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Domains of integral operators

by

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Abstract. It is shown that the proper domains of integral operators have separating duals but in general they are not locally convex. Banach function spaces which can occur as proper domains are characterized. Some known and some new results are given, illustrating the usefulness of the notion of proper domain.

1. Introduction. We consider linear integral operators of the form

$$(1.1) \quad Ku(t) = \int_S k(t, s)u(s) ds, \quad t \in T,$$

where S, T are σ -finite measure spaces with measures denoted by ds, dt , and k —the kernel of K —is a measurable function on $T \times S$.

As usual, we denote by $L^0(S)$ and $L^0(T)$ the spaces of all measurable, finite a.e. functions on S and T , respectively, and we consider K as an operator from $L^0(S)$ to $L^0(T)$. The *proper domain* of K , D_K , is the largest subspace of $L^0(S)$ for which the integral in (1.1) is defined as the Lebesgue integral and is finite a.e., i.e.,

$$D_K = \left\{ u \in L^0(S) : |K|u(t) := \int_S |k(t, s)||u(s)| ds < \infty \text{ a.e.} \right\}.$$

We insist on the adjective *proper* as distinguished from *extended*. The extended domain of an integral operator is the largest solid subspace of L^0 to which the operator K can be extended by continuity (from D_K). For details we refer to [AS] and to the remarks in Section 5.

The space L^0 is equipped with the complete metric vector topology of convergence in measure on all subsets of finite measure; this topology, when

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convenient, may be defined by the F -norm

$$(1.3) \quad \varrho_X(u) = \int_X \Phi(|u(x)|)\phi(x) dx, \quad X = S \text{ or } T,$$

where $\phi > 0$ a.e. is in $L^1(X)$ and $\Phi : [0, \infty) \rightarrow [0, 1]$ is a continuous, increasing and subadditive function such that $\Phi(0) = 0$. The usual choices of Φ are $\Phi(u) = u/(1+u)$, $\Phi(u) = \max\{u, 1\}$; we find it convenient to use also $\Phi(u) = 1 - e^{-u}$.

We recall that a subset A of L^0 is *solid* (in L^0) if for every function $u \in A$, A contains the order interval $[u] = \{v \in L^0 : |v| \leq |u| \text{ a.e.}\}$. A topological vector subspace of L^0 is *solid* (the term locally solid is also used in this context) if its topology is defined by a base of neighborhoods of the origin which are solid. Thus the adjective “solid” refers to both the space as a set and to its topology. An F -norm ϱ in L^0 is *solid* (or monotone) if $\varrho(v) \leq \varrho(u)$ whenever $|v| \leq |u|$ a.e. Clearly, (1.3) defines a solid F -norm on L^0 and the space L^0 is solid (by default).

The notion of solidity can be introduced similarly with L^0 replaced by any vector lattice; however, we do not need this more general setting.

We will need the following proposition.

PROPOSITION 1.1 ([Dr]). *Suppose that L is a solid metric vector space algebraically included in $L^0(S)$. Then the inclusion $L \subset L^0(S)$ is continuous.*

The proper domain D_K is equipped with the topology which is defined by the solid F -norm

$$(1.4) \quad \varrho_K(u) = \varrho_S(u) + \varrho_T(|K||u|).$$

With this topology, D_K is a solid complete vector subspace of $L^0(S)$. Also, $K : D_K \rightarrow L^0(T)$ is continuous. For the proof and for a further discussion, we refer to [AS].

We note at this point, to avoid possible confusion, that the topology of D_K is not the graph topology of the operator K (or $|K|$). The latter topology is usually not complete and the abstract completion of D_K in this topology cannot be identified with a subspace of L^0 . These difficulties appear already in the case of the operator with kernel $k(t, s) = 1$ and the space S which does not reduce to finitely many atoms (i.e., $Ku = \int_S u ds$ and the image of K , KD_K , consists of constant functions).

Our interest in the space D_K is motivated by the conviction that this space is basic for the understanding of the integral operator K . Because of its simplicity and generality, the use of the proper domain streamlines a number of results obtained by other—sometimes more sophisticated—means. Several examples to illustrate this claim are given in Section 5.

We remark that proper and extended domains can also be introduced for operators other than integral—we return to this theme in Section 5.

We now outline briefly the questions to be addressed in this paper.

Since L^0 is locally convex only if the underlying measure space is purely atomic, the topology of D_K cannot, in general, be expected to be locally convex: for instance, when $k = 0$ —a case excluded throughout this paper—we have $D_K = L^0$.

If T is purely atomic, then D_K is defined by finiteness of each of the seminorms $u \rightarrow \{|K||u|(t) : t \in T\}$. This family of seminorms defines the topology of D_K and this topology is locally convex. Since T is countable, D_K is then a countably weighted L^1 space.

In general, the definition (1.2) appears, at least formally, to be still equivalent to the condition of the finiteness of the family of weighted L^1 seminorms $u \rightarrow |K||u|(t)$, $t \in T$. If the quantifier “a.e.” in (1.2) was replaced by “for every t ” (as can be done in many concrete examples), then D_K would be locally convex by the definition.

The aim of this note is to clarify what, if anything, remains of these two propositions in a general situation.

On the positive side, we prove that D_K coincides piecewise with L^1 —a notion to be made precise below—and, as a consequence, D_K has a separating dual; this is a left-over convexity property. We also prove that D_K is a Banach space if and only if it is a weighted L^1 .

On the other hand, we show by means of three examples that even for some “very nice” kernels k , D_K need not be locally convex. This seems to emphasize a claim, already made in [R], that F -normed spaces (rather than Banach or locally convex spaces) provide a natural framework for the study of integral operators.

In the concluding remarks in Section 5, we elaborate somewhat on our motivation for the study of domains of integral operators by giving a few examples of the usefulness of this concept. We also make some simple remarks about the Zak transform (which is not an integral operator!); we characterize its domains—proper and extended. It turns out that the proper domain of the Zak transform has a trivial dual and thus does not inherit the property which for proper domains of integral operators is established in Section 2.

Throughout this paper, by a subset of a measure space we always mean a measurable subset, by a function we mean a measurable function and we use the symbol $|E|$ to denote the measure of the set E and 1_E to denote the characteristic function of E .

2. D_K is piecewise L^1 . If $D_K = \{0\}$ or if $D_K = L^0(S)$, then there is not much more that one can say about D_K and we refer to these two cases as trivial.

In the first case we say that K is *singular*.

The second case occurs if and only if for almost every $t \in T$, the set $\{s : k(t, s) \neq 0\}$ consists of finitely many atoms ([AS], Th. 5.2i) or is empty when S contains no atoms.

To avoid the two trivial cases or variants thereof (e.g. the partly singular case, see [AS]), we impose the following conditions on K .

$$(2.1) \quad \text{For } u \in D_K, |K||u| = 0 \text{ implies } u = 0.$$

$$(2.2) \quad \text{There is a function } f \in D_K \text{ such that } f > 0 \text{ a.e. on } S.$$

Condition (2.1) is easily translated into an equivalent condition on the kernel k .

$$(2.1') \quad |\{t \in T : k(t, s) \neq 0\}| > 0 \quad \text{for a.e. } s \in S.$$

Remarks. 1) (2.1) is equivalent to the condition that the semi- F -norm

$$(2.3) \quad D_K \ni u \rightarrow \varrho_T(|K||u|)$$

is actually an F -norm. Since with this F -norm D_K is a solid metric subspace of L^0 , it follows from Proposition 1.1 that the F -norm (2.3) defines on D_K a topology stronger than that of L^0 and hence the F -norms defined in (2.3) and in (1.4) are equivalent on D_K .

2) If K satisfies (2.2), then K is called *nonsingular* ([AS]). There is no simple translation of (2.2) into a corresponding property of the kernel k .

We now illustrate the significance of the conditions (2.1) and (2.2) in the context of convolution operators.

In this case, $T = S = \mathbb{R}$ with the Lebesgue measure (or more generally $T = S$ is a locally compact group with the Haar measure) and $k(t, s) = \kappa(t - s)$, where κ is a function of the single variable. In other words,

$$Ku(t) = \kappa * u(t) = \int_{\mathbb{R}} \kappa(t - s)u(s) ds.$$

PROPOSITION 2.1. *Suppose that K is an operator of convolution. Then*

(i) K satisfies (2.1) if and only if $\kappa \neq 0$ on a set of positive measure.

(ii) K satisfies (2.2) (K is nonsingular) if and only if $\kappa \in L^1_{loc}$. In this case $D_K \subset L^1_{loc}$.

(i) is obvious, since for every s the set appearing in (2.1') is the translate $s + S'$ of $S' = \{\sigma \in \mathbb{R} : \kappa(\sigma) \neq 0\}$.

For the sake of completeness, we recall a short proof of (ii). We may assume that $\kappa \geq 0$.

Because of the commutativity of the convolution product, the second statement of (ii) follows from the first. The "if" part is immediate—if κ is in L^1_{loc} , then every L^1 function with compact support is in D_K . On the other hand, if K is nonsingular, then we can find a function $f > 0$ a.e.

such that $Kf < \infty$ a.e. We can next find a function $g > 0$ a.e. such that $\int_{\mathbb{R}} g(t)Kf(t) dt < \infty$. By the Fubini theorem, the last integral can be written in the form $\int_{\mathbb{R}} (\int_{\mathbb{R}} (f(s - t)g(t) dt) \kappa(s) ds$. Replacing f by $\min(f, f_1)$ and g by $\max(g, 1)$, where $f_1 \in L^1$, $f_1 > 0$ a.e., we may assume that $f \in L^1$ and that $g \in L^\infty$; then the integral in parentheses defines a continuous function of s which is positive everywhere and it follows that $\kappa \in L^1_{loc}$.

In particular, when $\kappa(t) = t^{-1}$, the corresponding operator (the Hilbert transform) is singular (this is a motivation for our use of the terms singular and nonsingular).

Except for the situations when $k = 0$ or when $D_K = \{0\}$, both conditions (2.1) and (2.2) can be realized by the following modifications of the measure space S .

The case when $D_K \neq \{0\}$ but (2.2) does not hold is discussed in [AS]—in this case S can be replaced by the support of the space D_K , i.e., the complement of the maximal subset of S on which all functions in D_K vanish a.e.

If (2.1) or equivalently (2.1') fails, then S can be replaced by its subset $\{s \in S : |\{t \in T : k(t, s) \neq 0\}| > 0\}$, the support of the (possibly extended valued) measurable function $s \rightarrow |\{t \in T : k(t, s) \neq 0\}|$.

From now on we assume that these modifications have been made and that K satisfies the conditions (2.1) and (2.2).

Since the definition of D_K involves only the operator $u \rightarrow |K||u|$, we simplify notations by assuming that, unless otherwise stated, $k \geq 0$.

As usual, we denote by $u|E$ the restriction of a function u to a set E .

PROPOSITION 2.2. *Suppose that K satisfies (2.1). Then there is a sequence S'_n of subsets of S with $S'_n \uparrow S$ such that $u|S'_n \in L^1(S'_n)$ for every $u \in D_K$ and for every $n = 1, 2, \dots$*

Proof. Replacing the measures ds and dt with equivalent finite measures and suitably modifying k , we may assume that $|T|, |S| < \infty$.

For $n = 1, 2, \dots$ we let $S'_n = \{s \in S : |\{t : k(t, s) > 1/n\}| > 1/n\}$; by the hypothesis (in the form of (2.1')), the sequence S'_n increases to S (it may happen that S'_n is empty for a finite number of indices n). If $u \in D_K$, $u \geq 0$, then there exists a sequence $T_m \uparrow T$ (depending on u) such that $\int_{T_m} Ku(t) dt < \infty$ for $m = 1, 2, \dots$. Denote by $T_n(s)$ the set $\{t : k(t, s) > 1/n\}$.

For fixed n , we now choose m so that $|T \setminus T_m| < 1/(2n)$. Then $|T_m \cap T_n(s)| > 1/(2n)$ for every $s \in S'_n$, and by the Fubini theorem,

$$\int_{T_m} Ku(t) dt \geq \frac{1}{n} \int_{S'_n} \int_{T_m \cap T_n(s)} u(s) dt ds \geq \frac{1}{2n^2} \int_{S'_n} u(s) ds,$$

and $u|S'_n \in L^1(S'_n)$, as claimed.

PROPOSITION 2.2'. Suppose that K satisfies (2.2). Then there is a sequence S''_n of subsets of S with $S''_n \uparrow S$ such that for $n = 1, 2, \dots$ and for every u in $L^1(S''_n)$ the function u extended by 0 to $S \setminus S''_n$ is in D_K .

Proof. Let f be as in (2.2). Then there exists $g \in L^0(T)$, $g > 0$ a.e., such that $\int_T gKf dt < \infty$. It follows from the Fubini theorem that $K^*g(s) := \int_T k(t,s)g(t) dt < \infty$ a.e. (g is in the proper domain of the transposed operator K^*). We can now define $S''_n = \{s \in S : K^*g(s) < n\}$ (it may happen that S''_n is empty for a finite number of indices n). If $u \in L^1(S''_n)$, $u \geq 0$, then

$$n \int_{S''_n} u ds \geq \int_{S''_n} \int_T k(t,s)g(t) dt ds = \int_T g(t) dt \int_{S''_n} k(t,s)u(s) ds,$$

and $\int_{S''_n} k(t,s)u(s) ds < \infty$ a.e., as claimed.

Letting $S_n = S'_n \cap S''_n$, where S'_n and S''_n are the sets constructed in Propositions 2.2 and 2.2', we get the following.

THEOREM 2.3. Suppose that K satisfies (2.1) and (2.2). Then there exists a sequence S_n of subsets of S with $S_n \uparrow S$ such that $D_K|_{S_n} = L^1(S_n)$ for $n = 1, 2, \dots$. In particular, the continuous linear functionals on D_K separate its points.

Remarks. 1) In the context of Theorem 2.3, we can consider the function space $L^1(\{S_n\})$ defined by $L^1(\{S_n\}) = \{u \in L^0 : u|_{S_n} \in L^1(S_n) \text{ for } n = 1, 2, \dots\}$. With the obvious sequence of seminorms, this is a locally convex F -space which by Theorem 2.3 contains D_K continuously. Also, if $k(t,s) = 1_{S_t}(s)$, $t = 1, 2, \dots$, then $D_K = L^1(\{S_n\})$. However, as will become clear from the examples in Section 3, the inclusion $D_K \subset L^1(\{S_n\})$ is in general proper.

2) Replacing the increasing sequence S_n with the partition $S_1, S_2 \setminus S_1, S_3 \setminus S_2, \dots$ we can obtain an analogue of Theorem 2.3 involving a partition of S , rather than a sequence of subsets increasing to S .

As corollaries to Theorem 2.3, we mention the following results which support the opinion expressed in Section 1 concerning the usefulness of the concept of proper domain.

Theorem 2.3 implies that if $0 \leq p < 1$ and if S is nonatomic then L^p cannot be contained in the proper domain of an integral operator (except when $k = 0$ and $p = 0$). For $p = 0$ this was observed in [AS]; the argument given there is valid for $0 \leq p < 1$ (and also shows that L^p cannot be contained in the extended domain of an integral operator).

An independent proof of the preceding remark, for $0 < p < 1$, is given in [Sc].

These results were generalized by de Pagter [P] as follows.

Let M and L be solid vector lattices, $M \subset L^0(S)$, $L \subset L^0(T)$ and let K be an absolute integral operator, i.e., $|K| : M \rightarrow L$. Then the order continuous functionals (normal integrals) separate points of M .

The last statement of Theorem 2.3 supersedes de Pagter's result.

Moreover, for $p \geq 1$ we get the following remark. If S is not purely atomic and $p > 1$, then $D_K \neq L^p(S)$.

We will see in Section 4 that the preceding condition on S is not needed, except for the trivial case when S consists of finitely many atoms (hence L^0 is finite-dimensional).

We conclude this section with another observation concerning conditions (2.1) and (2.2).

As it is of general interest to see how various properties of K affect properties of the transposed operator K^* , we mention the following result.

K is nonsingular (i.e., satisfies (2.2)) if and only if the transposed operator K^* with kernel $k^*(s,t) = k(t,s)$ is nonsingular (see [AS]). If this is the case, then K^* satisfies (2.1) if and only if $Kf > 0$ a.e., where f is the function appearing in (2.2) (i.e., $f \in D_K$, $f > 0$ a.e.).

Here is a short proof. Sufficiency: Suppose that $Kf > 0$ a.e. and choose $g > 0$ a.e., $g \in D_{K^*}$, such that $\int_T gKf dt < \infty$. Suppose now that $K^*v = 0$ for some $v \geq 0$. Let $v_c = \min(cg, v)$, $c > 0$. We see then that $K^*v_c = 0$ and that $\int_T v_c Kf dt = 0$; hence $v_c = 0$ for all $c > 0$ and therefore $v = 0$. On the other hand, if K^* satisfies (2.1), then $S_t = \{s : k(t,s) > 0\}$ is of positive measure a.e. and

$$Kf(t) = \int_{S_t} k(t,s)f(s) ds > 0 \quad \text{whenever } |S_t| > 0.$$

3. Examples of non-locally convex domains. We present here three examples illustrating the fact that, in spite of the residual convexity property stated in Theorem 2.3, the proper domains of integral operators satisfying (2.1) and (2.2) need not, in general, be locally convex.

The first example is the simplest and the most direct. It uses the standard argument showing that the topology of L^0 (on a measure space which is not purely atomic) is not locally convex.

The second example deals with the convolution operator with a (smooth) L^1 kernel.

In the third example, we present a construction of an operator whose proper domain is ℓ^p , $0 < p < 1$. The construction is related to classical results on positive embeddings of ℓ^p in L^0 and may be of independent interest.

EXAMPLE 3.1. We take $S = \{1, 2, \dots\}$ with the counting measure and $T = [0, 1]$ with the Lebesgue measure. For a sequence of integers m_n such that $m_{n+1} - m_n \rightarrow \infty$ as $n \rightarrow \infty$, we consider the sequence of partitions $\{I_{m_{n+1}}, \dots, I_{m_{n+1}}\}$ of T into (half open) intervals of equal length $1/(m_{n+1} - m_n)$, $n = 1, 2, \dots$, and define $k(t, s) = 1_{I_n}(t)$. The topology of the sequence space D_K is defined by the F -norm $\{u(s)\} \rightarrow \varrho_T(K|u|)$, where ϱ_T is as in (1.3). It is easy to see that D_K is not locally convex: the sequences $u_i(s) = (m_{n+1} - m_n)\delta_{is}$, $q = m_n + 1, \dots, m_{n+1}$ (with δ_{is} denoting the Kronecker symbol), have F -norms as small as we please if n is sufficiently large, yet their arithmetic means

$$(m_{n+1} - m_n)^{-1}(u_{m_{n+1}} + \dots + u_{m_{n+1}})$$

are identically 1 on T .

Remarks. 1) The above example can be adapted to a nonatomic situation: if $S = \mathbb{R}$, then we let $K(t, s) = 1_{I_1}(t)1_{[t, t+1)}(s)$.

2) One natural choice of m_n is $m_n = 2^n$ (the Haar scale)—the example so obtained appears to be promising for a further study of D_K . It would be of interest to describe D_K and its topology directly in terms of its elements, rather than in terms of their images by K . We note here that D_K cannot be described by a simple growth condition and that it contains sequences of arbitrarily rapid growth, provided they contain sufficiently many zeros.

3) There are several ways of producing oscillating kernels with modulus equal to the kernel $k(t, s)$ defined above. One of them is to replace $k(t, s)$ by $(-1)^s k(t, s)$; another one (still assuming that $m_n = 2^n$) is to let $k_1(t, s) = 1$ on $[2^{-m_r}, 2^{-m_r} + 2^{-m-1})$ and $k_1(t, s) = -1$ on $[2^{-m_r} + 2^{-m-1}, 2^{-m}(r+1))$ (thus $k_1(t, s)$, $s = 1, 2, \dots$, is the orthogonal Haar sequence). In each of these two cases, the argument above shows that the extended domain of the operator K cannot be locally convex. This observation would be of some interest if we knew that the extended domain in one or in both of the two examples was larger than the proper domain. It is an interesting problem to determine the extended domains in these two (and in other similar) cases.

For the discussion of the next example we have to recall the following known result which will also be used in Section 4.

THEOREM MN (Maurey–Nikishin, [N]). *Suppose that $|T| < \infty$ and that C is a bounded convex set in $L_+^0(T) = \{u \in L^0(T) : u \geq 0\}$. Then for every $\varepsilon > 0$ there exist a subset T_ε of T and a number $M_\varepsilon > 0$ such that for all $f \in C$, we have $\int_{T_\varepsilon} f dt \leq M_\varepsilon$.*

For a σ -finite space T , Theorem MN implies the following:

THEOREM MN'. *For a set C as in Theorem MN, there exist sequences T_n and M_n with $|T_n| < \infty$, $T_n \uparrow T$, and $M_n > 0$ such that $\int_{T_n} f dt \leq M_n$ for all $f \in C$ and all n .*

EXAMPLE 3.2. Given three sequences $\delta_m \rightarrow 0$, $\varepsilon_m \rightarrow 0$, $\eta_m \rightarrow \infty$ as $|m| \rightarrow \infty$ ($m = 0, \pm 1, \pm 2, \dots$), we denote by I_n the characteristic function of the interval $[n - \delta_n, n + \delta_n]$ and consider (as in Section 2) the operator of convolution $Ku = \kappa * u$ with kernel $\kappa = \sum_{n=-\infty}^{\infty} I_n$. For an integer $r > 1$, define $f^r = \sum_{m=-\infty}^{\infty} \eta_m J_m$, where J_m denotes the characteristic function of the interval $[m/r - \varepsilon_m, m/r + \varepsilon_m]$. The sequences $\delta_m, \varepsilon_m, \eta_m$ will be chosen below.

The support of a convolution being the sum of the supports of its factors, $I_n * J_m$ is 0 outside of the interval

$$(3.2) \quad [n + m/r - \delta_n - \varepsilon_m, n + m/r + \delta_n + \varepsilon_m].$$

It follows that the (double) sum $\kappa * f^r(t)$ contains an infinite number of nonzero terms only if t is a fraction of the form q/r with q integer. Hence the sum $\kappa * f^r$ is finite outside of a countable set and $f^r \in D_K$. On the other hand, if I is any open interval containing a fraction q/r , then there is an index $n(I)$ such that for $|n| \geq n(I)$ the intervals in (3.2) with $n + m/r = q/r$ are all contained in I . The integral of a convolution being the product of the integrals of its factors, we have

$$\int_I \kappa * f^r dt \geq \sum_{|n| \geq n(I)} \eta_{q-nr} \varepsilon_{q-nr} \delta_n.$$

Choosing $\delta_n = 2^{-n} \varepsilon_n = 2^{-n}/r$ and $\eta_n = 2^{nr}$, we see that the sum on the right hand side is infinite for any choice of $n(I)$. It follows that $\kappa * f^r$ is not integrable over any interval containing a fraction of the form q/r .

We now show that the existence of such a function $f^r \in D_K$ implies that D_K cannot be locally convex.

To this end, let $u_\tau(s) = u(s + \tau)$, $u \in L^0(S)$, $\tau \in [0, 1]$. It is easy to see that the function $[0, 1] \ni \tau \rightarrow u_\tau \in L^0$ is continuous. Let now $u \in D_K$, $u \geq 0$. Since $K(u_\tau) = (Ku)_\tau$, the function $\tau \rightarrow u_\tau \in D_K$ is also continuous and it follows that the set $U = \{u_\tau : \tau \in [0, 1]\}$ is bounded in D_K (it is actually compact). Suppose now that D_K is locally convex. Then the convex hull of U , $c[U]$, is bounded in D_K , and so is its image $K(c[U])$. Applying Theorem MN to the set $K(c[U]) \subset L_+^0(T)$ (in fact we use the resulting estimate only for functions in KU), we conclude that there is a set T' of positive measure and a constant M such that $\int_{T'} Ku_\tau(t) dt \leq M$ for all $\tau \in [0, 1]$. Integrating with respect to τ and using the Fubini theorem, we conclude that $\int_{T'} \int_0^1 Ku(t + \tau) d\tau dt \leq M$. It follows that for every $u \in D_K$, Ku is integrable over an interval of length 1 and since any such interval contains a fraction of the form q/r , this leads to a contradiction when $u = f^r$.

Remarks. 1) Choosing a sequence $\lambda_r > 0$, $r = 2, 3, \dots$, such that $f = \sum \lambda_r f^r$ is convergent in D_K (here we use the completeness of D_K), we can construct a function $f \in D_K$ such that $Ku = \kappa * f$ is not integrable over any interval.

2) By the construction, the function κ belongs to L^1 . By “rounding off the corners of its graph” one can modify κ so as to make it arbitrarily smooth.

3) The above example should be compared with the observation recorded already in Section 1. The proper domain of the convolution operator with kernel κ which is bounded and has compact support is L^1_{loc} (which, of course, is locally convex).

In the third example we make use of p -stable independent random variables constructed by means of the following classical theorem of Bernstein which we quote here for the sake of completeness; for details we refer to [F].

A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is said to be *completely monotone* (c.m.) provided it is infinitely differentiable and satisfies the condition $(-1)^n \psi^{(n)}(\lambda) \geq 0$ for all $\lambda > 0$ and for all $n = 0, 1, 2, \dots$

THEOREM. *A function $\psi(\lambda)$, $\lambda \geq 0$, is the Laplace transform of a probability distribution F on $[0, \infty)$, $\psi(\lambda) = \int_0^\infty e^{-\lambda x} dF(x) = L(dF)(\lambda)$, if and only if ψ is completely monotone and $\psi(0) = 1$.*

If $\psi(\lambda) = e^{-\phi(\lambda)}$, then ψ is c.m. if the derivative ϕ' is c.m. This is, for instance, the case when $\phi(\lambda) = \lambda^p$, $0 < p \leq 1$.

This choice of ϕ gives rise to the following

EXAMPLE 3.3. For ψ and ϕ we use the Bernstein theorem to conclude the existence of a probability distribution F such that $\exp(-\lambda^p) = L(dF)(\lambda)$. By a standard argument we construct then a sequence $\{k(t, s) : s = 1, 2, \dots\}$ of nonnegative independent random variables on $[0, 1]$ with distribution F ; then $\int_0^1 e^{-\lambda k(t, s)} dt = \exp(-\lambda^p)$ for $s = 1, 2, \dots$. We consider the integral operator K with kernel $k(t, s)$, where $S = \{1, 2, \dots\}$ and $T = [0, 1]$. Let $\mathcal{J} = \{\{u(s)\} : u(s) \geq 0, \{s : u(s) > 0\} \text{ is finite}\}$. If $\{u(s)\} \in \mathcal{J}$, then, because of stochastic independence,

$$(3.4) \quad \int_0^1 (1 - e^{-Ku(t)}) dt = \int_0^1 (1 - e^{-\sum u(s)k(t, s)}) dt = 1 - \exp\left(\sum u(s)^p\right),$$

and it follows that the identity mapping $\ell^p \supset \mathcal{J} \rightarrow \mathcal{J} \supset D_K$ is an isometry if D_K is equipped with the appropriate F -norm (note that $u \rightarrow \varrho_T(|K||u|)$, with ϱ_T as in (1.3), is an F -norm on D_K because of (2.1)) with $\Phi(x) = 1 - e^{-|x|}$ and if ℓ^p is equipped with the F -norm appearing on the right hand side of (3.4). Since the set \mathcal{J} is dense in D_{K+} (by the dominated convergence

theorem) and also dense in ℓ^p_+ , it follows that $D_{K+} = \ell^p_+$. Both spaces being solid, we conclude that $D_K = \ell^p$.

Remarks. 1) It would be interesting to see if by some other choices of the function ϕ one could construct operators whose domains are sequence spaces other than ℓ^p .

2) The Bernstein theorem shows that the argument above fails for $p > 1$; in the next section we confirm that this example is truly exceptional, and that for $p > 1$ the space ℓ^p cannot occur as a domain of an integral operator.

4. Weighted L^1 spaces. In this section we characterize Banach spaces which can occur as proper domains of integral operators. Our conclusion is that, except for the curious example 3, the only L^p space which may occur as a proper domain is L^1 .

Throughout this section, L is a vector subspace of $L^0(S)$. We recall that the *Köthe dual* L' of L is defined by $L' = \{v \in L^0(S) : \int_S |vu| ds < \infty \text{ for all } u \in L\}$.

If L is a solid Banach space then, with the natural norm, L' is a solid Banach space as well. In this case L' is a closed subspace of the dual space L^* .

We say that L is a *countably weighted L^1* if there is a sequence $g_n \geq 0$, $g_n \in L'$, such that $L = \{u \in L^0 : \int_S g_n |u| ds < \infty \text{ for all } n\}$ and such that $\int_S g_n |u| ds = 0$ for all n implies that $u = 0$ (or equivalently that $\bigcup_n \{s : g_n(s) > 0\} = S$). If L is a countably weighted L^1 , then—with the obvious topology—it is a solid locally convex F -space.

For example, the space $L^1(\{S_n\})$ with $S_n \uparrow S$ and its special case L^1_{loc} are countably weighted L^1 spaces.

A solid Banach space L is a countably weighted L^1 if and only if L is an L^1 with weight.

The “if” part is obvious; to verify the “only if” part, we take the sequence $\{g_n\}$ as in the definition and define $g = \sum \alpha_n g_n$ where the sequence $\alpha_n > 0$ is so chosen as to make the series converge in L' . Then L is seen to be the space L^1 with the weight g .

THEOREM 4.1. *Let K satisfy (2.1) and (2.2). If D_K is a Banach space, then it is an L^1 with weight. Conversely, every L^1 with weight is the domain of some integral operator.*

Proof. The second part is immediate (the assumption that L is a Banach space is not needed). We can take T to be a single point and let $k(t, s) = g(s)$, the function appearing in the definition (or $T = \{1, 2, \dots\}$ and $k(t, s) = g_t(s)$).

To prove the first part, we consider B_+ , the set of all nonnegative functions in the unit ball in D_K . Since D_K is assumed to be a Banach space,

this is a bounded subset of D_K and $C = KB_+$ is a subset of L^0 satisfying the conditions of Theorem MN (recall that $k \geq 0$). We use this theorem to find sequences $T_n \uparrow T$ and $M_n > 0$ such that $\int_{T_n} K u dt \leq M_n$ for all n and for all $u \in B_+$. It follows that the functions $g_n(s) = \int_{T_n} k(t, s) dt$ (which are finite a.e. because of (2.2)) belong to the Köthe dual of D_K and, by the Fubini theorem, that the condition $\int g_n(s)|u(s)| ds < \infty$ for $n = 1, 2, \dots$ implies that $u \in D_K$. Also, if $\int_S g_n(s)|u(s)| ds = 0$ for all n , then $|K||u| = 0$ and, by (2.1), $u = 0$.

We now list some examples of function spaces which are not weighted L^1 and which therefore cannot occur as proper domains of integral operators.

Unless S consists of finitely many atoms, L^p (or ℓ^p) (possibly with weight) is an L^1 with weight if and only if $p = 1$. We already derived a similar conclusion from Theorem 2.3 under the additional assumption that S was nonatomic.

The amalgam space $\ell^p(L^q)$ is an L^1 with weight if and only if $p = q = 1$.

An Orlicz space is an L^1 with weight if and only if it is the L^1 space (i.e., the weight is equal to 1).

$L^1(\{S_n\})$ does appear as the proper domain of an integral operator (see Section 2) and L^1_{loc} is the proper domain of the convolution with any kernel κ which is in L^1 and has a compact support of positive measure.

QUESTION. If D_K is locally convex, is it a countably weighted L^1 ?

5. Concluding remarks. In the next few paragraphs we elaborate on the remarks we made in Section 1 about the importance of the notion of the proper domain.

The following result can be found in [AS].

THEOREM 5.1. *Let $L \subset L^0(S)$ and $L_1 \subset L^0(T)$ be F -spaces with both inclusions continuous. Suppose that $L \subset D_K$ (set theoretic inclusion) and that $KL \subset L_1$. Then the operator $K : L \rightarrow L_1$ is continuous.*

The theorem is an immediate consequence of the completeness of D_K and of the closed graph theorem. It remains valid whenever the closed graph theorem is valid for operators from L to L_1 .

The following are a few examples of known results which are special cases of Theorem 5.1. We formulate only the hypotheses under which these results were obtained, the conclusion in all of them being the same as in the theorem.

Banach ([B], 1922): L and L_1 are Banach spaces satisfying additional properties. This result was an inspiration to the introduction of the proper domain.

Korotkov ([K], 1983): L and L_1 are solid Banach spaces (Banach function spaces) with L_1 satisfying the so-called Gribanov condition (which is superfluous because of Proposition 1.1).

Halmos and Sunder ([HS], 1978): L and L_1 are Hilbert spaces.

Zaanen ([Z], Th. 96.9, 1978; see also [MN]): L and L_1 are ideals (i.e., solid vector subspaces of L^0) which are Banach lattices.

The continuity statements in [Z], [MN] and other literature on kernel operators in Riesz spaces impose usually the condition of regularity. A kernel operator (i.e., an integral operator) is *regular* (or an *absolute kernel operator*) as an operator from a solid space L into another solid space L_1 provided $L \subset D_K$ and $|K|(L) \subset L_1$. The point of Theorem 5.1 is that neither solidity of L and L_1 nor the regularity of K (when L and L_1 are solid) is needed for the conclusion.

We next give one example of a situation where the concept of the proper domain of an integral operator simplifies considerably the original proof of an interesting known result.

As usual, K^* denotes the operator with the transposed kernel $k^*(t, s) = k(s, t)$.

THEOREM (Sunder [Su], 1978). *Let $S = T$ and let K be an integral operator with a kernel $k \geq 0$ such that for some normed solid space L we have $L \subset D_K \cap D_{K^*}$ and K and K^* are bounded operators from L to L . Assume that there is a function $g \in L$ such that $g > 0$ a.e. Then K is a bounded operator from L^2 to L^2 (in particular $L^2 \subset D_K$) and $\|K\|_{L^2} \leq (\|K^*K\|_L)^{1/2}$.*

Proof. By Fubini's theorem ($\|Ku\|_2^2 = \langle Ku, Ku \rangle = \langle K^*Ku, u \rangle \leq \|K^*K\| \|u\|^2$, where $\langle u, v \rangle = \int uv$) it is sufficient to prove that K^*K is a bounded operator from L^2 to L^2 with an appropriate bound for the norm. This is so if K^*K satisfies the well known Schur test which we state here in the special form (see e.g. [Ga] for the general formulation):

Let $S = T$ and let K be an integral operator with positive symmetric kernel. Then $L^2 \subset D_K$ and $K : L^2 \rightarrow L^2$ is bounded if and only if there is a function $\phi \in D_K$ with $\phi > 0$ a.e. and a constant C such that $K\phi \leq C\phi$. Then $\|K\|_{L^2} \leq C$.

To construct such a function ϕ for the operator K^*K (which clearly has a symmetric kernel), we take an arbitrary $\varepsilon > 0$ and we let $\phi = \sum_{n=0}^{\infty} (\|K^*K\|_L + \varepsilon)^{-n} (K^*K)^n g$, where g is the function appearing in the hypotheses of the theorem. This is a series with nonnegative terms whose partial sums form a Cauchy sequence in L (note: we do not assume that L is complete). Proposition 1.1 implies that the series is convergent in $L^0(S)$. The same argument shows that the series $K^*K\phi$ is also convergent in $L^0(S)$.

and it follows that $\phi \in D_{K^*K}$ and that $K^*K\phi \leq (\|K^*K\| + \varepsilon)\phi$, which completes the proof.

The concept of proper (and extended) domain together with some of its properties can be extended to some operators in L^0 which are not necessarily integral. Rather than exploring this idea in a general setting, we restrict our attention to the Zak transform (see e.g. [D]). The point we are making is that the results which were established in the preceding sections fail in this case and are therefore peculiar to integral operators.

The Zak transform (also known as the Weil–Brezin map) plays a role in applications, e.g. in solid state physics and in signal processing. It is not an integral operator but some of the considerations concerning domains of integral operators still remain meaningful for this transform.

The Zak transform Z is defined on functions of the real variable and its range consists of functions defined on the square $Q = (0, 1) \times (0, 1)$; it is given by the formula

$$Zu(s, t) = \sum_{l=-\infty}^{\infty} u(s-l) \exp(2\pi i l t).$$

It is natural to define the proper domain of Z as follows:

$$D_Z = \left\{ u \in L^0(\mathbb{R}) : \sum_{l=-\infty}^{\infty} |u(s-l)| := |Z||u(s)| < \infty \text{ for a.e. } s \in (0, 1) \right\}.$$

Z can be defined in a similar manner in $L^0(\mathbb{R}^n)$, $n > 1$.

As in the case of integral operators, equipped with the F -norm $u \rightarrow \varrho_R(|Z||u|)$, D_Z is a solid F -space and for every F -subspace L of D_Z such that L is continuously included in $L^0(\mathbb{R})$, Z is continuous from L into $L^0(Q)$.

It is clear that D_Z contains all measurable functions with bounded supports; it is also clear that it contains the spaces $L^p(\mathbb{R})$ for $0 < p \leq 1$. Hence there are no continuous linear functionals on D_Z other than zero.

The usefulness of the operator Z lies in the fact that it can be extended to a unitary operator from $L^2(\mathbb{R})$ onto $L^2(Q)$. It is thus meaningful to look for a maximal extension of Z . For the lack of a more general class of spaces with the consistency property, we define such an extension in the class of solid spaces.

As in the case of integral operators, we define the complete solid group norm $\tilde{\varrho}_Z$ on $L^0(\mathbb{R})$ by $\tilde{\varrho}_Z(u) = \varrho_R(u) + \sup\{\varrho_Q(Zv) : v \in D_Z, |v| \leq |u|\}$, and we define the extended domain \tilde{D}_Z as the closure of D_Z in L^0 equipped with this norm.

Equivalently, one could define this extended domain as the completion of D_Z in the weakest solid topology making the operator $Z : D_Z \rightarrow L^0(\mathbb{R})$

continuous; such an abstract completion can then be realized as a subspace of $L^0(\mathbb{R})$.

As in the case of integral operators, \tilde{D}_Z contains continuously all the solid spaces to which Z can be extended by continuity.

We have the following characterization of the extended domain of Z :

$$\begin{aligned} \tilde{D}_Z &= L^0(\ell^2; \mathbb{R}) \\ &= L^0(\ell^2) := \left\{ u \in L^0(\mathbb{R}) : \sum_{l=-\infty}^{\infty} |u(s-l)|^2 =: u_1(s)^2 < \infty \text{ a.e.} \right\}, \end{aligned}$$

the topology of \tilde{D}_Z being defined by the F -norm $u \rightarrow \varrho_{(0,1)}(u_1)$.

To prove the inclusion of $L^0(\ell^2)$ in D_Z , consider any $u \in L^0(\ell^2)$; then for a.e. s in $(0, 1)$ the series $Zu(s, t)$ is convergent in $L^2((0, 1))$ and therefore it is convergent in $L^0(Q)$. It follows from the uniform boundedness principle that Z extends to a continuous operator from $L^0(\ell^2)$ into $L^0(Q)$ and, by the maximality of \tilde{D}_Z , we get the announced inclusion.

The proof of the reverse inclusion is similar to that used for integral operators: if $u \in \tilde{D}_Z$ then for any sequence $\varepsilon_n = \pm 1$ the sum $\sum \varepsilon_n 1_{[n, n+1)} u \in \tilde{D}_Z$ and from the known necessary condition for the unconditional convergence of series in L^0 , we see that $u \in L^0(\ell^2)$.

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σ -fragmented Banach spaces II

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Abstract. Recent papers have investigated the properties of σ -fragmented Banach spaces and have sought to find which Banach spaces are σ -fragmented and which are not. Banach spaces that have a norming M -basis are shown to be σ -fragmented using weakly closed sets. Zizler has shown that Banach spaces satisfying certain conditions have locally uniformly convex norms. Banach spaces that satisfy similar, but weaker conditions are shown to be σ -fragmented. An example, due to R. Pol, is given of a Banach space that is σ -fragmented using differences of weakly closed sets, but is not σ -fragmented using weakly closed sets.

1. Introduction. Let X be a normed vector space and let \mathcal{T} be a locally convex topology on X . We say that (X, \mathcal{T}) is σ -fragmented if, for each $\varepsilon > 0$,

$$(1) \quad X = \bigcup_{k=1}^{\infty} X_k,$$

each set X_k having the property that each of its non-empty subsets has a non-empty relatively \mathcal{T} -open subset of norm diameter less than ε . If \mathcal{S} is a family of subsets of X and the sets X_k in (1) can always be taken from the family \mathcal{S} , we say that (X, \mathcal{T}) is σ -fragmented using sets from \mathcal{S} . We shall be most interested in the case when X is a Banach space, \mathcal{T} is its weak topology and (X, \mathcal{T}) is σ -fragmented; in this case we shall say that X is a σ -fragmented Banach space.

In a series of papers [7–11] we have investigated the properties of σ -fragmented Banach spaces and shown that certain classes of Banach spaces are σ -fragmented and that Banach spaces of some other classes are not σ -fragmented; see also [6, 12, 16, 17]. In this note we show that some further classes of Banach spaces have this property.

We first consider a Banach space X that has an *extended Markushevich basis* $\{x_\gamma : \gamma \in \Gamma\}$. Then after a simple normalization (see [18, p. 673 and p. 691]) $\{x_\gamma : \gamma \in \Gamma\}$ is a family of points of X whose finite linear combina-