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σ-fragmented Banach spaces II

by

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Abstract. Recent papers have investigated the properties of σ-fragmented Banach spaces and have sought to find which Banach spaces are σ-fragmented and which are not. Banach spaces that have a norming M-basis are shown to be σ-fragmented using weakly closed sets. Zizler has shown that Banach spaces satisfying certain conditions have locally uniformly convex norms. Banach spaces that satisfy similar, but weaker conditions are shown to be σ-fragmented. An example, due to R. Pol, is given of a Banach space that is σ-fragmented using differences of weakly closed sets, but is not σ-fragmented using weakly closed sets.

1. Introduction. Let X be a normed vector space and let $\mathcal{T}$ be a locally convex topology on X. We say that $(X, \mathcal{T})$ is *σ-fragmented* if, for each $\varepsilon > 0$,

$$(1) \qquad\qquad X = \bigcup_{k=1}^{\infty} X_k,$$

each set X_k having the property that each of its non-empty subsets has a non-empty relatively $\mathcal{T}$-open subset of norm diameter less than ε. If $\mathcal{S}$ is a family of subsets of X and the sets X_k in (1) can always be taken from the family $\mathcal{S}$, we say that $(X, \mathcal{T})$ is *σ-fragmented using sets from $\mathcal{S}$*. We shall be most interested in the case when X is a Banach space, $\mathcal{T}$ is its weak topology and $(X, \mathcal{T})$ is σ-fragmented: in this case we shall say that X is a *σ-fragmented Banach space*.

In a series of papers [7–11] we have investigated the properties of σ-fragmented Banach spaces and shown that certain classes of Banach spaces are σ-fragmented and that Banach spaces of some other classes are not σ-fragmented; see also [6, 12, 16, 17]. In this note we show that some further classes of Banach spaces have this property.

We first consider a Banach space X that has an *extended Markushevich basis* $\{x_\gamma : \gamma \in \Gamma\}$. Then after a simple normalization (see [18, p. 673 and p. 691]) $\{x_\gamma : \gamma \in \Gamma\}$ is a family of points of X whose finite linear combina-

tions are norm dense in X, and there is an associated family $\{f_\gamma : \gamma \in \Gamma\}$, called the *coordinate functions*, in X^* with

$$\|f_\gamma\| = 1, \quad \gamma \in \Gamma,$$
$$f_\gamma(x_\gamma) = 1, \quad \gamma \in \Gamma,$$
$$f_\gamma(x_\delta) = 0, \quad \gamma \in \Gamma, \ \delta \in \Gamma, \ \gamma \neq \delta,$$

and

$$\bigcap \{\ker f_\gamma : \gamma \in \Gamma\} = \{0\}.$$

Such a basis is called an *M-basis*, for short.

Let F be the norm closure of the linear span of $\{f_\gamma : \gamma \in \Gamma\}$ in X^*. The formula

$$\||x\|| = \sup\{f(x) : f \in F, \ \|f\| \leq 1\}$$

defines a new norm on the linear space X satisfying $0 < \||x\|| \leq \|x\|$ for all $x \neq 0$ in X. Note that X is not necessarily complete under $\||\cdot\||$, the *norm associated with the M-basis*. The M-basis is said to be *norming* if the norm $\||\cdot\||$ is equivalent to the original norm $\|\cdot\|$, but we do not yet assume that this is the case.

We introduce a locally convex topology on X. For each $\varepsilon > 0$ and each finite subset Δ of Γ write

$$N(\Delta; \varepsilon) = \{x : |f_\delta(x)| < \varepsilon \text{ for } \delta \in \Delta\}.$$

We form a locally convex topology on X by taking these sets $N(\Delta; \varepsilon)$ as a base for the neighbourhoods of 0 in X. The condition $\bigcap \{\ker f_\gamma : \gamma \in \Gamma\} = \{0\}$ ensures that we have a Hausdorff topology. We call this topology the *coordinate topology*. Note that coordinate closed sets are automatically weakly closed.

We can now state our first theorem.

THEOREM 1. *Let X be a Banach space with a norming M-basis. Then there is a coordinate compact subset X_0 of X with $\bigcup_{n=1}^{\infty} nX_0$ norm dense in X. The space X with its coordinate topology, and so also with its weak topology, is σ-fragmented using coordinate closed sets, and so also using weakly closed sets.*

This result can be regarded as a refinement of a result of John and Zizler [10, p. 687] showing that a Banach space with a norming M-basis has an equivalent locally convex norm, and so by [8, Theorem 2.1] is σ-fragmented using differences of weakly closed sets.

Note that in the proof of Theorem 1 we study Banach spaces with M-bases that are not necessarily norming, and we obtain some information about the structure of such spaces.

In §3 we give an example of a Banach space with a locally uniformly convex norm that is not σ-fragmented using weakly closed sets. Our original example was a space of the form $C(K)$ with K derived from a tree in the spirit of Haydon [5]. Subsequently R. Pol showed us a simpler and better behaved tree. We are grateful to Professor Pol for allowing us to reproduce his example in the present paper.

Examples are given in [3, p. 261] of Banach spaces that have locally uniformly convex norms but admit no one-to-one bounded linear maps into any $c_0(\Gamma)$ and, a fortiori, have no M-bases. The example of Pol, which is important for our general theory, provides another example of a Banach space with a locally uniformly convex norm that has no M-basis.

Note that Haydon [5, see also 3, p. 325] has given an example of a Banach space (with a Kadec norm) that is σ-fragmented using differences of weakly closed sets, but has no equivalent strictly convex norm.

Our second theorem takes a form suggested by the following theorem of Zizler [19].

THEOREM Z. *Let X be a Banach space with a family $\{T_\gamma : \gamma \in \Gamma\}$ of bounded linear maps $T_\gamma : X \to X$ with the following properties.*

(i) *For each $x \in X$, the map on Γ given by $\gamma \to \|T_\gamma x\|$ belongs to $c_0(\Gamma)$.*

(ii) *If $x \in X$, then x belongs to the norm closure in X of the linear span of $\{T_\gamma x : \gamma \in \Gamma\}$.*

(iii) *For each $\gamma \in \Gamma$ the normed space $T_\gamma X$ admits an equivalent locally uniformly convex norm.*

Then X admits an equivalent locally uniformly convex norm.

We prove

THEOREM 2. *Let X be a Banach space with a family $\{T_\gamma : \gamma \in \Gamma\}$ of bounded linear maps $T_\gamma : X \to X$ with the properties (i) and (ii) of Theorem Z and also the property (iv) below.*

(iv) *For each $\gamma \in \Gamma$, the norm closure X_γ of $T_\gamma X$ is a σ-fragmented Banach space.*

Then X is σ-fragmented.

Further, if the Banach spaces in (iv) are σ-fragmented using weakly closed sets, then X will be σ-fragmented using differences of weakly closed sets.

It is easy to deduce from this theorem that the usual c_0 or ℓ^p, $1 \leq p < \infty$, sums,

$$c_0\{X_\gamma : \gamma \in \Gamma\} \quad \text{or} \quad \ell^p\{X_\gamma : \gamma \in \Gamma\},$$

of a family $\{X_\gamma : \gamma \in \Gamma\}$ of Banach spaces that are σ-fragmented (or σ-fragmented using weakly closed sets), are themselves σ-fragmented (or σ-fragmented using differences of weakly closed sets). These conclusions are similar to but not identical with Theorems 6.1 and 6.2 in [8].

Using a result of John and Zizler we obtain the following corollary.

COROLLARY 1. *Let X be a Banach space with a transfinite sequence $\{P_\gamma : 0 \le \gamma \le \Gamma\}$ of projections of X into itself satisfying:*

(i) $\sup\{\|P_\gamma\| : 0 \le \gamma \le \Gamma\} < \infty$;

(ii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ *for* $0 \le \alpha \le \beta \le \Gamma$ *and* $P_0 = 0$;

(iii) $P_\beta X$ *is the norm closure of the linear span of* $\{P_\gamma X : 0 \le \gamma < \beta\}$ *when β is a limit ordinal with $0 < \beta \le \Gamma$; and*

(iv) $P_\Gamma(X) = X$.

If for each γ with $0 \le \gamma < \Gamma$, $(P_{\gamma+1} - P_\gamma)X$ is σ-fragmented (or σ-fragmented using weakly closed sets), then X is σ-fragmented (or σ-fragmented using differences of weakly closed sets).

Note that the conditions (i)–(iv) in this corollary are similar to the condition that X should have a projective resolution of the identity, but contain no conditions on the density characters of the spaces $P_\gamma X$.

2. Proof of Theorem 1.

As we shall see, Theorem 1 will be a simple consequence of the following lemma about Banach spaces with M-bases that are not necessarily norming.

LEMMA 1. *Let X be a Banach space with an M-basis, and let $\|\|\cdot\|\|$ be the norm associated with the M-basis. Then there is a coordinate compact subset X_0 of X for which $\bigcup_{n=1}^\infty nX_0$ is $\|\cdot\|$-norm dense in X. The space X with its coordinate topology, and so also with its weak topology, is σ-fragmented by the norm $\|\|\cdot\|\|$ using coordinate closed sets.*

P r o o f. Recall that

$$\|\|x\|\| = \sup\{f(x) : f \in F, \ \|f\| \le 1\},$$

where F is the norm closure of the set F_0 of all finite linear combinations of the points f_γ, $\gamma \in \Gamma$, in X^*. Hence

$$\|\|x\|\| = \sup\{f(x) : f \in F_0, \ \|f\| \le 1\}.$$

Since for each $f \in F_0$ the map $x \to f(x)$ is coordinate continuous, the norm $\|\|\cdot\|\|$ is lower semi-continuous for the coordinate topology.

We transfer our attention to the subspace Ξ of $\mathbb{R}^\Gamma$ consisting of all points $\xi = (\xi_\gamma : \gamma \in \Gamma)$ of $\mathbb{R}^\Gamma$ of the form $\xi_\gamma = f_\gamma(x)$, $\gamma \in \Gamma$, for some x in X. Define the linear map $\varphi : X \to \Xi$ by

$$\varphi(x) = (f_\gamma(x) : \gamma \in \Gamma).$$

Then φ maps X bijectively to Ξ, since $\bigcap\{\ker f_\gamma : \gamma \in \Gamma\} = \{0\}$. We transform the norm $\|\|\cdot\|\|$ to Ξ by taking $\|\|\varphi(x)\|\| = \|\|x\|\|$ for each x in X. The coordinate topology on X corresponds exactly to the restriction to Ξ of the product topology on $\mathbb{R}^\Gamma$. Thus the norm $\|\|\cdot\|\|$ on Ξ is lower semi-continuous for the product topology.

We return to the space X for a while. We study the sets

$$X_n = \{x : |f_\gamma(x)| \le n \text{ for } \gamma \in \Gamma \text{ and}$$
$$|f_\gamma(x)| > 0 \text{ for at most } n \text{ elements } \gamma \text{ of } \Gamma\},$$

for $n = 1, 2, \ldots$. For each γ in Γ, we have $f_\delta(x_\gamma) = \delta_{\gamma\delta}$ so that $rx_\gamma \in X_1$ for all r with $|r| \le 1$. Also, for $n, m \ge 1$, the vector sum $X_n + X_m$ is contained in X_{n+m}. Since the linear span of $\{x_\gamma : \gamma \in \Gamma\}$ is norm dense in X, it follows that the set $\bigcup_{n=1}^\infty X_n$ is norm dense in X.

Further, on writing

$$X_0 = \bigcup_{n=1}^\infty \frac{1}{n^2} X_n,$$

we see that $\bigcup_{n=1}^\infty n^2 X_0$ is also dense in X under $\|\cdot\|$ and so under $\|\|\cdot\|\|$.

We study the images $\Xi_n = \varphi(X_n)$, $n \ge 0$, in Ξ. We have

$$\Xi_0 = \bigcup_{n=1}^\infty \frac{1}{n^2} \Xi_n,$$

and

$$\Xi_n = \{\xi : \xi \in \mathbb{R}^\Gamma, \ |\xi_\gamma| \le n \text{ for all } \gamma \in \Gamma \text{ and}$$
$$|\xi_\gamma| > 0 \text{ for at most } n \text{ elements } \gamma \text{ of } \Gamma\}$$

for $n \ge 1$. Note that $\bigcup_{n=1}^\infty \Xi_n$ and $\bigcup_{n=1}^\infty n^2 \Xi_0$ are dense in Ξ under $\|\|\cdot\|\|$.

We now verify that Ξ_n is closed in $\mathbb{R}^\Gamma$ for each $n \ge 1$. Suppose that $\eta = (\eta_\gamma : \gamma \in \Gamma)$ is any point of $\mathbb{R}^\Gamma$ not in Ξ_n. Perhaps

$$|\eta_\delta| > n$$

for some δ in Γ. Then η lies in the open set $\{\xi : |\xi_\delta| > n\}$ that does not meet Ξ_n. Otherwise, there will be $n + 1$ distinct elements, say $\delta(0), \delta(1), \ldots, \delta(n)$, in Γ with

$$|\eta_{\delta(i)}| > 0 \quad \text{for } 0 \le i \le n.$$

In this case η lies in the open set $\{\xi : |\xi_{\delta(i)}| > 0 \text{ for } 0 \le i \le n\}$ that does not meet Ξ_n. Thus Ξ_n is closed in $\mathbb{R}^\Gamma$. Since Ξ_n lies in the compact set $\{\xi : |\xi_\gamma| \le n, \ \gamma \in \Gamma\}$ of $\mathbb{R}^\Gamma$, it follows that Ξ_n is compact in $\mathbb{R}^\Gamma$ and so also in Ξ with its product topology.

Note that $\Xi_0 = \bigcup_{n=1}^\infty (1/n^2)\Xi_n$ contains the origin of Ξ and that any open set in the product topology that contains the origin also contains all but a finite number of the compact sets $(1/n^2)\Xi_n$, $n \ge 1$. Hence Ξ_0 is compact in Ξ with its product topology.

It now follows that X_0 is compact in the coordinate topology of X, so that X is generated by a coordinate compact set.

We now show that, for each $n \geq 1$, the set Ξ_n with its product topology is fragmented by the $\||\cdot\||$-norm on Ξ. To prove this we have to show that for each non-empty subset Θ of Ξ_n and each $\varepsilon > 0$, there is a non-empty subset of Θ that is relatively open in the product topology and of $\||\cdot\||$-norm diameter less than ε. Let $\varepsilon > 0$ be given and let Θ be a non-empty subset of Ξ_n. If Θ just contains the origin, its $\||\cdot\||$-diameter is zero. So we may suppose that Θ contains a non-zero point, and we can choose $\theta \neq 0$ in Θ so that θ has at least as many non-zero coordinates as any other point of Θ. Let θ_δ, $\delta = \delta(1), \ldots, \delta(k)$, with $1 \leq k \leq n$, be the non-zero components of θ. Consider the neighbourhood N_1 of θ defined by

$$N_1 = \{\xi : |\xi_{\delta(i)}| > 0 \text{ for } 1 \leq i \leq k\}.$$

All points ξ in $N_1 \cap \Theta$ have $|\xi_{\delta(i)}| > 0$, $1 \leq i \leq k$, and so, by the choice of Θ, have

$$\xi_\gamma = 0 \quad \text{for } \gamma \notin \{\delta(1), \ldots, \delta(k)\}.$$

Thus $N_1 \cap \Theta$ lies in a k-dimensional linear subspace of Ξ. On this subspace the product topology and the $\||\cdot\||$-norm topology coincide. Hence we can choose a neighbourhood N_2 of θ in the product topology, contained in N_1, with $\||\cdot\||$-diameter$(N_2 \cap \Theta) < \varepsilon$. Thus Ξ_n with its product topology is fragmented by the $\||\cdot\||$-norm.

Since the norm $\||\cdot\||$ is lower semi-continuous for the product topology, $\bigcup_{n=1}^{\infty} \Xi_n$ is $\||\cdot\||$-dense in Ξ and each set Ξ_n is fragmented by $\||\cdot\||$, it follows by Lemma 2.3 of [8] that Ξ with its product topology is σ-fragmented by $\||\cdot\||$ using sets closed in the product topology. It follows that X with its coordinate topology is σ-fragmented by the norm $\||\cdot\||$ using coordinate closed sets.

Proof of Theorem 1. The results follows immediately from Lemma 1 when the M-basis is norming so that the norms $\||\cdot\||$ and $\|\cdot\|$ are equivalent.

3. An example of Pol. Let T_0 and T_∞ be the sets of all finite and infinite sequences of natural numbers and write $T = T_0 \cup T_\infty$. We write $s \preccurlyeq t$ if $s, t \in T$ and either $s = t$ or s is an initial segment of t.

We include the empty sequence $\emptyset$ of zero length within T_0 as an initial segment of each sequence in T. Then T is a "tree" under the partial order $\preccurlyeq$. We use the natural interval notation:

$$[s, t] = \{r : s \preccurlyeq r \preccurlyeq t\}, \quad (s, t) = \{r : s \prec r \prec t\},$$

etc. We take the sets

$$[\emptyset, t], \quad t \in T, \quad \text{and} \quad T \setminus [\emptyset, t], \quad t \in T,$$

as a subbase for the open sets of a topology, the *interval topology*, on T. Note that for all $t \in T$, the family $\{(s, t] : s \prec t\}$ is a neighbourhood base for t in T. All the sets $[\emptyset, t]$, $t \in T$, are compact in this topology, and so T is locally compact.

Since T is locally compact, we can form the one-point compactification K of T by adjoining a single point, say ∞, to T. Let $C(K)$ be the Banach space of all continuous real-valued functions on K.

Since the third derived set of K is empty, a result of Deville [2, Théorème 2.5] shows that $C(K)$ has a locally uniformly convex renorming (see also Haydon & Rogers [6]). Hence $C(K)$ is σ-fragmented using differences of weakly closed sets [8, Th. 2.1]. Banach spaces of type $C(K)$, for much more general locally compact trees, have been studied by R. Haydon [4, 5].

Since each interval $[\emptyset, t]$, $t \in T$, is both open and compact in T, the characteristic function $\chi_{[\emptyset, t]}$ of this interval (tacitly augmented by the value 0 at ∞) belongs to $C(K)$. The set Ξ of all such functions $\chi_{[\emptyset, t]}$ is, in fact, pointwise closed in $C(K)$, but we do not need to use this.

We now suppose that $(C(K), \text{pointwise})$ is σ-fragmented using pointwise closed sets, and we seek a contradiction. Under this assumption, we can write $\Xi = \bigcup_{n=1}^{\infty} \Xi_n$ with each Ξ_n a relatively pointwise closed set in Ξ with the property that each non-empty subset of Ξ_n has a non-empty relatively pointwise open subset of diameter less than $1/2$.

Let $\tau : \Xi \to T$ be the map carrying the point ξ of Ξ that is the characteristic function of the interval $[\emptyset, t]$ to the point $\tau(\xi) = t$ of T. Then

$$T_\infty = \tau(\Xi) \cap T_\infty = \bigcup_{n=1}^{\infty} (\tau(\Xi_n) \cap T_\infty).$$

We identify T_∞ with $\mathbb{N}^{\mathbb{N}}$ with its usual product topology. In this topology T_∞ is of course a Baire space. Using the Baire category theorem we can choose n so that $\tau(\Xi_n) \cap T_\infty$ is dense in some open Baire interval of $T_\infty = \mathbb{N}^{\mathbb{N}}$, that is, a set of the form "all t in T_∞ with initial segment $t|m$ of length m coinciding with s, for some s in T_0 and some $m \geq 0$". So $\tau(\Xi_n) \cap T_\infty$ is dense in $[s, \infty) \cap T_\infty$, for some s in T_0, where we use $[s, \infty)$ to denote the generalized interval of all t in T with $s \preccurlyeq t$.

We now show that

$$[s, \infty) \cap T_0 \subset \tau(\Xi_n).$$

Otherwise there would be an r in $[s, \infty) \cap T_0$ not in $\tau(\Xi_n)$. So

$$\xi_r = \chi_{[\emptyset, r]} \in \Xi \setminus \Xi_n.$$

Since Ξ_n is pointwise closed in Ξ, we can choose a pointwise open neighbourhood of ξ_r in Ξ that does not meet Ξ_n. We may take this pointwise

open neighbourhood of ξ_r to be of the basic form $N \cap \Xi$ where

$$N = \{\xi : \xi(\theta) = 0,\ \xi(\varphi) = 1 \text{ for } \theta \in \Theta,\ \varphi \in \Phi\}$$

with Θ a finite subset of $T \setminus [\emptyset, r]$ and Φ a finite subset of $[\emptyset, r]$. Here we use the fact that the points of Ξ are all characteristic functions. For each θ in $\Theta \cap (r, \infty)$, there will be an immediate successor of r, say r_θ, with $r \prec r_\theta \prec \theta$. So we may choose an immediate successor, say u, of r, that avoids all the points r_θ, $\theta \in \Theta$. Now, for all v in $[u, \infty)$ we have $\chi_{[\emptyset, v]} \in N$, so that $\chi_{[\emptyset, v]} \notin \Xi_n$. Thus $[u, \infty)$ does not meet $\tau(\Xi_n)$. However, $[u, \infty) \cap T_\infty$ is a non-empty open subset of $[s, \infty) \cap T_\infty$ in the $\mathbb{N}^\mathbb{N}$ topology, and it does not meet $\tau(\Xi_n) \cap T_\infty$. This contradicts the density of $\tau(\Xi_n) \cap T_\infty$ in $[s, \infty) \cap T_\infty$, so that we must have $[s, \infty) \cap T_0 \subset \tau(\Xi_n)$.

The set

$$\Xi'_n = \{\chi_{[\emptyset, u]} : u \in [s, \infty) \cap T_0\}$$

is a non-empty subset of Ξ_n. By our assumption concerning Ξ_n, there is a relatively pointwise open subset G of Ξ that meets Ξ'_n in a non-empty set of diameter less than $1/2$. So we can choose a point $\xi' = \chi_{[\emptyset, u]}$, with $u \in [s, \infty) \cap T_0$ in $\Xi'_n \cap G$, and then a basic open neighbourhood

$$N = \{\xi : \xi(\theta) = 0,\ \xi(\varphi) = 1 \text{ for } \theta \in \Theta,\ \varphi \in \Phi\}$$

of ξ' contained in G, with Θ a finite subset of $T \setminus [\emptyset, u]$, and Φ a finite subset of $[\emptyset, u]$. Thus $\operatorname{diam}(\Xi'_n \cap N) < 1/2$. Since Θ is finite, it contains only a finite number of immediate successors of u. So we can choose an immediate successor w of u that is not in Θ. Now the points

$$\xi' = \chi_{[\emptyset, u]}, \qquad \xi'' = \chi_{[\emptyset, w]}$$

both lie in $\Xi'_n \cap N$ but $\xi'(w) = 0$ and $\xi''(w) = 1$, so that the diameter of $\Xi'_n \cap N$ must be at least 1. This contradiction shows that $(C(K), \text{pointwise})$ is not σ-fragmented using pointwise closed sets.

Since K is a scattered compact Hausdorff space, the pointwise topology and the weak topology coincide on all bounded subsets of $C(K)$. It follows that $(C(K), \text{weak})$ is not σ-fragmented using weakly closed sets.

4. Proof of Theorem 2. By a suitable scaling we may assume that $\|T_\gamma\| \le 1$ for all γ in Γ. We also assume that Γ is well-ordered.

We fix $\varepsilon > 0$ until the end of the proof.

For each $m \ge 1$ and each x in X, let

$$A_m(x) = \{\gamma \in \Gamma : \|T_\gamma(x)\| > 1/m\}.$$

Let $a_m(x) = |A_m(x)|$ and let

$$A_m(x) = \{\alpha(x, 1), \ldots, \alpha(x, a_m(x))\}$$

with $\alpha(x, 1) < \ldots < \alpha(x, a_m(x))$, suppressing the dependence of the α's on the parameter m for simplicity of notation. For $n \ge 1$ and $r = (r_1, \ldots, r_n) \ne 0$ a vector with n rational components, we introduce the sets

$$X(m, n, r) = \left\{x : a_m(x) = n \text{ and } \left\| x - \sum_{j=1}^n r_j T_{\alpha(x,j)} x \right\| \le \varepsilon\right\}.$$

We first verify that

$$X = \{0\} \cup \bigcup \{X(m, n, r) : m \ge 1,\ n \ge 1,\ 0 \ne r \in \mathbb{Q}^n\}.$$

Consider any $x \ne 0$ in X. By condition (ii) we can choose a finite rational linear combination of linearly independent vectors chosen from $\{T_\gamma x : \gamma \in \Gamma\}$, say $\sum_{j=1}^l s_j T_{\gamma_j} x$, with

$$\left\| x - \sum_{j=1}^l s_j T_{\gamma_j} x \right\| \le \varepsilon.$$

We may suppose that $l \ge 1$, that $|s_j| > 0$ for $1 \le j \le l$ and that $\|T_{\gamma_j} x\| > 0$ for $1 \le j \le l$.

Now we can choose m so large that $\|T_{\gamma_j} x\| > 1/m$ for $1 \le j \le l$. Then $A_m(x) \supset \{\gamma_j : 1 \le j \le l\}$, and taking $n = a_m(x)$ we have

$$\sum_{j=1}^l s_j T_{\gamma_j} x = \sum_{j=1}^n r_j T_{\alpha(x,j)} x$$

for suitable rationals r_j, $1 \le j \le n$, l of these being identical in some order with s_j, $1 \le j \le l$, the others being zero. This ensures that

$$\left\| x - \sum_{j=1}^n r_j T_{\alpha(x,j)} x \right\| \le \varepsilon,$$

so that $x \in X(m, n, r)$. Thus X is the union of the origin and the sets of this form.

We now show that for fixed $m \ge 1$, $n \ge 1$ and $r \in \mathbb{Q}^n$, the set $X(m, n, r)$ is the difference of two weakly closed sets. First note that for each x,

$$U = \{\xi : \|T_\gamma(\xi)\| > 1/m \text{ for } \gamma \in A_m(x)\}$$

defines a weakly open neighbourhood of x such that $A_m(\xi) \supset A_m(x)$ and so $a_m(\xi) \ge a_m(x)$ whenever $\xi \in U$. Hence, for each $k \ge 0$,

$$C_k = \{x : a_m(x) \le k\}$$

is weakly closed. Now

$$X(m, n, r) \subset \{x : a_m(x) = n\} = C_n \setminus C_{n-1}$$

for $n \geq 1$. Thus it is sufficient to prove that $X(m,n,r)$ is weakly relatively closed in $\{x : a_m(x) = n\}$.

Let $e \in \{x : a_m(x) = n\} \setminus X(m,n,r)$. Then

$$\left\| e - \sum_{j=1}^{n} r_j T_{\alpha(e,j)} e \right\| > \varepsilon.$$

Write

$$W = \left\{ \xi \in X : \left\| \xi - \sum_{j=1}^{n} r_j T_{\alpha(e,j)} \xi \right\| > \varepsilon \right\}.$$

Then W is a weakly open neighbourhood of e. As above, choose U to be a weakly open neighbourhood of e such that $A_m(\xi) \supset A_m(e)$ whenever $\xi \in U$. Then, if $\xi \in U \cap W \cap \{x : a_m(x) = n\}$, we must have $A_m(\xi) = A_m(e)$ and therefore

$$\left\| \xi - \sum_{j=1}^{n} r_j T_{\alpha(\xi,j)} \xi \right\| = \left\| \xi - \sum_{j=1}^{n} r_j T_{\alpha(e,j)} \xi \right\| > \varepsilon,$$

and $\xi \notin X(m,n,r)$. This proves that $X(m,n,r)$ is weakly relatively closed in $\{x : a_m(x) = n\}$, as required.

For each γ in Γ let X_γ be the norm closure of $T_\gamma X$ in X. By condition (iv), each X_γ, $\gamma \in \Gamma$, is σ-fragmented. Write

$$Y = c_0\{X_\gamma : \gamma \in \Gamma\}$$

for the c_0 sum of the Banach spaces X_γ, $\gamma \in \Gamma$. By Theorem 6.1 of [8] the space Y is σ-fragmented and is σ-fragmented using weakly closed sets in the case when each X_γ, $\gamma \in \Gamma$, is σ-fragmented using weakly closed sets. The map $R : X \to Y$ defined by

$$Rx = (T_\gamma x : \gamma \in \Gamma)$$

is a linear map from X to Y. Further, for x in X,

$$\|Rx\| = \sup\{\|T_\gamma x\| : \gamma \in \Gamma\} \leq \sup\{\|T_\gamma\| \cdot \|x\| : \gamma \in \Gamma\} \leq \|x\|.$$

So R is bounded.

For fixed $m \geq 1$, $n \geq 1$ and $r \in \mathbb{Q}^n$, $r \neq 0$, we take

$$\delta = \delta(m,n,r) = \varepsilon/\|r\|_1 \quad \text{with} \quad \|r\|_1 = |r_1| + \ldots + |r_n|.$$

Since Y is σ-fragmented, we can write $Y = \bigcup_{k=1}^{\infty} Y_k$, with $Y_k = Y(m,n,r,k)$, for $k \geq 1$, σ-fragmented down to δ, that is, with the property that each non-empty subset of Y_k has a non-empty relatively open subset of diameter less than δ. Further, when the spaces X_γ, $\gamma \in \Gamma$, are σ-fragmented using weakly closed sets, the sets Y_k can be taken to be weakly closed in Y.

We show that, for $k \geq 1$, the set $X(m,n,r) \cap R^{-1}(Y_k)$ is fragmented down to 7ε. Consider any non-empty subset C of this last set. Since $\emptyset \neq R(C) \subset Y_k$, there is a weakly open subset U of Y such that $R(C) \cap U \neq \emptyset$

but $\text{diam}(R(C) \cap U) < \delta$. Choose a point c of C in the weakly open set $V = R^{-1}(U)$ that necessarily meets C. Now we can choose a weakly open neighbourhood W of c with

$$A_m(\xi) \supset A_m(c) \quad \text{and} \quad a_m(\xi) \geq a_m(c) = n,$$

for all ξ in W. Consider any ξ in $X(m,n,r) \cap R^{-1}(Y_k) \cap V \cap W$. Then $a_m(\xi) = a_m(c) = n$ and $A_m(\xi) = A_m(c)$. Further, $R\xi \in Y_k \cap U$ and $Rc \in Y_k \cap U$, so that $\|R\xi - Rc\| < \delta$. Hence $\|T_\gamma \xi - T_\gamma c\| < \delta$ for γ in Γ and, in particular, for $\gamma \in A(\xi) = A(c)$. Thus

$$\left\| \sum_{j=1}^{n} r_j T_{\alpha(c,j)} \xi - \sum_{j=1}^{n} r_j T_{\alpha(c,j)} c \right\| < \|r\|_1 \delta \leq \varepsilon.$$

Since ξ and c belong to $X(m,n,r)$ we also have

$$\left\| \xi - \sum_{j=1}^{n} r_j T_{\alpha(\xi,j)} \xi \right\| \leq \varepsilon, \quad \left\| c - \sum_{j=1}^{n} r_j T_{\alpha(c,j)} c \right\| \leq \varepsilon,$$

and noting $\alpha(\xi,j) = \alpha(c,j)$ for $1 \leq j \leq n$, we have $\|\xi - c\| < 3\varepsilon$. Hence

$$\text{diam}(X(m,n,r) \cap R^{-1}(Y_k) \cap V \cap W) \leq 6\varepsilon,$$

and $X(m,n,r) \cap R^{-1}(Y_k)$ is fragmented down to 7ε.

Since, for fixed m, n, r, the sets $R^{-1}(Y_k)$, $k \geq 1$, cover X, and ε may be arbitrarily small, it follows that X is σ-fragmented. Further, when the spaces X_γ, $\gamma \in \Gamma$, are σ-fragmented using weakly closed sets, the space X is σ-fragmented using differences of weakly closed sets.
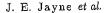
Proof of Corollary 1. For each γ, $0 \leq \gamma < \Gamma$, write $T_\gamma = P_{\gamma+1} - P_\gamma$. Then, by (ii), T_γ is a projection, and hence $T_\gamma(X) = \{x \in X : T_\gamma x = x\}$ is necessarily closed. From [13, Lemma 2], on noting that the condition

$$\sup\{\|T_\gamma\| : 0 \leq \gamma < \Gamma\} < \infty$$

suffices in place of the condition $\|T_\gamma\| = 1$, $0 \leq \gamma < \Gamma$, for the proof in [13], it follows that $\{T_\gamma : 0 \leq \gamma < \Gamma\}$ satisfies the conditions (i) and (ii) of Theorem 2. Hence the corollary follows from Theorem 2.

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Volume approximation of convex bodies by polytopes—a constructive method

by

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Abstract. Algorithms are given for constructing a polytope P with n vertices (facets), contained in (or containing) a given convex body K in $\mathbb{R}^d$, so that the ratio of the volumes $|K \setminus P|/|K|$ (or $|P \setminus K|/|K|$) is smaller than $f(d)/n^{2/(d-1)}$.

1. Introduction. This paper deals with constructive approximation of general convex bodies by polytopes, in the volume-difference sense. Specifically, given a convex body (compact, convex set with non-empty interior) K in $\mathbb{R}^d$, we intend to construct a polytope P contained in K (or containing K) so that the quotient of volumes

$$(1.1) \qquad \frac{|K \setminus P|}{|K|} \qquad \left(\text{or } \frac{|P \setminus K|}{|K|} \right)$$

will be small. (The notation $|A|$ for a measurable subset A of $\mathbb{R}^d$ is used here to denote the k-dimensional volume of A, where k is the dimension of the minimal flat containing A.)

There exists a large body of results concerning approximation of convex bodies by polytopes. We refer the reader to the surveys [5] and [6] by Gruber for information on this subject.

It was proved by Bronshteĭn and Ivanov [2] (cf. also results by Dudley [3] and Betke and Wills [1]) that for any convex body K contained in the Euclidean unit ball B_2^d of $\mathbb{R}^d$ and every sufficiently large positive integer n, there exists a polytope Q, containing K, with at most n vertices, whose distance from K in the Hausdorff metric is less than $c/n^{2/(d-1)}$, where c is an absolute constant. It is easy to check that the proof in [2] provides also