ are known in this case. Concluding, the lack of methods for establishing bijectivity of $i : g \to \Lambda(G)$

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is the first obstruction to developing a manifold-based Lie group theory in general.

Another obstruction seems to be of more objective nature. For transmitting the properties of g to G via $\Lambda(G)$ one applies the exponential map $\exp : g \simeq \Lambda(G) \ni \Phi \to \Phi(1) \in G$. For a Banach–Lie G this map, by the inverse map theorem, is a local diffeomorphism at 0, which is crucial for getting the desired conclusions. This, however, in general fails to be true in the non-Banach case. If for instance $G = \mathrm{Diff}^{\infty}(M)$ where M is a compact manifold, then G has a nice C^{∞} Fréchet–Lie group structure ([6], [10]) and $i : g \to \Lambda(G)$ is bijective. Moreover, the map $\exp : g \to G$ is C^{∞} with nonsingular differential at 0. Nevertheless exp is neither locally injective nor locally surjective at 0 (cf. [9]). Typically, $\exp(g)$ is a rather irregular first-category-like subset of G (cf. [11]). This is the main obstacle to applying the differential geometry methods, and at first sight almost excludes any possibility of transmitting properties from g to G. Fortunately enough the set $\exp(g)$ is always well placed inside G, namely it is a generating set for the connected component G_0 of e in G. For $G = \mathrm{Diff}^{\infty}(M)$ this rather nontrivial fact results from the Epstein–Herman–Thurston theorem (cf. [3], [5], [14]) which states that G_0 is a simple group.

In this paper we develop an algebraic approach to Lie group theory with a view to handling the second type of obstacle. We leave aside the question of the existence of one-parameter subgroups, and we restrict our attention to the case of groups for which one-parameter subgroups are abundant. Since we need no differentiability assumptions, on the level of definitions we adopt a topological setting. We start with a topological group G which has a "rich" family $\Lambda(G)$. The "richness" condition can be naturally phrased in the way making it possible to treat G as a quotient group of an "exponential $\mathbb{R}$-group" (abbreviated as $\mathbb{ER}$-group). Groups of this type admit an exterior binary operation of multiplication by real numbers and are generated by special ("exponential") elements. They can be viewed as noncommutative generalizations of topological vector spaces. Achieving some understanding of their structure is our main goal here since they seem to constitute a territory on which transition from (Lie) groups to Lie algebras and vice versa occurs. Summing up, we suggest a different scheme for Lie group theory: instead of treating groups equipped with a differential structure we propose the study of groups which are quotients of suitable regular "overlying" groups. The main task then is to find proper objects for this role. Regularity of their structure has to be the algebraic counterpart of the differential calculus in the manifold approach.

The paper is organized as follows. In the preliminary sections 2–4 we introduce objects dealt with later: in Section 2 we formulate the above mentioned "richness" condition and define the concept of a polynomial group over G; in Section 3 there is defined a natural Lie algebra structure on $\Lambda(G)$, and Section 4 introduces B-C-H groups and algebras. In Section 5 we put the objects discussed earlier into a common algebraic framework by introducing the concept of $\mathbb{ER}$-groups. The rest of the paper is devoted to the study of their structure. After describing in Section 6 important examples of $\mathbb{ER}$-groups arising in the context of Lie algebras, we introduce in Section 7 the basic procedure associating with a given $\mathbb{ER}$-group H its Lie algebra $L(H)$. Section 8 introduces C^{∞} $\mathbb{ER}$-groups which are hoped to be the proper "overlying objects" in the above mentioned approach to Lie group theory. In Section 9, Theorem 23 states that for a C^{∞} $\mathbb{ER}$-group its Lie algebra has good functorial properties. It is the basis for Theorem 28 of Section 10 which states that under an additional "analyticity condition" each C^{∞} $\mathbb{ER}$-group may be continuously and homomorphically embedded in the associated B-C-H group. This result opens the possibility of passing from topological assumptions about the discussed group to its full algebraic "analytic" description. The closing section 11 discusses perspectives of the suggested approach.

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2. Polynomial groups over a topological group, weakly exponential groups.

Given a topological group G (all the groups dealt with are supposed to be Hausdorff) let $C_0(\mathbb{R}, G)$ denote the family of all continuous G-valued functions on the real line $\mathbb{R}$ such that $f(0) = e$. The target space being a topological group, $C_0(\mathbb{R}, G)$ with pointwise multiplication and the compact-open topology is also a topological group. The real field structure on the source space reflects in the existence of the $\mathbb{R}$-product

$$(1)\qquad \mathbb{R} \times C_0(\mathbb{R}, G) \ni (s, f) \to s * f \in C_0(\mathbb{R}, G), \qquad (s * f)(t) = f(st).$$

The $\mathbb{R}$-product (1) is jointly continuous and it is related to the group multiplication in $C_0(\mathbb{R}, G)$ by the following equalities valid for $s, t \in \mathbb{R}$ and $f_1, f_2, f \in C_0(\mathbb{R}, G)$:

$$(2)\qquad \begin{aligned} &\text{(a)} && s * (t * f) = (st) * f,\\ &\text{(b)} && s * (f_1 f_2) = (s * f_1)(s * f_2),\\ &\text{(c)} && 1 * f = f, \quad 0 * f = e, \end{aligned}$$

where e is the unit of $C_0(\mathbb{R}, G)$.

Let $\Lambda(G)$ denote the family of all continuous one-parameter subgroups of G, and for $k \in \mathbb{N}$ let

$$\Lambda_k(G) = \{f \in C_0(\mathbb{R}, G) : f(t) = \phi(t^k) \text{ for some } \phi \in \Lambda(G)\}.$$

Note that the elements of $\Lambda_k(G)$ are distinguished among those of $C_0(\mathbb{R}, G)$ by the condition

$$(3) \qquad n * f = f^{n^k} \quad \text{if } n^k \in \mathbb{Z}$$

($\mathbb{Z}$ denotes the set of all integers and $\mathbb{N}$ the set of all positive integers). Note also that by (3) each $\Lambda_k(G)$ is a closed and $\mathbb{R}$-invariant subset of $C_0(\mathbb{R}, G)$.

The central place in our considerations will be occupied by subgroups of $C_0(\mathbb{R}, G)$ which contain $P_0(G)$, the subgroup of $C_0(\mathbb{R}, G)$ generated by $\Lambda(G)$, and are contained in $P(G)$, the subgroup of $C_0(\mathbb{R}, G)$ generated by $\bigcup_{j=1}^{\infty} \Lambda_j(G)$. Such groups will be considered in various topologies, always not weaker than the compact-open topology, and coinciding with it on $\bigcup_{j=1}^{\infty} \Lambda_j(G)$. They will be referred to as *polynomial groups* over G.

Let H be a polynomial group over G. Consider the evaluation map

$$(4) \qquad \operatorname{Exp} : H \ni f \to f(1) \in G,$$

which is clearly a continuous homomorphism. Note that if G is a Banach–Lie group then the restriction of (4) to $\Lambda(G)$ coincides with the classical exponential map. Moreover, (4) then maps bijectively each sufficiently small open neighbourhood of 0 in $\Lambda(G)$ onto an open neighbourhood of e in G. It follows that for G Banach–Lie the homomorphism (4) is open. In particular, if G is connected then

$$(5) \qquad G = H/\ker \operatorname{Exp}$$

topologically.

A topological group G is said to be *weakly exponential* if the mapping (4) is surjective for some (equivalently for each) polynomial group H over G. Clearly weakly exponential groups form the largest class of groups for which some abstract Lie group theory (understood as a skill of describing G in terms of $\Lambda(G)$) is possible.

3. The Trotter formulas and the natural Lie structure on $\Lambda(G)$. A starting point in our approach is the assumption that the considered topological group G is weakly exponential. We shall examine the possibility of endowing $\Lambda(G)$ with a topological Lie algebra structure appropriate for describing G. It seems reasonable to assume that a part of such a structure is a priori given by the compact-open topology and the $\mathbb{R}$-product restricted from $C_0(\mathbb{R}, G)$. It is interesting to note that in most cases when an "appropriate" structure does exist it is expressed by the following Trotter formulas:

$$(6) \quad \text{(a)} \qquad (\Phi_1 + \Phi_2)(t) = \lim_{n\to\infty} \left(\Phi_1\left(\frac{t}{n}\right)\Phi_2\left(\frac{t}{n}\right)\right)^n,$$

$$(6) \quad \text{(b)} \quad [\Phi_1, \Phi_2](t^2) = \lim_{n\to\infty} \left(\Phi_1\left(\frac{t}{n}\right)\Phi_2\left(\frac{t}{n}\right)\Phi_1\left(-\frac{t}{n}\right)\Phi_2\left(-\frac{t}{n}\right)\right)^{n^2},$$

where $\Phi_1 + \Phi_2$ and $[\Phi_1, \Phi_2]$ denote respectively the sum and the Lie bracket of $\Phi_1, \Phi_2 \in \Lambda(G)$ and where the limits are almost uniform, in particular, they can be treated as limits in $P(G)$.

The algebraic properties of the left-hand sides of (6)(a), (b) suggest the presence of a certain structure coded in the shape of the right-hand sides. The description of this structure is one of our goals here. The first step in this direction is to observe that (6)(a), (b), when expressed in terms of $P(G)$, extrapolates to the following sequence of formulas:

$$(6;k) \qquad d_k f = \lim_{n\to\infty} \left(\frac{1}{n} * f\right)^{n^k} \quad \text{for } f \in P(G),$$

where (6;1) reduces to (6)(a) when $f = \Phi_1\Phi_2$ while (6;2) reduces to (6)(b) when $f = \Phi_1\Phi_2\Phi_1^{-1}\Phi_2^{-1}$. The analysis of ranges and domains of d_k and of their algebraic properties is postponed to the next sections.

For brevity, the topological Lie algebra structure on $\Lambda(G)$ composed of the $\mathbb{R}$-product, compact-open topology and with the algebraic operations described by 6(a),(b) will be referred to as the *natural* structure.

6(a),(b) inspired efforts to create a Lie group theory based on these formulas (cf. [3]). It seems that these attempts failed to bring substantial progress.

We finish this section with an easy but important observation:

Remark 1. *Let G_i be a topological group with natural Lie algebra $\Lambda(G_i)$, $i = 1, 2$. Each continuous homomorphism $H : G_1 \to G_2$ induces a continuous homomorphism $h : \Lambda(G_1) \to \Lambda(G_2)$ such that $H \circ \operatorname{Exp} = \operatorname{Exp} \circ h$.*

Proof. Define $h(\phi) = H \circ \phi$.

4. The B-C-H groups. The Baker–Campbell–Hausdorff series

$$(7) \qquad \Theta(f, g) = \sum_{m=1}^{\infty} \Theta_m(f, g)$$

is a real power series of noncommuting formal variables f, g which is obtained as the composition $\Theta = L \circ Z$ where

$$L(z) = \log(1 + z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$$

and

$$Z(f, g) = e^f e^g - 1 = \sum_{j+k\geq 1} \frac{f^j}{j!} \frac{g^k}{k!}.$$

It is known (cf. [12]) that each homogeneous term Θ_m of order m in (7) is a Lie polynomial, i.e. it can be expressed as a finite linear combination of m-fold commutators of f and g. In particular, $\Theta_1(f,g) = f+g$ and $\Theta_2(f,g) = \frac{1}{2}(fg-gf)$.

Let L be a topological Lie algebra. Substituting elements of L for f and g in (7) and treating commutators in $\Theta_m(f,g)$ for $m=1,2,\ldots$ as Lie brackets in L we obtain the evaluated series $\Theta(f,g)$ with terms Θ_j in L.

DEFINITION 2. L is said to be a *B-C-H algebra* if for each pair of its elements the evaluated series $\Theta(f,g)$ converges and the function

$$(8)\qquad L\times L \ni (f,g) \to f\cdot g = \Theta(f,g) \in L$$

is jointly continuous.

It is known ([2], [12]) that in this case (8) defines a group operation on L, with unit 0 and the inverse of f equal to $-f$ for each $f\in L$.

The topological group obtained in this way will be called the *B-C-H group of* L and denoted by $\exp L$.

The following remark lists a few known properties of B-C-H groups (cf. [2], [12]) in a form suitable for us.

Remark 3. *Let L be a B-C-H algebra. Let* $\exp : L \to \exp L$ *be the identity map. Then*:

(a) *Each $f\in \Lambda(\exp L)$ is of the form f_x for some $x \in L$, where* $f_x(t) = \exp(tx)$.

(b) *The mapping $j : L \to \Lambda(\exp L)$ with $j(x) = f_x$ is a homeomorphism* (*where $\Lambda(\exp L)$ is endowed with the compact-open topology*).

(c) *The Lie algebra structure on $\Lambda(\exp L)$ obtained via the identification j is natural*, *i.e.*

$$(9)\quad \begin{aligned} &\text{(a)}\qquad f_{x+y}(t) = \lim_{n\to\infty}\Big(f_x\Big(\frac{t}{n}\Big)f_y\Big(\frac{t}{n}\Big)\Big)^n,\\ &\text{(b)}\qquad f_{[x,y]}(t^2) = \lim_{n\to\infty}\Big(f_x\Big(\frac{t}{n}\Big)f_y\Big(\frac{t}{n}\Big)f_x\Big(-\frac{t}{n}\Big)f_y\Big(-\frac{t}{n}\Big)\Big)^{n^2}.\end{aligned}$$

5. Exponential $\mathbb{R}$-groups. Led by examples of polynomial groups from Section 2, we now consider their abstract analogues. The axiomatization is partly justified by the necessity of considering various topologies on polynomial groups but mostly by the presence of different types of examples coming from Lie algebras. The axiomatic approach will exhibit the common algebraic background of both situations.

DEFINITION 4. A topological group H is said to be an $\mathbb{R}$-*group* if it admits a jointly continuous binary operation

$$\mathbb{R}\times H \ni (s,f) \to s*f \in H$$

satisfying the conditions (2)(a)–(c). Let

$$E_k(H) = \{f\in H : n*f = f^{n^k} \text{ if } n^k\in\mathbb{Z}\} \quad\text{and}\quad E(H) = \bigcup_{k=1}^{\infty} E_k(H).$$

The elements of $E_k(H)$ will be called k-*exponential*. An $\mathbb{R}$-group H is said to be *exponential* (resp. *polynomial*) if it is generated by $E_1(H)$ (resp. by $E(H)$), and it is said to be *topologically exponential* (resp. *topologically polynomial*) if it contains a dense exponential (resp. polynomial) $\mathbb{R}$-subgroup.

The assumption that an $\mathbb{R}$-group is topologically exponential has far-reaching consequences. An important example is provided by the following proposition which will be used as the initial step of Lemma 13 below:

PROPOSITION 5. *Let H be a topologically exponential $\mathbb{R}$-group. The following conditions are equivalent*:

(a) $E_1(H) = H$.

(b) *H is abelian.*

(c) *H is a topological vector space.*

The proof is easy and will be omitted.

The second type of exponential $\mathbb{R}$-groups is connected with Lie algebras of a special form:

DEFINITION 6. A topological Lie algebra K is said to be *product graded* if it is isomorphic as a topological vector space to the countable topological direct product of topological vector spaces M_j,

$$K = \prod_{j=1}^{\infty} M_j,$$

and if the Lie bracket in K satisfies the grading condition

$$(10)\qquad [M_k, M_j] \subset M_{k+j}.$$

(Here and in the sequel we write M_j for $(\underbrace{0,\ldots,0}_{j-1\text{ zeros}}, M_j, 0, 0, \ldots)$.)

Let $p_j : K \to M_j$ denote the projection onto the jth coordinate, $j = 1,2,\ldots$

PROPOSITION 7. *Let K be a product graded Lie algebra. Then K is a B-C-H algebra and $H = \exp K$ admits an $\mathbb{R}$-product defined by*

$$(11)\qquad s*h = \exp\Big(\sum_{j=1}^{\infty} s^j p_j(f)\Big)$$

for $s\in\mathbb{R}$ and $h = \exp f \in \exp H$ where $f = \sum_{j=1}^{\infty} p_j(f)$.

Proof. Let $f,g \in K$. The condition (10) implies that $p_k(\Theta_j(f,g)) = 0$ for $k \le j$, hence

$$p_k\Big(\sum_{m=1}^{k} \Theta_m(f,g)\Big) = p_k\Big(\sum_{m=1}^{p} \Theta_m(f,g)\Big)$$

for $p \ge k$ and the series (7) converges.

The product (11) clearly satisfies the conditions 2(a) and (c). To prove 2(b) observe that $p_j(\Theta(f,g))$ is a linear combination of iterated Lie brackets of coordinates either of f or of g, with the sum of relevant indices equal to j. Hence each term of $p_j(\Theta((s*f),(s*g)))$ is the corresponding term of $p_j(\Theta(f,g))$ multiplied by s^j.

Since each coordinate of (7) and of (11) depends on a finite number of coordinates of the involved elements via the operations in L, the products (7) and (11) are jointly continuous.

In the following we shall refer to the groups $\exp K$ for product graded K as *algebraic* $\mathbb{R}$*-groups.*

6. Algebraic $\mathbb{R}$-groups. In this section $K = \prod_{j=1}^{\infty} M_j$ is a product graded Lie algebra and $H = \exp K$ is its B-C-H group. For $h_1, h_2 \in H$ set $\{h_1,h_2\} = h_1^{-1}h_2^{-1}h_1h_2$ and let $H = \overline{H}_1 \supset \overline{H}_2 \supset H_3 \supset \ldots$ be the closed central descending series of H (i.e. $\overline{H}_n$ is the closure of the subgroup generated by all the n-fold commutators). Similarly, let $K = \overline{K}_1 \supset \overline{K}_2 \supset \overline{K}_3 \supset \ldots$ be the closed central descending series of K. Denote by $p_j : K \to M_j$ the coordinate projections, $j = 1,2,\ldots$, and let

$$K^n = \prod_{k=n}^{\infty} M_j, \quad H^n = \exp K^n \quad \text{for } n = 1,2,\ldots \tag{12}$$

Proposition 8. (a) $E_n(H) = \exp M_n$, $n = 1,2,\ldots$

(b) *For each $h \in H^n$ the limit*

$$d_n h = \lim_{k\to\infty} \frac{1}{k} * h^{k^n}$$

exists and $d_n \circ \exp = \exp \circ p_n$. In particular, $d_n : H^n \to E_n(H)$ is a continuous retraction.

(c) *The mapping $\alpha : K \to H$ which assigns to $f = \sum_{n=1}^{\infty} p_n(f)$ the infinite product*

$$\alpha(f) = \lim_{m\to\infty} \prod_{n=1}^{m} \exp p_n(f)$$

is a homeomorphism. Moreover, $\alpha(K^j) = H^j$ for $j = 1,2,\ldots$ (We write products from left to right, e.g. $\prod_{n=1}^{m} a_n = a_1 \cdot a_2 \cdot \ldots \cdot a_m$. The reverse order yields another map with analogous properties.)

Proof. (a) Let $h \in \exp M_n$, i.e. $h = \exp f$ with $p_j(f) = 0$ for $j \ne n$. Then for $k \in \mathbb{Z}$,

$$k * h = \exp(k^n f) = (\exp f)^{k^n} = h^{k^n},$$

i.e. $h \in E_n(H)$. Conversely, if $h \in E_n(H)$ and $h = \exp f$ then for $k \in \mathbb{Z}$,

$$\exp\Big(\sum_{j=1}^{\infty} k^j p_j(f)\Big) = k * \exp f = (\exp f)^{k^n} = \exp(k^n f) = \exp\Big(\sum_{j=1}^{\infty} k^n p_j(f)\Big)$$

and hence $p_j(f) = 0$ for $j \ne n$.

(b) Let $h \in H^n$, i.e. $h = \exp f$ for some $f = \sum_{j=n}^{\infty} p_j(f) \in K^n$. Then

$$d_n h = \lim_{k\to\infty} \frac{1}{k} * (\exp f)^{k^n} = \exp \lim_{k\to\infty} k^n \Big(\sum_{j=n}^{\infty} \frac{1}{k^j} p_j(f)\Big)$$

$$= \exp \lim_{k\to\infty} \sum_{j=n}^{\infty} \frac{1}{k^{j-n}} p_j(f) = \exp p_n(f).$$

(c) Since the nth coordinate of $\prod_{j=1}^{m} \exp p_j(f)$ depends only on factors with indices $\le n$, the limit defining $\alpha(f)$ converges, and $\alpha(f)$ depends continuously on f. Observe also that α has a continuous inverse β,

$$\beta(h) = \sum_{j=1}^{\infty} \exp^{-1} \circ d_j \circ R_{j-1}(h),$$

where

$$\begin{cases} R_0(h) = h, \\ R_j(h) = (d_{j-1}R_{j-1}(h))^{-1} \cdot R_{j-1}(h), \quad j = 1,2,\ldots \end{cases} \tag{13}$$

Corollary 9. *Each $h \in H$ has a unique representation in the form of an infinite product*

$$h = \prod_{j=1}^{\infty} \widetilde{d}_j(h)$$

with $\widetilde{d}_j(h) \in E_j(H)$ depending continuously on h.

Proof. Set $\widetilde{d}_j(h) = d_j R_{j-1}(h)$.

Proposition 10. (a) *H is topologically exponential if M_1 generates K topologically.*

(b) *If H is topologically exponential then $K^n = \overline{K}_n$ and $H^n = \overline{H}_n$ for $n \in \mathbb{N}$.*

Proof. (a) Let K_0 be the Lie subalgebra of K generated by M_1, and let H_0 be the subgroup of H generated by $E_1(H)$. Denote by $\overline{K}_0$ and $\overline{H}_0$ their respective closures. Clearly $H_0 \subset \exp \overline{K}_0$, hence if H_0 is dense, then so is K_0. To prove the converse observe that the density of K_0 implies the density of $\alpha(K_0)$, and hence to prove that H_0 is dense it is sufficient to approximate the elements of $\exp p_j(K_0)$ for $j = 1, 2, \ldots$ by elements of H_0. Note that $p_j(K_0)$ is spanned by elements of the form $y = [x_1, [x_2, \ldots, [x_{j-1}, x_j]] \ldots]$ for $x_1, \ldots, x_j \in M_1$ and it is easy to verify that

$$\exp[x_1, [x_2, \ldots, [x_{j-1}, x_j]] \ldots] = d_j(\{\exp x_1, \{\exp x_2, \ldots, \{\exp x_{j-1}, \exp x_j\}\} \ldots\}).$$

Since $\{\exp x_1, \{\exp x_2, \ldots, \{\exp x_{j-1}, \exp x_j\}\} \ldots\} \in H_0 \cap \exp K^j$ the left hand side is aproximable by H_0. To complete the proof we only need to observe that by (12) the map $\exp^{-1} \circ d_j : H^j \to M_j$ is a homomorphism, thus we can approximate the elements of $p_j(K_0)$ by taking appropriate products in H_0.

(b) By the grading condition (10), $K_n \subset K^n$ and $H_n \subset H^n$. Since both K^n and H^n are closed, also $\overline{K}_n \subset K^n$ and $\overline{H}_n \subset H^n$. On the other hand, since K_0 is linearly spanned by $\bigcup_{j=1}^{\infty}(K_0 \cap M_j)$ and it is dense in K, $K_0 \cap M_j$ is dense in M_j for each j. Simultaneously $K_0 \cap M_j \subset \overline{K}_n$ for $j \geq n$ by (10). It follows that $K^n \subset \overline{K}_n$ for $n = 1, 2, \ldots$

To prove that $\overline{H}_n = H^n$ observe that by Corollary 9, H^n is topologically generated by $\bigcup_{j=n}^{\infty} \exp M_j$, thus it is sufficient to prove that $\exp M_j \subset \overline{H}_n$ for $j \geq n$, and this is done in the same way as in part (a) of this proposition.

COROLLARY 11. *For $n = 1, 2, \ldots$,*

(a) *$\overline{H}_n / \overline{H}_{n+1}$ is in 1-1 correspondence with M_n.*

(b) *If $a_{m,j} = \exp x_{m,j} \cdot r_{m,j}$ with $x_{m,j} \in M_m$ and $r_{m,j} \in \overline{H}_{m+1}$ for $j = 1, 2$, then*

$$(14) \qquad \begin{aligned} a_{m,1} \cdot a_{m,2} &= \exp(x_{m,1} + x_{m,2}) \cdot r_m, \\ \{a_{m,1}, a_{n,2}\} &= \exp[x_{m,1}, x_{n,2}] \cdot r_{m+n} \end{aligned}$$

where $r_k \in \overline{H}_{k+1}$ for $k = m$ and $k = m + n$.

Proof. (a) results from Proposition 10(b). Formulas (14) follow from Proposition 8(c) and the B-C-H formula.

7. The product graded Lie algebra of an exponential $\mathbb{R}$-group. Let H be a topologically exponential $\mathbb{R}$-group. For $j \in \mathbb{N}$ set $M_j = \overline{H}_j / \overline{H}_{j+1}$. Since $\{\overline{H}_k, \overline{H}_m\} \subset \overline{H}_{k+m}$ for $k, m \in \mathbb{N}$, each M_j is abelian. It is known ([2], [12]) that for every topological group H the topological product

$$L(H) = \prod_{j=1}^{\infty} M_j$$

can be endowed with a structure of a topological Lie ring with additive structure given by coordinatewise multiplication and with Lie bracket obtained as the common extension of the family of biadditive maps

$$[\cdot, \cdot]_{k,m} : M_k \times M_m \to M_{k+m}, \qquad k, m \in \mathbb{N},$$

where $[\overline{a}, \overline{b}]_{k,m} = \overline{\{a, b\}}$ and the bar denotes the corresponding quotient class.

In this section we prove that if H is an exponential $\mathbb{R}$-group, then the Lie ring structure on $L(H)$ extends to a real product graded Lie algebra. The proof proceeds in a few steps.

LEMMA 12. *Let H be a topologically exponential $\mathbb{R}$-group. If $h \in \overline{H}_k$ then*

$$(15;k) \qquad \begin{aligned} &\text{(a)} \qquad n * h = h^{n^k} \bmod \overline{H}_{k+1} \quad \textit{for } n \in \mathbb{N}, \\ &\text{(b)} \qquad (-1) * h = h^{-1} \bmod \overline{H}_{k+1} \end{aligned}$$

Proof. Since both sides of (15;k)(a),(b) depend continuously on h, with no loss of generality we may assume that H is exponential and $h \in H_k$. Note also that for $a, b \in H_k$ and $n \in \mathbb{Z}$ one has $a^n b^n = (ab)^n \bmod H_{k+1}$, thus in the proof of (15;k) we may restrict attention to the generating elements of the form

$$h_k = \{a_1, \{a_2, \ldots, \{a_{m-1}, a_m\}\} \ldots\},$$

where $a_i \in E_1(H)$, $m \geq k$ and $\{a, b\} = a^{-1}b^{-1}ab$ for $a, b \in H$.

Proceeding by induction observe first that $h_1 \in E_1(H)$, and hence (15; k)(a),(b) hold. Let now $h_k = \{a_1, h_{k-1}\}$ where by the induction hypothesis $n * h_{k-1} = h_{k-1}^{n^{k-1}} r_{k,n}$ for some $r_{k,n} \in H_k$. Since for $a, b, c \in H$ such that at least one of the three belongs to H_{k-1} we have

$$\{a, bc\} = \{a, b\}\{a, c\} \bmod H_{k+1},$$

and in particular $\{a, b^n\} = \{a, b\}^n \bmod H_{k+1}$, it follows that $n * h_k = \{n * a_1, n * h_{k-1}\} = \{a_1^n, h_{k-1}^{n^{k-1}} r_k\} = \{a_1, h_{k-1}^{n^{k-1}}\}^n = h_k^{n^k} \bmod H_k$.

The proof of (15;k)(b) is similar.

Lemma 12 implies that the quotient $\mathbb{R}$-product on M_k satisfies

$$(16;k) \qquad n * a = a^{\operatorname{sgn}(n) \cdot |n|^k} \quad \text{for } a \in M_k \text{ and } n \in \mathbb{Z}.$$

Thus introducing on each M_k a new $\mathbb{R}$-product $\mathbb{R} \times M_k \ni (s, a) \to s \cdot a \in M_k$ by

$$(17) \qquad s \cdot a = (\sqrt[k]{|s|}) * a^{\operatorname{sgn} s} \quad \text{for } s \in \mathbb{R} \text{ and } a \in M_k$$

we obtain

$$(18;k) \qquad m \cdot a = a^m, \quad m = \operatorname{sgn}(n) \cdot |n|^k \quad \text{with } n \in \mathbb{Z}.$$

LEMMA 13. *If for each a in an $\mathbb{R}$-group K there exists $k \in \mathbb{N}$ such that* $(18;k)$ *holds then K is a topological vector space.*

Proof. Suppose $a \in K$ satisfies $(18;k)$ and let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be sequences of positive integers such that

$$\lim_{n\to\infty} q_n^{-1} p_n = 0.$$

Then

$$\lim_{n\to\infty} q_n^{-1} \cdot a^{p_n} = e, \tag{19}$$

$$\lim_{n\to\infty} q_n^{-1} \cdot a^{q_n} = a. \tag{20}$$

To prove (19) note that by the Waring theorem (cf. [7]) for each $k \in \mathbb{N}$ there exists $s \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ we can choose $r_{n,1}, r_{n,2}, \ldots, r_{n,s} \in \mathbb{N} \cup \{0\}$ such that

$$p_n = r_{n,1}^k + r_{n,2}^k + \ldots + r_{n,s}^k.$$

Thus

$$q_n^{-1} \cdot a^{p_n} = q_n^{-1} \cdot (a^{r_{n,1}^k} \cdot a^{r_{n,2}^k} \cdot \ldots \cdot a^{r_{n,s}^k}) = \prod_{i=1}^{s} (q_n^{-1} \cdot r_{n,i}^k) \cdot a.$$

Clearly $\lim_{n\to\infty} q_n^{-1} \cdot r_{n,i}^k = 0$ for $i = 1, \ldots, s$, hence each factor of the right hand side tends to e.

To prove (20) note that $q_n \to \infty$ and choose positive integers r_n, $n = 1, 2, \ldots$, satisfying

$$r_n^k \le q_n < (r_n + 1)^k, \quad n = 1, 2, 3, \ldots$$

Then

$$\lim_{n\to\infty} q_n^{-1}(r_n+1)^k = 1 \quad \text{and} \quad \lim_{n\to\infty} q_n^{-1}((r_n+1)^k - q_n) = 0.$$

Hence by (19),

$$\begin{aligned}\lim_{n\to\infty} q_n^{-1} \cdot a^{q_n} &= (\lim_{n\to\infty} q_n^{-1} \cdot a^{q_n})(\lim_{n\to\infty} q_n^{-1} \cdot a^{(r_n+1)^k - q_n}) \\ &= \lim_{n\to\infty} q_n^{-1} \cdot a^{(r_n+1)^k} = \lim_{n\to\infty} q_n^{-1}(r_n+1)^k \cdot a = a.\end{aligned}$$

This proves (20).

Let now $m \in \mathbb{N}$ and $a \in K$. Then by (20),

$$\begin{aligned} m \cdot a &= m \cdot (\lim_{n\to\infty} (nm)^{-1} \cdot a^{nm}) = \lim_{n\to\infty} n^{-1} \cdot a^{nm} \\ &= (\lim_{n\to\infty} n^{-1} \cdot a^n)^m = a^m. \end{aligned}$$

Hence by Proposition 5, K is a topological vector space.

THEOREM 14. *Let H be an exponential $\mathbb{R}$-group. The topological Lie ring structure on $L(H)$ can be compatibly completed by the $\mathbb{R}$-product given coordinatewise by* (17), *yielding a real product graded Lie algebra.*

Proof. The product given coordinatewise by (17) is compatible with the additive structure of $L(H)$. Since

$$na = \underbrace{a + \ldots + a}_{n \text{ summands}} \quad \text{for } n \in \mathbb{N} \text{ and } a \in L(H),$$

the biadditivity of the bracket implies $[n \cdot a, b] = n \cdot [a, b]$. It follows that $[\lambda \cdot a, b] = \lambda \cdot [a, b]$ for λ rational, and by continuity this extends to all real λ. Antisymmetry of the bracket yields homogeneity also in the second argument.

8. C^∞ and analytic $\mathbb{R}$-groups. The Lie algebra $L(H)$ is usually too poor to determine H (this is e.g. the case when $\overline{H}_k = \overline{H}_{k+1} \neq \{e\}$ for some $k \in \mathbb{N}$). In this section we shall impose conditions on H which exclude such situations and enable us to relate $L(H)$ to $\Lambda(H)$. Consider first the case when the situation is the best possible.

PROPOSITION 15. *Let $K = \prod_{j=1}^{\infty} M_j$ be a product graded Lie algebra and let $H = \exp K$. Then $L(H) = K$. In particular, $L(H)$ is isomorphic to $\Lambda(H)$ equipped with the natural Lie algebra structure.*

Proof. Let $L(H) = \prod_{j=1}^{\infty} \overline{M}_j$ with $\overline{M}_j = \overline{H}_j / \overline{H}_{j+1}$. By Corollary 11(a), M_j is in 1-1 correspondence with $\overline{M}_j$, $j = 1, 2, \ldots$, thus so are K and $L(H)$, and by Corollary 11(b) the Lie ring structures of K and $L(H)$ coincide. By (11) and (17) the same holds for multiplication by real numbers.

The second statement results from Remark 3.

DEFINITION 16. Let H be a topologically exponential $\mathbb{R}$-group. H is said to be a *C^∞ $\mathbb{R}$-group* if for each $k \in \mathbb{N}$ there exists a continuous map $\exp : M_k \to E_k(H)$ such that $\mathrm{id} \circ \exp : M_k \to \overline{H}_k$ splits the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{H}_{k+1} & \xrightarrow{\mathrm{id}} & \overline{H}_k & \xrightarrow{\pi_k} & M_k & \longrightarrow & 0 \\ & & & & \uparrow \mathrm{id} & \swarrow \exp & & & \\ & & & & E_k(H) & & & & \end{array}$$

Moreover, H is said to satisfy the *analyticity condition* if

$$\bigcap_{k=1}^{\infty} \overline{H}_k = \{e\}; \tag{21}$$

and H is *analytic* if it is C^∞ and satisfies (21).

The next proposition and the following examples justify the above terminology.

PROPOSITION 17. *Let H be a topologically exponential $\mathbb{R}$-group. Consider the limits $(6;k)$, $k=0,1,2,\ldots$, and assume (inductively) that:*

(a) *The limit $(6;k)$ exists for $f\in\ker d_{k-1}$.*
(b) *The map $d_k:\ker d_{k-1}\to H$ is continuous.*
(c) *d_k satisfies the condition*

$$d_k(f\cdot g)=d_k(d_kf\cdot d_kg), \tag{22;k}$$

$k=1,2,\ldots$, provided the relevant limits are well defined.

Then H is a C^∞ $\mathbb{R}$-group. If, moreover,

$$\bigcap_{k=1}^{\infty}\ker d_k=\{e\}, \tag{23}$$

then H is analytic.

Proof. A not difficult induction shows that $E_k(H)\subset\ker d_{k-1}$. The form of the right hand side of $(6;k)$ implies that $d_kf\in E_k(H)$ and that $d_k:\ker d_{k-1}\to E_k(H)$ is a retraction for $k=1,2,\ldots$; moreover,

$$d_k(gfg^{-1})=d_k(f)\quad\text{ if } d_k(f) \text{ exists.} \tag{24}$$

Defining (inductively) the binary operation $f+g:=d_k(f\cdot g)$ on $E_k(H)$, $k=1,2,\ldots$, one obtains, by $(22;k)$, an abelian group multiplication such that $d_k:\ker d_{k-1}\to E_k(H)$ is a homomorphism. Thus $\ker d_k$ is a subgroup of $\ker d_{k-1}$ and both sides of $(22;k+1)$ are defined on the whole of $\ker d_k$. From (24) it now follows that $\ker d_k\supset H_{k+1}$ and by continuity $\ker d_k\supset\overline{H}_{k+1}$, $k=1,2,\ldots$

On the other hand, $(15;k)$ yields for $h\in\overline{H}_k$, $k=1,2,\ldots$,

$$h=\left(\frac{1}{n}*h^{n^k}\right)\left(\frac{1}{n}*\Delta_k\right)\quad\text{ with } \Delta_k\in\overline{H}_{k+1}.$$

Let $k=1$. Letting $n\to\infty$, we get

$$h=d_1h\cdot\Delta_1h\quad\text{ with } \Delta_1\in\overline{H}_2.$$

Thus

$$H=E_1(H)\cdot\overline{H}_2\quad\text{ and }\quad\ker d_1\subset\overline{H}_2.$$

Hence $\ker d_1=\overline{H}_2$ and since $\ker d_1\cap E_1(H)=\{e\}$, also $E_1(H)\cap\overline{H}_2=\{e\}$. Proceeding by induction and assuming $\ker d_{k-1}=\overline{H}_k$ we similarly get for $k=1,2,\ldots$,

$$\ker d_k=\overline{H}_{k+1}, \tag{25;k}$$
$$\overline{H}_k=E_k(H)\cdot\overline{H}_{k+1}, \tag{26;k}$$
$$E_k(H)\cap\overline{H}_{k+1}=\{e\}. \tag{27;k}$$

Thus $d_k:\overline{H}_k\to E_k(H)$ is a homomorphism with $\ker d_k=\overline{H}_{k+1}$ and it defines the quotient homomorphism $\exp: M_k\to E_k(H)$, which by $(26;k)$ and $(27;k)$ splits the diagram $(20;k)$.

Note also that by $(25;k)$ the condition (21) takes the form (23).

EXAMPLE 18. Let G be a Banach–Lie group with Lie algebra g. Similarly to $C_0(\mathbb{R},G)$ one can consider the group $C_0^\infty(\mathbb{R},G)$ composed of all C^∞ functions from $\mathbb{R}$ to G such that $f(0)=e$. Clearly $C_0^\infty(\mathbb{R},G)$ is an $\mathbb{R}$-group containing $P(G)$ as an $\mathbb{R}$-subgroup. Let $P_0^\infty(G)$ be the C^∞ closure of $P_0(G)$ in $P(G)$. Introducing exponential coordinates one can assign to each $f\in P(G)$ the function $\widehat{f}:\mathbb{R}\to g$ where $\widehat{f}=\exp^{-1}\circ f$ (more precisely, $\widehat{f}$ is only a germ at 0). By the property

$$\exp(nX)=(\exp X)^n$$

of the exponential map, noting that $\widehat{f}(0)=0$ we get

$$\begin{aligned}\exp^{-1}(d_kf(t))&=\exp^{-1}\lim_{n\to\infty}f\left(\frac{t}{n}\right)^{n^k}\\&=\lim_{n\to\infty}n^k\widehat{f}\left(\frac{t}{n}\right)=t^k\lim_{n\to\infty}\frac{\widehat{f}(t/n)-\widehat{f}(0)}{(t/n)^k},\end{aligned}$$

hence d_kf exists if $\widehat{f}$ is k times differentiable at 0 with $(d^j\widehat{f}/dt^j)(0)=0$ for $j<k$, and then

$$d_kf=\exp\left[k!\left(\frac{d^k\widehat{f}}{dt^k}(0)\right)\cdot t^k\right].$$

Moreover, $f(t/n)^{n^k}$ then converges to d_kf also in C^∞ topology. It follows that for $P_0^\infty(G)$ the operations d_k do exist on proper domains, and they satisfy the conditions $(22;k)$ and are continuous. Since $\widehat{f}$ is analytic for $f\in P(G)$, the condition (23) is also satisfied. Summing up, $P_0^\infty(G)$ with the C^∞ topology is an analytic $\mathbb{R}$-group.

Observe that d_n's are not necessarily continuous for $P_0^\infty(G)$ considered in the compact-open topology. The following example is due to J. Grabowski to whom the author is grateful for his permission to include it here.

EXAMPLE 19. Consider $H=P_0^\infty(G)$ for G the connected 2-dimensional noncommutative Lie group of all 2×2 real matrices of the form

$$\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix},$$

where $a>0$. By the previous example and Proposition 17, if d_1 were continuous we would have $E_1(H)\cap\overline{H}_2=\{e\}$. To see that this is not the case for the compact-open topology, take basis elements X and Y of the Lie algebra

g of G of the form

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and consider the corresponding one-parameter groups $\phi_X(t) = e^{tX}$ and $\phi_Y(t) = e^{tY}$.

A standard computation shows that

$$\{\phi_X, \phi_{\beta Y}\} = e^{t\beta(e^t-1)Y}, \quad \text{i.e.} \quad \{\phi_{\alpha X}, \phi_{\beta Y}\}(t) = \begin{pmatrix} 1 & \beta(e^t-1)t \\ 0 & 1 \end{pmatrix}.$$

It follows that H_2 contains the family Γ of all functions of the form

$$f(t) = \begin{pmatrix} 1 & t\sum_{k=1}^n \beta_k(e^t-1)^k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & tW_n(e^t-1) \\ 0 & 1 \end{pmatrix},$$

where W_n is an arbitrary polynomial satisfying $W_n(0) = 0$. By the Stone–Weierstrass theorem the function $\phi_Y \in E_1(H)$ belongs to the almost uniform closure of Γ. Note also that all the functions from Γ satisfy $f'(0) = 0$. Since $\phi'_Y(t) = Y \neq 0$ the almost uniform convergence does not imply here the convergence of the corresponding derivatives at 0.

Remark 20. Essentially the same arguments as in Example 18 prove that for a Fréchet–Lie group G admitting an exponential chart at unity, the group $P_0^\infty(G)$ equipped with the C^∞ topology is a C^∞ $\mathbb{R}$-group. This applies e.g. to some "current groups", i.e. groups of all functions of a given smoothness class (C^k or C^∞) from a compact manifold M into a Lie group G.

EXAMPLE 21. Let G be the B-C-H group associated with a product graded Lie algebra L (this type essentially includes the groups of invertible elements of formal power series algebras). We shall indicate how $P(G)$ can be given the natural C^∞ topology.

Observe first that according to Remark 3(a) the map $\exp^{-1} : G \to L$ provides a global exponential chart on G, and each f in $P(G)$ has the form of a finite pointwise B-C-H product with factors of the form $\phi(t) = \exp(t^k x)$ with $k \in \mathbb{N}$ and $x \in L$. Therefore

$$f(t) = \exp\Big(\sum_{j=1}^{\infty} a_j t^j\Big),$$

where $a_j \in L^{m_j}$ and $m_j \to \infty$ as $j \to \infty$. It follows that the series converges in L and the function $\widehat{f} : \mathbb{R} \to L$ where $\widehat{f} = \exp^{-1} \circ f$ is analytic (in particular, it is C^∞) with derivatives of the form

$$\widehat{f}^{(n)} = \sum_{j=0}^{\infty} (n+j)(n-1+j)\ldots(1+j)a_{n+j}t^j.$$

We endow $P(G)$ with the topology of almost uniform convergence with all derivatives of the representing functions in the global chart. In the same way as in Example 18, this topology can be proved to be C^∞. (Clearly $P(G)$ also satisfies the analyticity condition.)

9. Exponential map and induced homomorphisms. Let H be a C^∞ $\mathbb{R}$-group. Put $M(H) = \bigcup_{k=1}^\infty M_k \subset L(H)$ and consider the continuous map $\pi : E(H) \to M(H)$ where $\pi|_{E_k(H)} = \pi_k \circ \mathrm{id}$. By Definition 16, H is C^∞ iff π is a homeomorphism. Let $\exp = \pi^{-1} : M(H) \to H$ be the inverse map.

PROPOSITION 22. *For each $\overline{x} \in M(H)$ the function $f_{\overline{x}}(t) = \exp t\overline{x}$ belongs to $\Lambda(H)$.*

Proof. Let $\overline{x} \in M_k$ for $k \in \mathbb{N}$. By (20; k), $\overline{x} = x \cdot \overline{H}_{k+1}$ for some $x \in E_k(H)$ and by (17), $t\overline{x} = (\sqrt[k]{|t|} * x^{\operatorname{sgn} t}) \cdot \overline{H}_{k+1}$. Thus

$$f_{\overline{x}}(t) = \exp(t\overline{x}) = \sqrt[k]{|t|} * x^{\operatorname{sgn} t}. \tag{28}$$

This implies that $f_{\overline{x}}(-t) = f_{\overline{x}}(t)^{-1}$ and since $x \in E_k(H)$, for $m_1, m_2, q \in \mathbb{N}$ one gets

$$f_{\overline{x}}\Big(\frac{m_1}{q} + \frac{m_2}{q}\Big) = \frac{1}{\sqrt[k]{|q|}} * x^{m_1+m_2} = \Big(\frac{1}{\sqrt[k]{|q|}} * x^{m_1}\Big)\Big(\frac{1}{\sqrt[k]{|q|}} * x^{m_2}\Big) = f_{\overline{x}}\Big(\frac{m_1}{q}\Big) \cdot f_{\overline{x}}\Big(\frac{m_2}{q}\Big).$$

Since f_x is continuous this means that $f_{\overline{x}} \in \Lambda(H)$.

THEOREM 23. *Let P be a C^∞ polynomial $\mathbb{R}$-group and H be an analytic $\mathbb{R}$-group. Let $L(P) = \prod_{n=1}^\infty M_n$ and $L(H) = \prod_{n=1}^\infty \widetilde{M}_n$ be the corresponding product graded Lie algebras. For $M = \bigcup_{n=1}^\infty M_n$ and $\widetilde{M} = \bigcup_{n=1}^\infty \widetilde{M}_n$ let $\exp_P : M \to P$ and $\exp_H : \widetilde{M} \to H$ be the corresponding exponential maps. Then:*

(a) *For each graded Lie algebra homomorphism $\phi : L(P) \to L(H)$ there exists a unique $\mathbb{R}$-homomorphism $\Phi : P \to H$ such that*

$$\Phi \circ \exp_P = \exp_H \circ \phi. \tag{29}$$

(b) *The homomorphism ϕ is induced by Φ, i.e. for $n \in \mathbb{N}$,*

$$\phi \circ \pi_n = \varrho_n \circ \Phi, \tag{30}$$

where $\pi_n : \overline{P}_n \to M_n$ and $\varrho_n : \overline{H}_n \to \widetilde{M}_n$ are the quotient homomorphisms.

(c) *If ϕ is injective then $\ker \Phi \subset \bigcap_{n=1}^\infty \overline{P}_n$. If ϕ is surjective and H is polynomial then Φ is also surjective.*

The proof involves a few auxiliary constructions. Since M is the union of linear subspaces of $L(P)$, it can be endowed with the $\mathbb{R}$-product restricted from $L(P)$. For $x \in M$ define $|x|$ by setting $|x| = k$ if $x \in M_k(P)$. Let J be a pure $\mathbb{R}$-group (meaning that J has no topology and an $\mathbb{R}$-product is defined on J satisfying the conditions (2)(a)–(c)). A map $\alpha : M \to J$ will be called $\mathbb{R}$-*graded exponential* if for each $x \in M$,

$$(31)\quad \begin{aligned} &\text{(a)} && \alpha(t^{|x|}x) = t * \alpha(x) \quad \text{for } t \in \mathbb{R},\\ &\text{(b)} && \mathbb{R} \ni t \to \alpha(tx) \in J \text{ is a homomorphism.} \end{aligned}$$

PROPOSITION 24. *There exist a pure $\mathbb{R}$-group F and an $\mathbb{R}$-graded exponential map $j : M \to F$ such that:*

(i) *$j(M)$ generates F.*

(ii) *For each $\mathbb{R}$-graded exponential map $\alpha : M \to J$ there exists an $\mathbb{R}$-group homomorphism $\beta : F \to J$ such that $\alpha = \beta \circ j$.*

Proof. The proof is standard and we only sketch it. Let F_0 be the free group over $M\backslash\{0\}$. Let $j_0 : M \to F_0$ be the extension of the canonical embedding of $M\backslash\{0\}$ obtained by assigning to 0 the unit of F_0. Introducing an $\mathbb{R}$-product on F_0 by the formula

$$\lambda * (j_0(x_1) \ldots j_0(x_n)) = j_0(\lambda^{|x_1|}x_1) \ldots j_0(\lambda^{|x_n|}x_n)$$

we get a pure $\mathbb{R}$-group with j_0 satisfying (31)(a). Let I be the ideal of F_0 generated by all the elements of the form $j_0(\lambda x) j_0(\mu x) j_0((\lambda + \mu)x)^{-1}$ with $x \in M$. Put $F = F_0/I$ and let $j = p \circ j_0$ where $p : F_0 \to F$ is the quotient homomorphism.

Let X be a linear space over $\mathbb{R}$. Given a pure $\mathbb{R}$-group A and a positive integer k, a homomorphism $\gamma : A \to X$ is said to be a *k-homomorphism* if $\gamma(t * a) = t^k \gamma(a)$ for each $a \in A$ and $t \in \mathbb{R}$.

LEMMA 25. *Let A be an $\mathbb{R}$-subgroup of F. For $k \in \mathbb{N}$ let Q_k denote the $\mathbb{R}$-subgroup of F generated by all elements of the form*

$$(32)\quad c = \{j(x_1), \{j(x_2), \ldots, \{j(x_{i-1}), j(x_i)\}\} \ldots\} \quad \text{where} \quad \sum_{m=1}^{i} |x_m| \ge k.$$

For each $a \in A$ there exist $b \in A \cap Q_k$ such that $\gamma(a) = \gamma(b)$ for each k-homomorphism γ defined on A.

Proof. Let $a \in A$. Define recurrently

$$(33)\qquad a_1 = a, \quad a_i = (2 * a_{i-1}) a_{i-1}^{-2^{i-1}} \quad \text{for } i = 2, 3, \ldots$$

We claim that $a_j \in A \cap Q_j$. Clearly $a_j \in A$ for each j. We shall prove that for $c \in Q_{j-1}$,

$$(34; j)\qquad 2 * c = c^{2^{j-1}} \bmod Q_j.$$

Since $c_1^n c_2^n = (c_1 c_2)^n \bmod Q_j$ for $c_1, c_2 \in Q_{j-1}$ and since each $c \in Q_{j-1}$ is a product of elements of the form (32) with $k = j - 1$, in the proof of $(34; j)$ we may restrict ourselves to elements of that form. Proceeding by induction assume first that $j - 1 = 1$ and consider a single element $j(x)$ with $x \in M(H)$. Then by (31)(a),(b),

$$2 * j(x) = j(2^{|x|} \cdot x) = j(x)^{2^{|x|}}.$$

Since $j(x)^{2^{|x|-1}}$ always belongs to Q_2 we see that $2 * j(x) = j(x)^2 \bmod Q_2$. Assume now that c is of the form (32) with $k = j - 1$ and note that if $\sum_{m=1}^{i} |x_m| > j - 1$ then both sides of (34) are in Q_j so with no loss of generality we may assume that $\sum_{m=1}^{i} |x_m| = j - 1$. Then $c = \{j(x_i), d\}$ where $d \in Q_k$ with $k = j - 1 - |x_i|$. By the induction assumption $2 * d = d^{2^k} \cdot f$ where $f \in Q_{k+1}$. Then by similar arguments to the proof of Lemma 12 we get

$$\begin{aligned} 2 * c = \{2 * j(x_i), 2 * d\} &= \{j(x_1)^{2^{|x_i|}}, d^{2^k} \cdot f\} = \{j(x_1)^{2^{|x_1|}}, d^{2^k}\} \\ &= \{j(x_1), d\}^{2^k \cdot 2^{|x_1|}} = c^{2^{j-1}} \bmod Q_j. \end{aligned}$$

This proves that $a_j \in Q_j$.

To finish the proof observe that $\gamma(a_j) = (2^k - 2^{j-1})\gamma(a_{j-1})$ for $j = 1, 2, \ldots$ It follows that $b = \lambda^{-1/k} a_k$ with $\lambda = \prod_{j=1}^{k-1}(2^k - 2^j)$ has the needed properties.

Proof of Theorem 23. Since $\exp(M(P))$ generates P, the homomorphism $\Phi : P \to H$ satisfying (29) is unique if it exists. To construct such a Φ consider the maps $\alpha_1 : M \to P$ and $\alpha_2 : M \to H$ where $\alpha_1 = \exp_P$ and $\alpha_2 = \exp_H \circ \phi$. By Proposition 22, α_1 and α_2 are $\mathbb{R}$-graded exponential maps. Let $\beta_1 : F \to P$ and $\beta_2 : F \to H$ be $\mathbb{R}$-homomorphisms such that $\alpha_i = \beta_i \circ j$ for $i = 1, 2$ (Proposition 24). The central part of the proof is contained in the following statement:

LEMMA 26. *Let $\pi_0 = \varrho_0 = 0$. For each $a \in F$ and $n \in \mathbb{N}$,*

$$(35; n)\qquad (\pi_{n-1} \circ \beta_1(a) = 0) \Rightarrow (\varrho_{n-1} \circ \beta_2(a) = 0),$$

$$(36; n)\qquad (\pi_{n-1} \circ \beta_1(a) = 0) \Rightarrow (\phi \circ \pi_n \circ \beta_1(a) = \varrho_n \circ \beta_2(a)).$$

Assuming that Lemma 26 holds we first conclude the proof of Theorem 23. Observe that Lemma 26 yields $\ker \beta_1 \subset \ker \beta_2$. Indeed, if $a \in \ker \beta_1$ then $\pi_n \circ \beta_1(a) = 0$ for each $n \in \mathbb{N}$, hence $\beta_2(a) \in \bigcap_{n=1}^{\infty} \overline{H}_n = \{e\}$ by $(35; n)$. Thus the formula $\Phi(\beta_1(a)) = \beta_2(a)$ defines an $\mathbb{R}$-homomorphism $\Phi : P \to H$ satisfying (29) and such that $\Phi(P_n) \subset H_n$. Let $\widetilde{\phi} : L(P) \to L(H)$ be defined by (30). Then by (20; k), for $x \in M_n$ and $p = \exp_p x = \beta_1 j(x)$,

$$\widetilde{\phi}(x) = \widetilde{\phi} \circ \pi_n(p) = \varrho_n \circ \Phi(p) = \varrho_n \circ \Phi(\exp_p x) = \varrho_n \beta_2 j(x) = \phi \pi_n \beta_1 j(x) = \phi \pi_n \exp(x) = \phi(x).$$

This implies that $\phi = \widetilde{\phi}$ and so ϕ is induced by Φ. Note also that if ϕ is injective then (36; n) for $n = 1, 2, \ldots$ yields the implication

$$(\varrho_n \beta_2(a) = 0) \Rightarrow (\pi_n \beta_1(a) = 0),$$

hence $\ker \Phi \subset \bigcap_{n=1}^{\infty} \overline{P}_n$. Surjectivity of ϕ implies that $\phi(M(P)) = M(H)$ and since $\exp M(H)$ generates H, (29) implies that Φ is surjective.

Proof of Lemma 26. The implication (35; 1) is obvious. To prove (36; 1) consider $a \in F$ of the form $a = j(x_1) \cdot \ldots \cdot j(x_k)$ with $x_i \in M_{|x_i|}$. Then

$$\beta_1(a) = \exp(x_1) \ldots \exp(x_n) \quad \text{and} \quad \phi \circ \pi_1 \circ \beta_1(a) = \phi\Big(\sum_{|x_i|=1} x_i\Big).$$

Similarly

$$\beta_2(a) = \exp \phi(x_1) \ldots \exp \phi(x_n) \quad \text{and} \quad \varrho_1 \circ \beta_2(a) = \sum_{|x_i|=1} \phi x_i.$$

Since ϕ is a homomorphism the implication (36; 1) holds.

Assume now that (35; n) and (36; n) hold for $n \le k$ and let $\pi_k \circ \beta_1(a) = 0$. Then $\pi_{k-1}\beta_1(a) = 0$, hence (36; k) applies with $\pi_k \circ \beta_1(a) = 0$, yielding $\varrho_k \circ \beta_2(a) = 0$, i.e. (35, $k+1$) holds. To prove (36; $k+1$) put $A = \beta_1^{-1}(\overline{P}_{k+1})$. Clearly $Q_{k+1} \subset A$. Consider first the case when $a \in Q_{k+1}$. Then a is represented as a finite product of factors each of the form (32) with $\sum_{m=1}^{i} |x_m| \ge k+1$. Since β_1, β_2, ϕ, ϱ_k and π_k are homomorphisms, when proving (36; $k+1$) we may restrict ourselves to a single element c of this form. Then

$$\phi \circ \pi_{k+1} \circ \beta_1(c) = \begin{cases} 0 & \text{if } \sum_{m=1}^{i} |x_m| > k+1, \\ \phi([x_1, [x_2, \ldots, [x_{i-1}, x_i]] \ldots]) & \text{if } \sum_{m=1}^{i} |x_m| = k+1. \end{cases}$$

Similarly,

$$\varrho_k \circ \beta_2(c) = \varrho_k(\{\exp \phi(x_1), \{\exp \phi(x_2), \ldots, \{\exp x_i, \exp x_{i+1}\}\} \ldots\}) = \begin{cases} 0 & \text{if } \sum_{m=1}^{i} |x_m| > k+1, \\ [\phi(x_1), [\phi(x_2), \ldots, [\phi(x_i), \phi(x_{i+1})]] \ldots] & \text{if } \sum_{m=1}^{i} |x_m| = k+1. \end{cases}$$

This gives (36; $k+1$) for $a \in Q_{k+1}$.

Passing to the general case, observe that by (17) both $\phi \circ \pi_{k+1} \circ \beta_1$ and $\varrho_{k+1} \circ \beta_2$ are $(k+1)$-homomorphisms defined on A and with values in N_{k+1}. Thus by Lemma 25 for each $a \in A$ there exists $b \in Q_{k+1}$ such that $\phi \circ \pi_{k+1} \circ \beta_1(a) = \phi \circ \pi_{k+1} \circ \beta_1(b)$ and $\varrho_{k+1} \circ \beta_2(a) = \varrho_{k+1} \circ \beta_2(b)$. This reduces the general case to the one just considered.

10. Embedding theorem. The conclusion of Theorem 23 is purely algebraic and so it is too weak for some applications. The situation improves for some particular ϕ, e.g. for ϕ the identity isomorphism.

PROPOSITION 27. *Let P be a C^∞ $\mathbb{R}$-group with the corresponding product graded Lie algebra $L(P)$. Let P' be the dense polynomial $\mathbb{R}$-subgroup of P generated by $E(P)$. Then $L(P) = L(P')$.*

Proof. Clearly P' is a C^∞ $\mathbb{R}$-group and $E_n(P') = E_n(P)$. Since $\overline{P'_n} = P' \cap \overline{P}_n$ for each n, the map $j_n : \overline{P'_n}/\overline{P'_{n+1}} \to \overline{P}_n/\overline{P}_{n+1}$ is well defined and by (27) it is an isomorphism of linear spaces. The direct product J of j_n's provides the desired product graded isomorphism.

THEOREM 28. *Let P be a C^∞ $\mathbb{R}$-group with the product graded Lie algebra $L(P)$. Let $H = \exp L(P)$. There exists a continuous $\mathbb{R}$-homomorphism $\Phi : P \to H$ with $\ker \Phi = \bigcap_{n=1}^{\infty} \overline{P}_n$. In particular, each analytic $\mathbb{R}$-group embeds continuously in a B-C-H group.*

Proof. Let P' be a dense polynomial subgroup of P (cf. Proposition 27). Then $L(P) = L(P') = L(H)$. Thus for ϕ being the identity map, Theorem 23 provides an $\mathbb{R}$-homomorphism $\Phi_0 : P' \to H$ such that $\Phi_0 \circ \exp'_P(x) = \exp_H \circ \phi(x)$ for $x \in M(P)$. We shall prove that Φ_0 extends to a continuous homomorphism defined on the whole of P. For this purpose consider the mapping $\Phi : P \to H$ given by

$$\Phi(p) = \alpha\Big(\sum_{j=1}^{\infty} \phi \circ \pi_j \circ R^P_{j-1}(p)\Big) \tag{37}$$

where $\alpha : L(H) \to H$ is the homeomorphism introduced in Proposition 8(c) and R^P_j for $j = 0, 1, 2 \ldots$ is defined recurrently by

$$R^P_0(p) = p, \quad R^P_j(p) = (d^P_j \circ R^P_{j-1}(p))^{-1} \cdot R^P_{j-1}(p) \quad \text{for } j = 2, 3, \ldots, \tag{38}$$

where $d^P_j = \exp \circ \pi_j : \overline{P}_j \to E_j(P)$ is a continuous retraction. Since $\exp : M(P) \to E(P)$ and α are homeomorphisms, Φ is continuous. We claim that $\Phi(p) = \Phi_0(p)$ for $p \in P'$. By Corollary 9 this is equivalent to

$$\phi \circ \pi_j \circ R^P_{j-1}(p) = \varrho_j \circ R^H_{j-1} \circ \Phi_0(p), \quad j = 1, 2, \ldots, \tag{39}$$

where the functions R^H_j are defined in (13) and $\varrho_j = \exp_H^{-1} \circ d^H_j$. To prove (39) observe that ϕ and Φ_0 satisfy (29) and (30) and thus

$$\Phi_0 \circ d^P_j = \Phi_0 \circ \exp_P \circ \pi_j = \exp_H \circ \phi \circ \pi_j = \exp_H \circ \varrho_j \circ \Phi_0 = d^H_j \circ \Phi_0.$$

Hence the recurrent formulas (12) and (38) yield

$$\Phi_0 \circ R^P_j = R^H_j \circ \Phi_0 \quad \text{for } j = 0, 1, 2, \ldots$$

and therefore

$$\phi \circ \pi_j \circ R^P_{j-1}(p) = \varrho_j \circ \Phi_0 \circ R^P_{j-1}(p) = \varrho_j \circ R^H_{j-1} \circ \Phi_0(p)$$

for $p \in P' \subset P$. This implies that Φ extends Φ_0. Since Φ_0 is a homomorphism defined on a dense subgroup, Φ is also a homomorphism.

To conclude the proof observe that since α and ϕ are bijective, (37) and (38) yield the equivalences

$$(p \in \ker \Phi) \equiv (\pi_j \circ R^H_{j-1} = 0,\ j = 1, 2, \ldots) \equiv \Big(p \in \bigcap_{j=1}^{\infty} \overline{P}_j\Big).$$

11. Comments. The aim of this paper is to provide a step towards understanding mutual relations of ("Lie") groups and Lie algebras. It is our belief that to achieve this one should complete the scene with a third type of objects—special type $\mathbb{R}$-groups. Such a group, say H, is to be generated topologically by its subset $E(H)$ and should embed continuously in the corresponding B-C-H group $\exp L(H)$. The "Lie group", say G, appears in this scheme as related to H by the quotient homomorphism $\pi : H \to G \simeq H/\Gamma$ (where Γ is a closed subgroup of H) in such a way that the induced map $\widehat{\pi} : \Lambda(H) \to \Lambda(G)$ (as in Remark 1) is surjective. (For commutative G this relation has been studied in [4].) Observe also that if H is obtained as $P(G)$ for some Banach–Lie G then $L(H)$ is isomorphic to the Lie algebra of all formal power series of one variable, with coefficients in g, the Lie algebra of G. Generalization of this observation should provide us with the third object, the Lie algebra g corresponding to G. The relations among all three members of the triplet (G, H, g) constitute in our opinion a Lie group theory formulated in the general algebraic-topological setting. We intend to provide more arguments justifying this scheme in a forthcoming paper.

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