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On the non-existence of norms for some algebras of functions

by

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Abstract. Let $C(\Omega)$ be the algebra of all complex-valued continuous functions on a topological space Ω where $C(\Omega)$ contains unbounded functions. First it is shown that $C(\Omega)$ cannot have a Banach algebra norm. Then it is shown that, for certain Ω, $C(\Omega)$ cannot possess an (incomplete) normed algebra norm. In particular, this is so for $\Omega = \mathbb{R}^n$ where $\mathbb{R}$ is the reals.

1. Introduction. Throughout this paper let $C(\Omega)$ be the algebra of all complex-valued functions on a topological space Ω *where* $C(\Omega)$ *contains unbounded functions*. Also, let A be a subalgebra of $C(\Omega)$ containing the identity function e, for each $f \in A$, also the conjugate function $\bar{f}$, and *where* A *contains unbounded functions*.

Our initial observation (Proposition 2.1) is that A cannot be given a Banach algebra norm. Then, by example, we note that A may or may not possess an (incomplete) normed algebra norm. The case $A = C(\Omega)$ is especially interesting. It seems reasonable to conjecture that it never has a normed algebra norm.

We verify this conjecture for $C(\Omega)$ if Ω is a locally compact Hausdorff space provided that $C(\Omega)$ has a function $h(p)$ where $\{p \in \Omega : h(p) = a\}$ is compact for each complex number a. In particular, $C(\Omega)$ has no normed algebra norm if Ω is a closed unbounded locally compact subset of $\mathbb{R}^n$.

The algebra of all rational functions of a complex variable is an example of an algebra over the complex field which has no normed algebra norm. An unrelated example was given in [1, Theorem 5.4].

There is considerable literature on the algebra of all real-valued continuous functions on a topological space, see [3]. Our subject matter was not treated there.

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2. On norms for A. We begin with Banach algebra norms.

PROPOSITION 2.1. *An algebra of complex-valued functions containing an unbounded function cannot have a Banach algebra norm.*

Proof. The values of an unbounded function must be in the spectrum of that function. However, in a Banach algebra the spectrum of each element is bounded.

Among the examples of algebras A which have an incomplete normed algebra norm is the algebra of all polynomials with complex coefficients in the real variable t; one simply chooses as a norm of $p(t)$

$$\|p\| = \sup\{|p(t)| : 0 \le t \le 1\}.$$

We now turn to our first example where there is no such norm.

THEOREM 2.2. *$C(\Omega)$ does not have a normed algebra norm if Ω is a T_1-space with a dense set of isolated points.*

Proof. Suppose otherwise, that $C(\Omega)$ has the normed algebra norm $\|f\|$. Let Δ be the dense set of isolated points of Ω. For each $q \in \Delta$ the characteristic function δ_q of the set $\{q\}$ is in $C(\Omega)$. For $f \in C(\Omega)$ we have $f\delta_q = f(q)\delta_q$. Thus

$$|f(q)|\,\|\delta_q\| = \|f\delta_q\| \le \|f\|\,\|\delta_q\|.$$

Therefore $|f(q)| \le \|f\|$ for all $q \in \Delta$. As Δ is dense we see that f is a bounded function. But as $C(\Omega)$ contains an unbounded function there can be no normed algebra norm for $C(\Omega)$.

In particular, the conclusion of Theorem 2.2 holds for $C(\mathbb{Z})$ where $\mathbb{Z}$ is the set of integers in the discrete topology.

We adopt the following notation. Let $\Phi(A)$ be the set of all non-zero multiplicative linear functionals on A. We say that $\gamma \in \Phi(A)$ is a *point-evaluator* if there exists $p \in \Omega$ such that $\gamma(f) = f(p)$ for all $f \in A$. We say that A is *inverse-closed* if the inverse f^{-1} lies in A for any $f \in A$ such that $f(p)$ is never zero. Also, if A has a normed algebra norm we denote by A^c the completion of A in that norm.

In the sequel, when A has a normed algebra norm we associate with A the following subset Γ of Ω. Γ is the set of all $p \in \Omega$ so that the mapping $\gamma_p : f \to f(p)$ is a continuous mapping of A as a normed algebra onto the complex field.

We derive some elementary properties of Γ. For Lemmas 2.3 and 2.4 we assume that A has a normed algebra norm $\|f\|$.

LEMMA 2.3. *Γ is a closed subset of Ω and $\Gamma \ne \Omega$.*

Proof. For each $p \in \Gamma$, γ_p extends to a multiplicative linear functional Ψ_p on A^c. Thus, for $f \in A$, we have $|f(p)| = |\Psi_p(f)| \le \|f\|$. Therefore, if q is in the closure of Γ we have $|f(q)| \le \|f\|$. Hence $q \in \Gamma$. This argument shows that every $f \in A$ is bounded on Γ. As A has unbounded functions we see that $\Gamma \ne \Omega$.

LEMMA 2.4. *Suppose that each $\gamma \in \Phi(A)$ is a point-evaluator. Then Γ is non-empty and every $f \in A$ which vanishes on Γ lies in the radical of A^c.*

Proof. Since A^c is a commutative Banach algebra with an identity there must be a non-zero multiplicative linear functional Ψ on A^c. The restriction γ of Ψ to A is a non-zero multiplicative linear functional on A which is continuous on A as Ψ is continuous on A^c. By hypothesis γ is a point-evaluator $f \to f(p)$ which is continuous, so that $p \in \Gamma$. Note that if $f \in A$ vanishes on Γ then $\Psi(f) = 0$. As Ψ can be any multiplicative linear functional on A^c we see that f is in the radical of A^c.

As is well known [6, Th. 2.3.4], every $f \in A$ in the radical of A^c satisfies $\lim \|f^n\|^{1/n} = 0$. We shall use this fact later.

Lemma 2.4 shows that we need a criterion to ensure that every $\gamma \in \Phi(A)$ is a point-evaluator.

LEMMA 2.5. *Suppose that A is inverse-closed. If A also contains a function h where $\{p \in \Omega : h(p) = h(p_0)\}$ is compact for each $p_0 \in \Omega$ then each $\gamma \in \Phi(A)$ is a point-evaluator.*

Proof. For each $f \in A$ let $Z(f) = \{p \in \Omega : f(p) = 0\}$ and let $\gamma \in \Phi(A)$. We note that $Z(f)$ is not void if $\gamma(f) = 0$. Let $f_j \in A$, $j = 1, \ldots, n$, where each $\gamma(f_j) = 0$. We claim that $\bigcap_{j=1}^n Z(f_j)$ is not void. For suppose otherwise. Then, for each $p \in \Omega$, there is some f_j where $f_j(p) \ne 0$. Hence $g = f_1\bar{f}_1 + \ldots + f_n\bar{f}_n$ is never zero and $\gamma(g) = 0$ contrary to the above remark.

Next, $v = h - \gamma(h)e$ is in the kernel of γ and $Z(v) = \{p \in \Omega : h(p) = \gamma(h)\}$, which is compact and not void. The sets $Z(v) \cap Z(f)$ as f ranges over all $f \in A$ with $\gamma(f) = 0$ form a collection of closed subsets of $Z(v)$ with the finite intersection property. Hence there is some $q \in Z(v)$ where $f(q) = 0$ for all $f \in A$ satisfying $\gamma(f) = 0$. Now let $k \in A$. We have $\gamma(k - \gamma(k)e) = 0$, so that $\gamma(k) = k(q)$.

We remind the reader of our standing hypothesis that $C(\Omega)$ has unbounded functions.

THEOREM 2.6. *Let Ω be a locally compact Hausdorff space. Suppose that $C(\Omega)$ has a function h where $\{p \in \Omega : h(p) = h(p_0)\}$ is compact for each $p_0 \in \Omega$. Then $C(\Omega)$ cannot have a normed algebra norm.*

Proof. Suppose that $C(\Omega)$ has a normed algebra norm $\|f\|$. Let E be the subalgebra of $C(\Omega)$ consisting of all $f \in C(\Omega)$ which vanish at infinity.

The preceding lemmas apply to $C(\Omega)$. Let $p_0 \in \Omega$, $p_0 \notin \Gamma$. There exists a neighborhood U of p_0 disjoint from Γ. By [4, Th. 6.78] there is a neighborhood V of p_0 where $\overline{V}$ is compact and $\overline{V} \subset U$. By a version of Urysohn's lemma [4, Th. 6.80] there exists a continuous function $g(p)$ from Ω to $[0,1]$ so that $g(p_0) = 1$ and $g(p) = 0$ for all $p \notin V$. Then $g(p)$ has compact support, so $g \in E$. Now $g(p)$ vanishes on Γ. Then we see, by Lemma 2.4, that g lies in the radical of the completion of $C(\Omega)$ in the norm $\|f\|$. Therefore $\lim \|g^n\|^{1/n} = 0$.

But $\|f\|$ defines a normed algebra norm on E. Hence a theorem of Kaplansky [5, Th. 6.2] tells us that

$$\|f\| \geq \sup\{|f(p)| : p \in \Omega\}$$

for all $f \in E$. Each $g^n \in E$, so that $\|g^n\| \geq 1$ for each positive integer n. This contradicts $\lim \|g^n\|^{1/n} = 0$. Hence $C(\Omega)$ has no normed algebra norm.

The existence of the function h in Theorem 2.6 was to ensure that every non-zero multiplicative functional on $C(\Omega)$ is a point-evaluator. Thus we have the following.

THEOREM 2.7. *Let Ω be a locally compact Hausdorff space. If every non-zero multiplicative functional on $C(\Omega)$ is a point-evaluator then $C(\Omega)$ has no normed algebra norm.*

COROLLARY 2.8. *Let Ω be any unbounded closed locally compact subspace of $\mathbb{R}^n$. Then $C(\Omega)$ has no normed algebra norm.*

Proof. For each $p = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ we set $h(p) = x_1^2 + \ldots + x_n^2$. The set $\{p \in \Omega : h(p) = h(p_0)\}$ is a bounded and closed subset of $\mathbb{R}^n$. We can now apply Theorem 2.6.

THEOREM 2.9. *Let Ω be a zero-dimensional Hausdorff space. Suppose that there exists a function h in $C(\Omega)$ where $\{p \in \Omega : h(p) = h(p_0)\}$ is compact for each $p_0 \in \Omega$. Then $C(\Omega)$ does not possess a normed algebra norm.*

Proof. Suppose that $C(\Omega)$ has a normed algebra norm. By Lemmas 2.3–2.5 the set $\Gamma \neq \Omega$. Therefore (see [3, p. 247]) there is a closed and open set Δ in Ω which is disjoint from Γ. There is $g \in C(\Omega)$ where $g(p) = 1$, $p \in \Delta$ and $g(p) = 0$ for $p \notin \Delta$. As $g(\Gamma) = 0$ we see that g lies in the radical of the completion of $C(\Omega)$ by Lemma 2.4. This is impossible as g is a non-zero idempotent.

COROLLARY 2.10. *If Ω is an unbounded zero-dimensional subspace of the complex plane then $C(\Omega)$ does not have a normed algebra norm.*

Proof. The function $h(z) = z$ used with Theorem 2.9 provides this conclusion.

References

[1] W. G. Bade and P. C. Curtis Jr., *Homomorphisms of commutative Banach algebras*, Amer. J. Math. 82 (1960), 589–608.

[2] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, 1973.

[3] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

[4] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer, New York, 1965.

[5] I. Kaplansky, *Normed algebras*, Duke Math. J. 16 (1949), 399–418.

[6] C. E. Rickart, *General Theory of Banach Algebras*, Van Nostrand, Princeton, 1960.

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