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Spectral multipliers for a distinguished Laplacian on certain groups of exponential growth

by

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Dedicated to the memory of our friend Mario Raimondo

Abstract. We prove that on Iwasawa AN groups coming from arbitrary semisimple Lie groups there is a Laplacian with a nonholomorphic functional calculus, not only for $L^1(AN)$, but also for $L^p(AN)$, where $1 < p < \infty$. This yields a spectral multiplier theorem analogous to the ones known for sublaplacians on stratified groups.

0. Introduction. In this paper, we consider two Laplacian-like operators, both denoted by Δ here:

(i) Let G be a connected Lie group, and $\check{X}_j$, $j = 1, \dots, J$, be *right*-invariant vector fields on G, which generate the Lie algebra of G. Then $-\sum_{j=1}^{J} \check{X}_j^2$ is known as a sublaplacian on G.

(ii) Suppose that M is a complete Riemannian manifold; then there is a canonical second-order differential operator on M known as the Laplace–Beltrami operator. We let $\mathcal{L}_0$ be minus this operator to make it formally nonnegative. The L^2-spectrum of $\mathcal{L}_0$ is contained in an interval $[b, \infty)$, and we take Δ to be $\mathcal{L}_0 - b$.

In these examples, Δ is hypoelliptic and elliptic respectively. In both cases, Δ is formally self-adjoint and nonnegative on L^2, where this space is constructed relative to the *left*-invariant Haar measure and the Riemannian measure in examples (i) and (ii) respectively. Consequently, Δ admits a

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spectral resolution,

$$\Delta = \int_0^\infty \xi \, dE(\xi),$$

where the $E(\xi)$ are self-adjoint projections, and from the spectral theorem, if m is a bounded Borel function on $[0, \infty)$, then the operator $m(\Delta)$, given by

$$m(\Delta) = \int_0^\infty m(\xi) \, dE(\xi),$$

is well-defined and bounded on L^2. A problem that has received much attention over the last twenty odd years is that of finding sufficient conditions on m that ensure that $m(\Delta)$ is also bounded on L^p, for some p different from 2, or a range of such p. It is impossible for us to give a complete bibliography here, but let us mention the work of N. J. Weiss [32], R. R. Coifman and G. Weiss [11], A. Bonami and J.-L. Clerc [6] (on compact Lie groups), of L. De Michele and Mauceri [13], [14], Hulanicki and E. M. Stein [23] (see also G. B. Folland and Stein [15]), M. Christ [9], Mauceri and S. Meda [24], D. Müller and Stein [26] (on nilpotent Lie groups), Clerc and Stein [10], L. Vretare [31], R. J. Stanton and P. A. Tomas [28], J.-Ph. Anker and N. Lohoué [5], Anker [2], [3] (on symmetric spaces), M. E. Taylor [29] (on Riemannian manifolds), and G. Alexopoulos [1] (for groups of polynomial growth).

In most of the work on this problem, in the Lie group or Riemannian manifold environment, a dichotomy seemed to be emerging, based on the growth of the volume of balls as their radius becomes large. Two paradigmatic results are the following.

THEOREM 0.1 (L. Hörmander [21], S. Mikhlin [25]). *Suppose that Δ denotes minus the usual Laplacian on $\mathbb{R}^n$. If $\kappa > [\![n/2]\!]$, $[\![\cdot]\!]$ denoting the integer part function, $m \in C^\kappa(\mathbb{R}^+)$, and*

$$\sup_{\xi \in \mathbb{R}^+} \left| \xi^k \left(\frac{\partial}{\partial \xi} \right)^k m(\xi) \right| < \infty \quad \forall k \in \{0, 1, \ldots, \kappa\},$$

then $m(\Delta)$ extends to a bounded operator on $L^p(\mathbb{R}^n)$ whenever $1 < p < \infty$, and to an operator of weak type $(1,1)$.

To state our next theorem we need a preliminary definition: for any positive real κ, $S_\kappa = \{\zeta \in \mathbb{C} : |\mathrm{Im}(\zeta)| < \kappa\}$.

THEOREM 0.2 (Taylor [29]). *Suppose that Δ denotes minus the Laplace–Beltrami operator on a noncompact n-dimensional complete Riemannian manifold M, with "bounded C^∞ geometry", and a lower bound on the Ricci curvature, of the form $\mathrm{Ric}_M \geq (1-n)\kappa^2$. If $m : \mathbb{C} \to \mathbb{C}$ is even, bounded and holomorphic in the strip $S_{\kappa/2}$, and if*

$$\sup_{\xi \in \mathbb{R}^+} (1+\xi)^k \left| \left(\frac{\partial}{\partial \xi} \right)^k m(\xi) \right| < \infty \quad \forall k \in \{0, 1, \ldots\},$$

then $m(\Delta^{1/2})$ extends to a bounded operator on $L^p(M)$ provided that $1 < p < \infty$.

Note that the hypotheses imply that the growth of the volume of a ball in M is at most exponential: indeed, for x_0 in M, the volume $|B(x_0, r)|$ of the ball of radius r centred at x_0 satisfies $|B(x_0, r)| \leq Ce^{\kappa r}$ for all positive r.

It is certainly true that, in general, some holomorphy of m is necessary for $m(\Delta)$ to be bounded on $L^p(M)$; Clerc and Stein [10] establish this for noncompact symmetric spaces.

In view of these results, and others like them, it seemed not unreasonable to expect that when the volume of balls grows exponentially, then holomorphy of m is a necessary condition for $m(\Delta)$ to be bounded on some L^p with p different from 2. The recent result of W. Hebisch [18] therefore came as a surprise; basing his computations on some formulae in Cowling, Giulini, G. I. Gaudry, and Mauceri [12] (the formulae can also be found in the work of P. Bougerol [7], a paper we discovered after publishing our own), Hebisch showed that for a particular right-invariant Laplacian on the Iwasawa AN component of a complex semisimple Lie group, a necessary and sufficient condition on m so that $m(\Delta)$ be bounded on $L^1(AN)$ is that $m(\Delta')$ be bounded on $L^1(\mathbb{R}^n)$, where Δ' denotes the standard Laplacian on the Euclidean space $\mathbb{R}^n$ of the same dimension as AN. The formulae used by Hebisch are specific to complex semisimple Lie groups.

The point of this paper is to show that this result of Hebisch may be extended in several ways: we prove that on Iwasawa AN groups coming from *arbitrary* semisimple Lie groups there is a Laplacian with a nonholomorphic functional calculus, not only for $L^1(AN)$, but also for $L^p(AN)$, where $1 < p < \infty$.

1. Notation and preliminaries.

In this paper, we use the variable constant convention, in which C denotes a constant which may not be the same in different lines.

Let G be a connected, noncompact, semisimple Lie group, with Lie algebra $\mathfrak{g}$. We need a few basic facts about G and its harmonic analysis. These can be found in S. Helgason's book [19].

Denote by θ a Cartan involution of $\mathfrak{g}$, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the associated Cartan decomposition. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$; this determines a root space decomposition: $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$, Σ denoting the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$. After choosing an ordering of the roots, we

have an Iwasawa decomposition of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$, where

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha.$$

Write $G = NAK$ for the corresponding Iwasawa decomposition of G, and S for the solvable group NA, which is identifiable, as a manifold, with the symmetric space G/K. The image of the G-invariant Riemannian measure on G/K under this identification is the left-invariant Haar measure on S and the Riemannian metric on G/K corresponds to a left-invariant metric on S. In the following we shall systematically identify the metric, functions, distributions and differential operators on G/K with the corresponding objects on S. By B_r and $\overline{B}_r$ we denote the open ball of radius r centred at the identity of S, defined relative to the left-invariant metric, and its closure.

Denote by m_α the multiplicity $\dim(\mathfrak{g}_\alpha)$ of the root α, and define ϱ by the usual formula, $\varrho = \frac{1}{2}\sum_{\alpha \in \Sigma^+} m_\alpha \alpha$. We denote by n the dimension of S, by l its real rank, i.e., the real dimension of A, and by ν the "pseudodimension" $2d+l$, where d is the cardinality $|\Sigma_0^+|$ of the set of indivisible positive roots. Notice that $\nu = n$ if G is complex, $\nu = 3$ if G has rank one, and $\nu > n$ if G is a normal real form, since then $n = d + l$. For every x in G, we denote by $A(x)$ the $\mathfrak{a}$-component of x in the decomposition $G = N \exp(\mathfrak{a})\, K$.

For each complex-valued linear form λ on $\mathfrak{a}$, the elementary spherical function φ_λ is given by the integral formula

$$\varphi_\lambda(x) = \int_K e^{(i\lambda+\varrho)(A(kx))}\, dk \quad \forall x \in S.$$

The spherical Fourier transform of a K-invariant function in $C_c(S)$ is then defined by the formula

$$\widetilde{f}(\lambda) = \int_S f(x)\varphi_{-\lambda}(x)\, dx \quad \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

Harish-Chandra proved an inversion formula and a Plancherel formula for the spherical Fourier transform, namely

$$f(x) = \int_{\mathfrak{a}^*} \widetilde{f}(\lambda)\, \varphi_\lambda(x)\, d\mu(\lambda) \quad \forall x \in S$$

for "nice" K-invariant functions f on G, and

$$\|f\|_2 = \Big[\int_{\mathfrak{a}^*} |\widetilde{f}(\lambda)|^2\, d\mu(\lambda) \Big]^{1/2} \quad \forall f \in L^2(S),$$

where the Plancherel measure μ is given by the formula $d\mu(\lambda) = c_G |\mathbf{c}(\lambda)|^{-2} d\lambda$, $\mathbf{c}$ denoting the Harish-Chandra $\mathbf{c}$-function. From the Gindikin–Karpelevich formula for $\mathbf{c}$, it is easy to deduce that

$$|\mathbf{c}(\lambda)|^{-2} \le C \begin{cases} |\lambda|^{\nu-l} & \text{if } |\lambda| \le 1, \\ |\lambda|^{n-l} & \text{if } |\lambda| > 1. \end{cases} \tag{1.1}$$

We shall use the integral formula for the Cartan decomposition:

$$\int_S f(x)\, dx = C \int_{\mathfrak{a}^+} \int_K f(k \exp(H))\, dk\, D(H)\, dH \quad \forall f \in L^1(S),$$

where dk is the normalized Haar measure on K, $\mathfrak{a}^+$ is the positive Weyl chamber in $\mathfrak{a}$, and

$$D(H) = \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^{m_\alpha} \le C \Big(\frac{|H|}{1+|H|} \Big)^{n-l} e^{2\varrho(H)} \quad \forall H \in \mathfrak{a}^+.$$

One corollary of this result is the following trivial estimate for the measure of B_r:

$$|B_r| \le C r^n e^{2|\varrho| r} \quad \forall r \in \mathbb{R}^+. \tag{1.2}$$

Here is another. Since the basic spherical function φ_0 satisfies the estimate

$$\varphi_0(\exp H) \le C(1+|H|)^d e^{-\varrho(H)} \quad \forall H \in \mathfrak{a}^+,$$

we have

$$\int_{B_r} |\varphi_0(x)|^2\, dx \le C \begin{cases} r^n & \text{if } r \le 1, \\ r^\nu & \text{if } r > 1. \end{cases} \tag{1.3}$$

To define our distinguished Laplacian Δ, we recall that if B denotes the Killing form on $\mathfrak{g}$, then the bilinear form $\langle \cdot, \cdot \rangle$, given by the formula

$$\langle X, Y \rangle = -B(X, \theta Y) \quad \forall X, Y \in \mathfrak{g},$$

is an inner product on $\mathfrak{g}$, for which the decomposition

$$\mathfrak{s} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \oplus \mathfrak{a}$$

is orthogonal. We choose an orthonormal basis $\{H_1, \dots, H_l, X_1, \dots, X_m\}$ of $\mathfrak{s}$, also denoted by $\{Y_1, \dots, Y_n\}$, adapted to this decomposition, and view its elements as left-invariant vector fields on S in the usual way. For a smooth function f on S, ∇f denotes the vector-valued function $(Y_1 f, \dots, Y_n f)$. We denote by $\check{Y}$ the *right*-invariant vector field on S which agrees with the left-invariant vector field Y at the identity. Then

$$\Delta = -\Big(\check{H}_1^2 + \dots + \check{H}_l^2 + \frac{1}{2}(\check{X}_1^2 + \dots + \check{X}_m^2) \Big).$$

The operator Δ is essentially self-adjoint on $C_c^\infty(S)$ with respect to the left-invariant Haar measure and has a special relationship with the Laplace–Beltrami operator.

Let δ be the modular function for S; δ is the N-biinvariant function whose restriction to A is given by the formula $\delta(a) = \exp(-2\varrho(\log a))$ for all a in A. We denote by $\mathcal{L}_0$ minus the Laplace–Beltrami operator on G/K and by $\mathcal{L}$ the shifted operator $\mathcal{L}_0 - \langle \varrho, \varrho \rangle I$, whose L^2-spectrum is $[0, \infty)$.

Let τ be the inversion $S \ni x \mapsto x^{-1} \in S$ and $\widetilde{\mathcal{L}} = \tau \mathcal{L} \tau$. Then, trivially, $\delta^{1/2} \widetilde{\mathcal{L}} \delta^{-1/2} = \Delta$ and, if f is a K-invariant function, then $\widetilde{\mathcal{L}} f = \mathcal{L} f$.

PROPOSITION 1.2. *If m is a bounded measurable function on $[0, \infty)$ and k and $\mathcal{K}$ denote the distributional kernels of the operators $m(\Delta)$ and $m(\widetilde{\mathcal{L}})$, respectively, then $\delta^{1/2} k = \mathcal{K}$. We note that $\mathcal{K}$ is K-invariant, so $\mathcal{K}$ is the kernel of $m(\mathcal{L})$.*

Proof. This result was (essentially) proved by Bougerol [7]. See also the papers of Cowling, Gaudry, Giulini, and Mauceri [12], Giulini and Mauceri [17], and Hebisch [18]. ∎

Notice that, since Δ is right-invariant while $\mathcal{L}$ is left-invariant, $m(\Delta) f = k * f$ and $m(\mathcal{L}) = f * \mathcal{K}$, for every test function f.

Our next result is, in part, a corollary of Proposition 1.1.

LEMMA 1.2. *Suppose that L denotes either $\Delta^{1/2}$ or $\mathcal{L}^{1/2}$. Then the kernel corresponding to the operator $\cos(tL)$ is supported in the ball $\overline{B}_t$, for all positive t.*

Proof. By Proposition 1.1, the kernel for $\cos(t\Delta^{1/2})$ may be obtained from that for $\cos(t\mathcal{L}^{1/2})$ by multiplying by $\delta^{-1/2}$. Consequently, it suffices to prove that the kernel of $\cos(t\mathcal{L}^{1/2})$ has the claimed support property. This has apparently been known for some time to experts in Fourier integral operators (see, e.g., Taylor [29]), but it may also be easily deduced from the Paley–Wiener theorem for K-biinvariant functions on G. For completeness, we explain how. The spherical Fourier transform of the kernel w_t of $\cos(t\mathcal{L})$ is the function $\widetilde{w}_t : \lambda \mapsto \cos(t\sqrt{\langle \lambda, \lambda \rangle})$ on $\mathfrak{a}^*$ (this is well defined because cos is even), which extends analytically to an entire function on $\mathfrak{a}_{\mathbb{C}}^*$ such that

$$|\widetilde{w}_t(\lambda_1 + i\lambda_2)| \le e^{t|\operatorname{Im}\sqrt{\langle \lambda_1 + i\lambda_2, \lambda_1 + i\lambda_2 \rangle}|} \le e^{t|\lambda_2|} \quad \forall \lambda_1, \lambda_2 \in \mathfrak{a}^*.$$

If f is C^∞, K-biinvariant, and supported in $\overline{B}_\varepsilon$, then the Paley–Wiener theorem implies that $\cos(t\mathcal{L}^{1/2}) f$ is supported in $\overline{B}_{t+\varepsilon}$, for any small positive ε. Taking an approximate identity for convolution of K-biinvariant functions with shrinking supports then establishes the claim. ∎

The following technical lemma allows us to tame the exponential growth of the volume of the balls. Hereafter, χ_E denotes the characteristic function of a set E in S.

LEMMA 1.3. *Let E be a K-invariant measurable subset of S, and f a function in $L^2(S)$ such that $\delta^{1/2} f$ is K-invariant. Then*

$$\|\chi_E f\|_2 = \|\chi_E \delta^{1/2} f\|_2. \tag{1.4}$$

Moreover,

$$\|\chi_E f\|_1 \le \|\chi_E \varphi_0\|_2 \|\chi_E f\|_2. \tag{1.5}$$

Proof. Write g for the function $\delta^{1/2} f$. The L^2-norm of $\chi_E f$ is the L^2-norm of $\chi_E g$ with respect to the right-invariant Haar measure $\delta(x)^{-1}\,dx$, i.e., the L^2-norm of $(\chi_E g)^\vee : x \mapsto (\chi_E g)(x^{-1})$ with respect to the left Haar measure. Since $\chi_E g$ is K-invariant, the latter coincides with the L^2-norm of $\chi_E g$. This proves (1.4).

By the K-invariance of g, E and the invariant measure on G/K, we have

$$\int_E |f(x)|\,dx = \int_E |f(kx)|\,dx = \int_E |\delta(kx)^{-1/2}||g(kx)|\,dx = \int_E |\delta(kx)^{-1/2}||g(x)|\,dx,$$

for any k in K. Integrating over K gives

$$\|\chi_E f\|_1 = \int_E \int_K |\delta(kx)^{-1/2}|\,dk\,|g(x)|\,dx = \int_E \varphi_0(x)|g(x)|\,dx$$

by Harish-Chandra's integral formula for the spherical functions. Thus (1.5) follows from Schwarz' inequality and (1.4). ∎

2. Statement of the multiplier theorem. We fix once and for all a function ψ in $C_c^\infty(\mathbb{R}^+)$, supported in $(1/2, 2)$, such that

$$\sum_{j=-\infty}^{\infty} \psi(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^+. \tag{2.1}$$

We denote by $H^s(\mathbb{R})$ the L^2-Sobolev space of order s on $\mathbb{R}$. If m is a function on $\mathbb{R}^+$ which is locally in $H^s(\mathbb{R})$ on $(1, \infty)$, we define $\|m\|_{(s)}$ thus:

$$\|m\|_{(s)} = \sup_{t \ge 1} \|\psi(\cdot) m(t\cdot)\|_{H^s}.$$

The aim of this paper is to prove the following result.

THEOREM 2.1. *Fix s_0 and s_∞ in $\mathbb{R}^+$ such that*

$$s_0 > \frac{\nu+1}{2} \quad \text{and} \quad s_\infty > \max\left\{\frac{\nu+1}{2}, \frac{n+1}{2}\right\}.$$

Let m be a function on $[0, \infty)$ such that

(i) *on the interval $[0, 2]$, m coincides with a function in $H^{s_0}(\mathbb{R})$;*
(ii) *m is locally in $H^{s_\infty}(\mathbb{R})$ on $[1, \infty)$ and $\|m\|_{(s_\infty)} < \infty$.*

Then $m(\Delta)$ is bounded on $L^p(S)$, for $1 < p < \infty$, and is of weak type $(1, 1)$.

To prove the multiplier theorem we decompose m into a sum: $m = m_0 + m_\infty$, where m_0 is obtained from m by multiplying by a smooth function on $[0,\infty)$, equal to 1 on $[0,1]$ and 0 off $[0,2]$. Then the "local multiplier" m_0 is supported in $[0,2)$ and the "global multiplier" m_∞ is supported in $(1,\infty)$. Let k_0 and k_∞ be the kernels of the operators $m_0(\Delta)$ and $m_\infty(\Delta)$ respectively. In Section 3 we prove that the kernel k_0 of the local multiplier is in $L^1(S)$, using the functional calculus based on the heat kernel; it follows that $m_0(\Delta)$ is bounded on $L^p(S)$ for all p in $[1,\infty]$. In Section 4 we prove that k_∞ is integrable away from the identity and is a Calderón–Zygmund kernel near the identity. These estimates of k_∞ are obtained by exploiting the property of finite propagation speed of the fundamental solution of the wave equation, following J. Cheeger, M. Gromov, and Taylor [8], and using small-time estimates for the fundamental solution of the heat equation due to N. Th. Varopoulos [30] and Anker [3]. In the last section we conclude the proof of the multiplier theorem by means of a covering lemma.

3. Estimate of the kernel of the local multiplier. We denote by p_t and q_t the "heat kernels" associated with the operators Δ and $\mathcal{L}$, i.e., the kernels of the operators $e^{-t\Delta}$ and $e^{-t\mathcal{L}}$, for positive t. For complex w we define

$$E_w = e^{iwp_1} - \delta_e = \sum_{j=1}^{\infty} \frac{(iw)^j}{j!} p_j,$$

and we denote by $\mathcal{E}_w$ the corresponding kernel $e^{iwq_1} - \delta_e$ for the operator $\mathcal{L}$. Then, by Proposition 1.1, $\delta^{1/2} E_w = \mathcal{E}_w$, and $\mathcal{E}_w$ is K-invariant.

LEMMA 3.1. *For every complex w, E_w is in $L^1(S)$. Moreover,*

$$\|E_u\|_1 \le C(1 + |u|^2 \log|u|)^{\nu/4} \quad \forall u \in \mathbb{R}.$$

Proof. Since $\|p_t\|_1 = 1$ for all positive t, the series defining E_w converges in $L^1(S)$ for every w in $\mathbb{C}$. To estimate the L^1-norm of E_u for real u, we consider separately the integrals over the ball B_r and over its complement B_r^c. By Lemma 1.3 and (1.3),

$$\|\chi_{B_r} E_u\|_1 \le \|\chi_{B_r}\varphi_0\|_2 \|E_u\|_2 \le C(1+r)^{\nu/2} \|\mathcal{E}_u\|_2 \quad \forall u \in \mathbb{R}\ \forall r \in \mathbb{R}^+.$$

The L^2-norm of $\mathcal{E}_u$ can be found from the Plancherel formula:

$$\|\mathcal{E}_u\|_2 = \Big[\int_{\mathfrak{a}^*} |\widetilde{\mathcal{E}}_u(\lambda)|^2 \, d\mu(\lambda) \Big]^{1/2} \quad \forall u \in \mathbb{R},$$

where $\widetilde{\mathcal{E}}_u$, the spherical Fourier transform of $\mathcal{E}_u$, is the function $\lambda \mapsto \exp(iue^{-\langle\lambda,\lambda\rangle}) - 1$ on $\mathfrak{a}^*$, and so

$$|\widetilde{\mathcal{E}}_u(\lambda)| \le \min\{|u| e^{-\langle\lambda,\lambda\rangle}, 2\} \quad \forall u \in \mathbb{R}\ \forall \lambda \in \mathfrak{a}^*.$$

Using this estimate and the asymptotic behaviour of the Plancherel measure (1.1), we deduce that

$$\|\chi_{B_r} E_u\|_1 \le C(1+r)^{\nu/2} \frac{|u|}{1+|u|} (\log(3+|u|))^{\nu/4} \quad \forall u \in \mathbb{R}\ \forall r \in \mathbb{R}^+.$$

To estimate the L^1-norm of E_u outside B_r we observe that, following Hulanicki [22], by the submultiplicativity of the function $x \mapsto e^{|x|}$ (where $|x|$ denotes the distance to the identity), there exists a constant κ such that

$$\int_S p_t(x) e^{|x|} \, dx \le \kappa^t \quad \forall t \in \mathbb{R}^+.$$

Therefore

$$\begin{aligned} \|\chi_{B_r^c} E_u\|_1 &\le e^{-r} \int_{B_r^c} |E_u(x)| e^{|x|} \, dx \\ &\le e^{-r} \sum_{j=1}^{\infty} \frac{|u|^j}{j!} \int_S p_j(x) \, e^{|x|} \, dx \le C \, e^{|u|\kappa - r} \quad \forall u \in \mathbb{R}\ \forall r \in \mathbb{R}^+. \end{aligned}$$

When $r = |u|\kappa$, we obtain the desired estimate. ∎

PROPOSITION 3.2. *Suppose that $s > (\nu+1)/2$. Let m_0 be a function in $H^s(\mathbb{R})$ with compact support. Then the kernel k_0 of the operator $m_0(\Delta)$ is in $L^1(S)$. Consequently, $m_0(\Delta)$ is bounded on $L^p(S)$ for all p in $[1,\infty]$.*

Proof. Let f be a function in $H^s(\mathbb{R})$ such that $f(0) = \int_{\mathbb{R}} \widehat{f}(u) \, du = 0$. Then

$$f(v) = \int_{\mathbb{R}} (e^{iva} - 1) \widehat{f}(u) \, du \quad \forall v \in \mathbb{R},$$

and the kernel of the operator $f(e^{-\Delta})$ is $f(p_1) = \int_{\mathbb{R}} E_u \, \widehat{f}(u) \, du$. Therefore by Lemma 3.1,

$$\begin{aligned} \|f(p_1)\|_1 &\le \int_{\mathbb{R}} \|E_u\|_1 |\widehat{f}(u)| \, du \\ &\le C \int_{\mathbb{R}} (1+|u|^2 \log|u|)^{\nu/4} |\widehat{f}(u)| \, du \\ &\le C \Big[\int_{\mathbb{R}} (1+|u|^2 \log|u|)^{\nu/2} (1+|u|)^{-2s} \, du \Big]^{1/2} \\ &\quad \times \Big[\int_{\mathbb{R}} (1+|u|)^{2s} |\widehat{f}(u)|^2 \, du \Big]^{1/2} \\ &\le C \|f\|_{H^s}. \end{aligned}$$

Thus to prove the proposition it is enough to choose a function f in $H^s(\mathbb{R})$ such that $f(e^{-t}) = m_0(t)$ for all t in $\mathbb{R}^+$. The function f equal to $m_0(-\log\cdot)$ on $\mathbb{R}^+$ and 0 elsewhere will do.

It is standard that $m_0(\Delta)$ is L^p-bounded because the corresponding kernel lies in $L^1(S)$; see, e.g., Corollary 20.14 of E. Hewitt and K. A. Ross [20]. ∎

4. Estimates on the kernel of the global multiplier. In this section we derive estimates on the kernel of the "global multiplier" m_∞. The basic tools are the property of finite propagation speed of the operator $\cos(t\Delta^{1/2})$, and some estimates on the heat kernel q_t for small positive t.

By renaming s_∞ as s, and m_∞ as m, we may and shall assume in this section and the next that $2s > \max\{n+1, \nu+1\} > 1$, that $\operatorname{supp}(m) \subset [1,\infty)$, and that

$$\|m\|_{(s)} = \sup_{t\geq 1} \|\psi(\cdot)m(t\cdot)\|_{H^s} < \infty.$$

Our aim is now to estimate the kernel of m (in this section) and then to use these estimates (in the next section) to show that $m(\Delta)$ is bounded on $L^p(S)$ when $1 < p < \infty$ and is of weak type $(1,1)$. Let $m_{(J)}$ denote $\sum_{j=1}^J \psi(2^{-j}\cdot)m$. It suffices to obtain estimates for the operator $m_{(J)}(\Delta)$, uniform in J, for then limiting arguments give the result in general. Thus in what follows, we assume that m has compact support.

LEMMA 4.1. *Suppose that h in $H^s(\mathbb{R})$ is even, and that $0 < r < R < \infty$. Then for every function u in $L^2(S)$,*

$$\|\chi_{B_R^c} h(L)u\|_2 \leq C\Big(\Big[\int_{R-r}^{\infty} |\widehat{h}(t)|^2 t^{2s}\,dt\Big]^{1/2} (R-r)^{(1-2s)/2}\|\chi_{B_r}u\|_2 + \|\widehat{h}\|_1\|\chi_{B_r^c}u\|_2\Big).$$

Proof. The proof is a slight modification of an argument of Cheeger, Gromov, and Taylor [8]. From the inverse Fourier transform formula,

$$h(L)u = \frac{1}{\pi}\int_0^\infty \operatorname{Re}(\widehat{h}(t))\cos(tL)(\chi_{B_r}u + \chi_{B_r^c}u)\,dt.$$

By the property of finite propagation speed (Lemma 1.2),

$$\operatorname{supp}(\cos(tL)(\chi_{B_r}u)) \subseteq \overline{B}_{t+r}.$$

Therefore, from the L^2-boundedness of the operator $\cos(tL)$,

$$\Big[\int_{B_R^c} |h(L)(u\chi_{B_r})(x)|^2\,dx\Big]^{1/2} \leq \frac{1}{\pi}\int_{R-r}^\infty |\operatorname{Re}(\widehat{h}(t))|\|\chi_{B_r}u\|_2\,dt = \frac{1}{\pi}\int_{R-r}^\infty t^{-s}(|\operatorname{Re}\widehat{h}(t)|t^s)\,dt\,\|\chi_{B_r}u\|_2$$

$$\leq \frac{1}{\pi}\Big(\frac{(R-r)^{1-2s}}{2s-1}\Big)^{1/2}\Big[\int_{R-r}^\infty |\operatorname{Re}(\widehat{h}(t))|^2 t^{2s}\,dt\Big]^{1/2}\|\chi_{B_r}u\|_2.$$

Furthermore, from the L^2-boundedness of $\cos(tL)$,

$$\Big[\int_{B_R^c} |h(L)(\chi_{B_r^c}u)(x)|^2\,dx\Big]^{1/2} \leq \frac{1}{\pi}\int_0^\infty |\operatorname{Re}\widehat{h}(t)|\|\chi_{B_r^c}u\|_2\,dt,$$

and the lemma follows. ∎

Recall that for positive t, q_t and p_t denote the kernels of the operators $e^{-t\mathcal{L}}$ and $e^{-t\Delta}$ respectively.

LEMMA 4.2. *Suppose that $0 < \gamma < 1/4$. The following inequalities hold, uniformly for r positive:*

(i) $$\|\chi_{B_r^c}p_t\|_2 \leq C_\gamma t^{-n/4}e^{-\gamma r^2/t} \qquad \forall t \in (0,1],$$

(ii) $$\|\chi_{B_r^c}|\nabla p_t|\|_2 \leq C_\gamma t^{-(n+2)/4}e^{-\gamma r^2/t} \qquad \forall t \in (0,1].$$

Proof. The first part of this proof involves proving variations of the estimates announced, with q_t in place of p_t. To prove these analogues, we use the estimate (1.2) for the volume of a ball:

$$V(r) = |B_r| \leq Cr^n e^{\alpha r} \qquad \forall r \in \mathbb{R}^+,$$

where $\alpha = 2|\varrho|$, as well as the following pointwise estimates, which are consequences of the sharp pointwise estimates of Anker [3]:

$$q_t(x) \leq Ct^{-n/2}e^{-|x|^2/(4t)} \qquad \forall x \in S\ \forall t \in (0,1),$$

$$|\nabla q_t(x)| \leq Ct^{-(n+2)/2}e^{-|x|^2/(4t)} \qquad \forall x \in S\ \forall t \in (0,1).$$

An inequality like the first of these estimates was earlier proved by Varopoulos [30] (p. 354), though the number 4 in the denominator of the exponent must be replaced by a larger real number. The argument below does not depend on having the exact constant in the exponential, so Varopoulos' inequality could also be applied in this situation.

To prove the analogue of (i), we integrate in polar coordinates, then by parts: for any t in $(0,1)$,

$$\|\chi_{B_r^c}q_t\|_2^2 \leq Ct^{-n}\int_r^\infty e^{-u^2/(2t)}V'(u)\,du = Ct^{-n}\Big[\int_r^\infty \frac{u}{t}e^{-u^2/(2t)}V(u)\,du - e^{-r^2/(2t)}V(r)\Big] \leq Ct^{-n-1}\int_r^\infty u^{n+1}e^{\alpha u - u^2/(2t)}\,du$$

$$= Ct^{-n/2}e^{t\alpha^2/2}\int_{r'}^{\infty}(v+t^{1/2}\alpha)^{n+1}e^{-v^2/2}\,dv$$
$$\le Ct^{-n/2}e^{t\alpha^2/2}\int_{r'}^{\infty}(|v|^{n+1}+t^{(n+1)/2}\alpha^{n+1})e^{-v^2/2}\,dv,$$

where $v = t^{-1/2}u - t^{1/2}\alpha$, and $r' = t^{-1/2}r - t^{1/2}\alpha$. Hence if $r \le 2t^{1/2}\alpha$ then $r' \le \alpha$ and $\|\chi_{B_r^c}q_t\|_2^2 \le Ct^{-n/2}$, while if $r > 2t^{1/2}\alpha$ then $r' \ge \alpha$ and

$$\|\chi_{B_r^c}q_t\|_2^2 \le Ct^{-n/2}e^{t\alpha^2/2}\int_{r'}^{\infty}v^{n+1}e^{-v^2/2}\,dv,$$
$$\le Ct^{-n/2}e^{\alpha r}(r^2/t)^{n/2}e^{-r^2/(2t)}$$
$$\le Ct^{-n/2}e^{\alpha(r^2/t)^{1/2}}(r^2/t)^{n/2}e^{-r^2/(2t)}.$$

The desired estimate for $\|q_t\|_2$ follows. The estimate for $\||\nabla q_t|\|_2$ is similar. Note that when $r = 0$, these estimates simplify to estimates for $\|q_t\|_2$ and $\|\|\nabla q_t|\|_2$, which are readily deduced from the Plancherel formula.

We now derive the estimates desired. Since $\delta^{1/2}p_t = q_t$, and q_t is K-invariant, by Proposition 1.1, $\|\chi_E p_t\|_2 = \|\chi_E q_t\|_2$ for all positive t and K-invariant sets E, by Lemma 1.3. Inequality (i) follows. Moreover, $|\nabla q_t|$ is also K-invariant, and δ is a homomorphism of S into $\mathbb{R}^+$, so that

$$|\nabla p_t| = |\nabla(\delta^{-1/2}q_t)| \le |\nabla\delta^{-1/2}|q_t + \delta^{-1/2}|\nabla q_t|$$
$$= |\nabla\delta^{-1/2}(e)|\delta^{-1/2}q_t + \delta^{-1/2}|\nabla q_t|,$$

and the right hand side of this inequality is again the product of $\delta^{-1/2}$ with a K-invariant function, so that

$$\|\chi_E|\nabla p_t|\|_2 \le |\nabla\delta^{-1/2}(e)|\|\chi_E\delta^{-1/2}q_t\|_2 + \|\chi_E\delta^{-1/2}|\nabla q_t|\|_2,$$

from which inequality (ii) follows. ∎

The first estimate of this corollary could also be proved directly, by using Varopoulos-type estimates for p_t. However, there are no good pointwise estimates available (to our knowledge) for $|\nabla p_t|$ on a group of exponential growth such as S. Thus the second estimate seems to require this somewhat devious proof.

Define the function $h_{(t)}$ on $\mathbb{R}$ by the formula

$$h_{(t)}(\tau) = m(\tau^2)\,\psi(t^{-1}\tau^2)\,e^{t^{-1}\tau^2} \qquad \forall\tau\in\mathbb{R}\ \forall t\in\mathbb{R}^+.$$

LEMMA 4.3. *If $t \ge 1$, then*

$$\|\widehat{h}_{(t)}\|_1 \le C\|m\|_{(s)}, \qquad \|\cdot|^s\widehat{h}_{(t)}\|_2 \le Ct^{(1-2s)/4}\|m\|_{(s)}.$$

Proof. Since composition with the map $\tau \mapsto \tau^2$ is a bounded linear operator from the subspace of $H^s(\mathbb{R})$ of functions supported in $[1/2, 2]$ to the subspace of $H^s(\mathbb{R})$ of functions supported in $[1/\sqrt{2}, \sqrt{2}]$, we have

$$\|h_{(t)}(t^{1/2}\cdot)\|_{H^s} \le C\|m(t\cdot)\psi(\cdot)\exp(\cdot)\|_{H^s} \le C\|m\|_{(s)}.$$

Thus the first estimate follows from the dilation invariance of the L^1-norm of the Fourier transform and the estimate $\|\widehat{f}\,\|_1 \le C\|f\|_{H^s}$, valid since $s > 1/2$. The second estimate is proved similarly, using homogeneity. ∎

We shall denote by h_j the function $h_{(2^j)}$, for integral j. Let L be the operator $\Delta^{1/2}$. Then, from (2.1),

$$m(\Delta) = \sum_{j=0}^{\infty} h_j(L)e^{-2^{-j}\Delta}$$

and the kernel k associated with $m(\Delta)$ is given by $\sum_{j=0}^{\infty}k_j$ where $k_j = h_j(L)p_{2^{-j}}$ and the series converges distributionally on S. To estimate k we decompose $S\setminus\{e\}$ as the disjoint union of the dyadic annuli A_p, defined to be $B_{R(p+1)}\setminus B_{R(p)}$, where $R(p) = 2^{p/2}$.

LEMMA 4.4. *For all nonnegative integers j and integers p,*

$$\|\chi_{A_p}k_j\|_2 \le CI_s(j,p)\|m\|_{(s)}, \qquad \|\chi_{A_p}|\nabla k_j|\|_2 \le CJ_s(j,p)\|m\|_{(s)},$$

where

$$I_s(j,p) = 2^{((p+j)(1-2s)+nj)/4} + 2^{nj/4}e^{-2^{p+j-4}},$$
$$J_s(j,p) = 2^{((p+j)(1-2s)+(n+2)j)/4} + 2^{(n+2)j/4}e^{-2^{p+j-4}}.$$

Proof. The estimate is a straightforward consequence of Lemmata 4.1, 4.2 (with γ equal to $1/8$), and 4.3. Indeed, by set inclusion and Lemma 4.1,

$$\|\chi_{A_p}k_j\|_2 \le \|\chi_{B^c_{R(p)}}h_j(L)p_{2^{-j}}\|_2$$
$$\le C\Big(\Big[\int_{R(p)-R(p-1)}^{\infty}|\widehat{h}_j(t)|^2t^{2s}\,dt\Big]^{1/2}$$
$$\times (R(p)-R(p-1))^{(1-2s)/2}\|\chi_{B_{R(p-1)}}p_{2^{-j}}\|_2$$
$$+ \|\widehat{h}_j\|_1\|\chi_{B^c_{R(p-1)}}p_{2^{-j}}\|_2\Big)$$
$$\le C\Big(\Big[\int_0^{\infty}|\widehat{h}_j(t)|^2t^{2s}\,dt\Big]^{1/2}$$
$$\times R(p)^{(1-2s)/2}\|p_{2^{-j}}\|_2 + \|\widehat{h}_j\|_1\|\chi_{B^c_{R(p-1)}}p_{2^{-j}}\|_2\Big),$$

and using the results of Lemmata 4.2(i) and 4.3 yields the first inequality desired.

Recall that there are left-invariant vector fields $Y_1, \dots, Y_n$ such that, for smooth functions f, $\nabla f = (Y_1 f, \dots, Y_n f)$. Since Δ is right-invariant, $Y_m k_j = h_j(L) Y_m p_{2-j}$, for $m = 1, \dots, n$. The argument above shows that

$$\begin{aligned}\|\chi_{B^c_{R(p)}} & h_j(L) Y_m p_{2-j}\|_2 \\ &\le C\Big(\Big[\int_0^\infty |\widehat{h}_j(t)|^2 t^{2s}\,dt\Big]^{1/2} \\ &\quad \times R(p)^{(1-2s)/2}\|Y_m p_{2-j}\|_2 + \|\widehat{h}_j\|_1 \|\chi_{B^c_{R(p-1)}} Y_m p_{2-j}\|_2\Big),\end{aligned}$$

for each m; square and add (over m), then take the square root to obtain the second inequality. ∎

PROPOSITION 4.5. *For y in S, denote by $A(y)$ the set $\{x \in S : 2|y| \le |x| \le 1\}$. The kernel k is locally integrable on $S \setminus \{e\}$, and satisfies the following estimates*:

$$\int_{B_1^c} |k(x)|\,dx \le C\|m\|_{(s)}, \tag{4.1}$$

$$\int_{A(y)} |k(xy) - k(x)|\,dx \le C\|m\|_{(s)}. \tag{4.2}$$

Proof. By Lemmata 1.3 and 4.4,

$$\begin{aligned}\int_{A_p} |k(x)|\,dx &\le \|\chi_{A_p}\varphi_0\|_2 \sum_{j=0}^\infty \|\chi_{A_p} k_j\|_2 \\ &\le C\|\chi_{A_p}\varphi_0\|_2 \sum_{j=0}^\infty I_s(j,p)\|m\|_{(s)} \\ &\le C\|\chi_{A_p}\varphi_0\|_2 \|m\|_{(s)} \sum_{j=0}^\infty [2^{((p+j)(1-2s)+nj)/4} + 2^{jn/4} e^{-2^{p+j-4}}] \\ &\le C\|\chi_{A_p}\varphi_0\|_2 \|m\|_{(s)} [2^{p(1-2s)/4} + e^{-2^{p-4}}],\end{aligned}$$

since $s > (n+1)/2$. Now we use (1.3) and sum over p:

$$\int_{B_1^c} |k(x)|\,dx = \sum_{p=0}^\infty \int_{A_p} |k(x)|\,dx \le C\|m\|_{(s)} \sum_{p=0}^\infty 2^{p\nu/4}[2^{p(1-2s)/4} + e^{-2^{p-4}}].$$

Since $s > (\nu+1)/2$ the series converges, which proves (4.1).

The same argument shows that if $p_0 < 0$, then

$$\sum_{p=p_0}^{-1} \int_{A_p} |k(x)|\,dx \le C\|m\|_{(s)} \sum_{p=p_0}^{-1} 2^{pn/4}[2^{p(1-2s)/4} + e^{-2^{p-4}}]$$

$$\le C 2^{p_0(n+1-2s)/4}\|m\|_{(s)},$$

whence k is locally integrable on $S \setminus \{e\}$.

For the proof of (4.2), we may assume that $|y| < 1/2$. Define p_y to be $[\![2\log_2(2|y|)]\!]$. Then

$$\begin{aligned}\int_{A(y)} |k(xy) - k(x)|\,dx &\le \sum_{p=p_y}^{-1} \sum_{j=0}^\infty \int_{A_p} |k_j(xy) - k_j(x)|\,dx \\ &= \sum_{p=p_y}^{-1} \Big(\sum_{j=-p}^\infty + \sum_{j=0}^{-p-1} \Big) \int_{A_p} |k_j(xy) - k_j(x)|\,dx \\ &= S_1 + S_2,\end{aligned}$$

say. We treat these two sums separately.

Since $|x| - |y| \le |xy| \le |x| + |y|$ for any x, y in S, and since the modular function is bounded and bounded away from 0 on compact sets,

$$\begin{aligned}\int_{A_p} |k_j(xy) - k_j(x)|\,dx &\le \delta(y)^{-1} \int_{A_p y} |k_j(x)|\,dx + \int_{A_p} |k_j(x)|\,dx \\ &\le C \sum_{q=p-2}^{p+2} \int_{A_q} |k_j(x)|\,dx + \int_{A_p} |k_j(x)|\,dx\end{aligned}$$

if $2|y| \le |x| \le 1$. Consequently,

$$\begin{aligned}S_1 &\le C \sum_{p=p_y}^{-1} \sum_{j=-p}^\infty \sum_{q=p-2}^{p+2} \int_{A_q} |k_j(x)|\,dx \\ &\le C \sum_{p=p_y}^{-1} \sum_{j=-p}^\infty \sum_{q=p-2}^{p+2} \|\chi_{A_q}\varphi_0\|_2 \|\chi_{A_q} k_j\|_2 \\ &\le C \sum_{p=p_y}^{-1} \sum_{j=-p}^\infty \sum_{q=p-2}^{p+2} 2^{qn/4} I_s(j,q)\|m\|_{(s)} \le C\|m\|_{(s)},\end{aligned}$$

much as in the proof of (4.1).

Now we estimate the second sum. Let $s \mapsto y_s$ be the (constant speed) geodesic in S such that $y_0 = e$ and $y_1 = y$. Then $|y_s| \le |y|$, $|\dot{y}_s| = |y|$, and

$$k_j(xy) - k_j(x) = \int_0^1 \dot{y}_s \cdot \nabla k_j(xy_s)\,ds,$$

so

$$\int_{A_p} |k_j(xy) - k_j(x)|\,dx \le |y| \int_0^1 \int_{A_p} |\nabla k_j(xy_s)|\,dx\,ds$$
$$\le |y| \int_0^1 \delta(y_s)^{-1} \int_{A_p y_s} |\nabla k_j(x)|\,dx\,ds$$
$$\le C|y| \sum_{q=p-2}^{p+2} \int_{A_q} |\nabla k_j(x)|\,dx,$$

because $|x| - |y_s| \le |xy_s| \le |x| + |y_s|$, and $2|y| \le |x| \le 1$.

We conclude much as before, but with a subtle difference: we choose s' in $((n+1)/2, (n+3)/2)$ such that $s' \le s$, and note that $\|m\|_{(s')} \le C\|m\|_{(s)}$. We use the estimates which arise using s' rather than s in the preceding lemmata. Thus

$$S_2 \le C|y| \sum_{p=p_y}^{-1} \sum_{j=0}^{-p} \sum_{q=p-2}^{p+2} \|\chi_{A_q}\varphi_0\|_2 \|\chi_{A_q}\nabla k_j\|_2$$
$$\le C|y| \sum_{p=p_y}^{-1} \sum_{j=0}^{-p} \sum_{q=p-2}^{p+2} 2^{qn/4} J_{s'}(j,q) \|m\|_{(s')}$$
$$\le C|y| \sum_{p=p_y}^{-1} \sum_{j=0}^{-p} \sum_{q=p-2}^{p+2} 2^{qn/4}$$
$$\times [2^{((q+j)(1-2s')+(n+2)j)/4} + 2^{(n+2)j/4} e^{-2^{q+j-4}}] \|m\|_{(s)}$$
$$\le C\|m\|_{(s)},$$

as required. ∎

5. The boundedness of the global multiplier. In this section, we show that, with the notation established in the previous section, $m(\Delta)$ is bounded on $L^p(S)$ if $1 < p < \infty$, and that $m(\Delta)$ is of weak type $(1,1)$. Theorem 2.1 follows immediately.

Recall that k is the kernel corresponding to $m(\Delta)$, and define k_0 and k_∞ to be $\chi_{B_1} k$ and $\chi_{B_1^c} k$ respectively. Then, trivially, $k = k_0 + k_\infty$; further, from Lemma 4.5, k_∞ lies in $L^1(S)$, and so convolution with k_∞ is a bounded operator on all the spaces $L^p(S)$, where $1 \le p \le \infty$. It now suffices to show that convolution with k_0 is an operator of weak type $(1,1)$. Indeed, if this is so, then on the one hand, $m(\Delta)$, i.e., convolution with k, is also an operator of weak type $(1,1)$, and on the other, $m(\Delta)$ is bounded on $L^2(S)$ by spectral theory. Marcinkiewicz' interpolation theorem establishes that $m(\Delta)$ is also

bounded on $L^p(S)$ if $1 < p \le 2$, and the boundedness for p in $(2,\infty)$ follows by duality.

To handle k_0, we can reduce matters to a purely local question, by appealing to a suitable covering lemma, and applying standard partition of unity arguments. To treat the local problem, we use Coifman and Weiss' notion of space of homogeneous type.

We denote by $L^1(\overline{B}_1)$ the subspace of $L^1(S)$ of all functions with support in $\overline{B}_1$.

LEMMA 5.1. *The following weak type inequality holds*:

$$|\{x \in S : |k_0 * f(x)| \ge \lambda\}| \le C\frac{\|f\|_1}{\lambda} \quad \forall \lambda \in \mathbb{R}^+ \ \forall f \in L^1(\overline{B}_1).$$

Proof. Let $d : S \times S \to [0,\infty)$ denote the left-invariant distance function associated with the canonical left-invariant Riemannian metric on S.

At least formally,

$$k_0 * f(x) = \int_{B_2} k_0(xy^{-1})\, f(y)\, \delta(y)^{-1}\, dy \quad \forall x \in S \ \forall f \in L^1(\overline{B}_1),$$

and so $k_0 * f$ is supported in $\overline{B}_2$. Further, the modular function δ is bounded and bounded away from 0 on compact sets, so that $f \in L^1(\overline{B}_1)$ if and only if $\delta^{-1} f$ is. Finally, equipped with this distance function, and the left Haar measure, B_2 is a homogeneous space in the sense of Coifman and Weiss [11]. Consequently, in view of Coifman and Weiss' [11] Théorème III.2.4, it is enough to prove that there exist constants C and C_1 such that

$$\int_{L_C(y,y_0)} |k_0(xy^{-1}) - k_0(xy_0^{-1})|\,dx \le C_1 \quad \forall y, y_0 \in \overline{B}_1, \tag{5.1}$$

where $L_C(y,y_0) = \{x \in S : d(x,y_0) \ge Cd(y,y_0)\}$. However,

$$\int_{L_C(y,y_0)} |k_0(xy^{-1}) - k_0(xy_0^{-1})|\,dx \tag{5.2}$$
$$\le \sup_{z \in \overline{B}_1} \delta(z) \int_{L_C(y,y_0)y_0^{-1}} |k_0(x'y_0y^{-1}) - k_0(x')|\,dx'$$

and $x' \in L_C(y,y_0)y_0^{-1}$ if and only if

$$|y_0^{-1}x'y_0| = d(x'y_0, y_0) \ge Cd(y,y_0) = |y^{-1}y_0| = |y_0^{-1}(y_0y^{-1})y_0|.$$

There exist constants α and β such that, for all y_0 and z in the compact set $\overline{B}_4$,

$$\alpha^{-1}|z| \le |y_0^{-1}zy_0| \le \beta|z|.$$

Then if $C = 2\alpha\beta$, it follows that if $x' \in L_C(y,y_0)y_0^{-1}$ and the integrand of the right hand side of (5.2) is not trivially 0, then $|x'| \ge 2|y_0y^{-1}|$, so

Lemma 4.5 implies the existence of a constant C_1 with the required property, and proves (5.1). ∎

LEMMA 5.2. *There exist a sequence (x_j) of elements of S and an integer m such that*

$$S = \bigcup_{j=1}^{\infty} B_1 x_j,$$

and each point of S belongs to at most m of the sets $B_2 x_i$.

Proof. See Pier [27], Anker [4] or Gaudry, Qian and Sjögren [16]. ∎

It is now a simple application of a partition of unity to prove the desired result. The presence of δ requires some slight variations on the standard method, but these are simple; see, e.g., Gaudry, Qian and Sjögren [16] for details.

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