

The cancellation law
for inf-convolution of convex functions

by

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Abstract. Conditions under which the inf-convolution of f and g

$$f \square g(x) := \inf_{y+z=x} (f(y) + g(z))$$

has the cancellation property (i.e. $f \square h \equiv g \square h$ implies $f \equiv g$) are treated in a convex analysis framework. In particular, we show that the set of strictly convex lower semicontinuous functions $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ on a reflexive Banach space such that $\lim_{\|x\| \rightarrow \infty} f(x)/\|x\| = \infty$ constitutes a semigroup, with inf-convolution as multiplication, which can be embedded in the group of its quotients.

1. Introduction. Let X be a real locally convex topological vector space and $f, h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper functions, i.e. there exist points where f and h are finite. The *inf-convolution* of f and h at $x \in X$ is defined by

$$(1.1) \quad f \square h(x) := \inf_{y+z=x} (f(y) + h(z)).$$

Many properties of inf-convolution are already known (see e.g. [2, 8]). They have very deep consequences in a variety of nonlinear problems (see [1, 2, 5], see also the Moreau–Yosida approximation); moreover, they seem to be useful tools in integration of subdifferentials (see e.g. [15]). However, the cancellation property for inf-convolution has not been obtained so far for the simple reason that it is not true in general. A simple counterexample for strictly convex functions has been communicated to me by S. Rolewicz.

COUNTEREXAMPLE. Let $f, g, h : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ be as follows:

$$f(t) := \exp(t), \quad g(t) := \exp(2t), \quad h(t) := \exp(-t) \quad \text{for every } t \in \mathbb{R}.$$

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It is easy to calculate that $f \square h(x) = 0$ and $g \square h(x) = 0$ for every $x \in \mathbb{R}$. Thus

$$f \square h \equiv g \square h \text{ does not imply } f \equiv g.$$

This counterexample shows that the cancellation law is not valid in the set of all strictly convex functions on the real line. Thus the problem arises of what properties of functions guarantee the cancellation property. In this paper we provide such conditions (see Theorems 4.2, 4.3, 4.6), but it is still an open question if they are necessary. The conditions, when expressed in set-valued analysis language, mean that the subdifferential mapping ∂h is surjective and the domains of the multifunctions $\partial f, \partial g$ are sufficiently dense in the domains of f and g , respectively. If X is a reflexive Banach space and the functions involved are convex and lower semicontinuous then the Ky Fan inequality allows us to reduce the assumptions on h to the following one: h is strictly convex and $h(x)/\|x\| \rightarrow \infty$ as $\|x\| \rightarrow \infty$. If X is a Banach space and the functions are convex and lower semicontinuous then one can assume that h is uniformly convex.

It is also of interest that we get a new characterization of reflexivity of Banach spaces (see Proposition 3.5). Namely, if for every lower semicontinuous, convex, proper function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ the condition $f(x)/\|x\| \rightarrow \infty$ as $\|x\| \rightarrow \infty$ implies that the operator ∂f is surjective, then X is reflexive.

2. Some facts from convex analysis. The differential properties of convex functions can be quite strange even in finite dimensions (see e.g. [10, 13, 17]). When we are concerned with convex (even continuous) functions on infinite-dimensional spaces we encounter objects which are nowhere Fréchet differentiable or nowhere Gateaux differentiable (for beautiful examples on l^1, l^∞ we refer to Examples 1.14 and 1.21 of [9]). Additionally, the infimum operation is involved which destroys differential properties radically, so it is hard to expect that the “classical differential calculus” can handle our objects. Luckily, there is a subdifferential calculus for convex functions, which is our crucial tool in obtaining the cancellation property. Below we gather some facts on subdifferentials; more details can be found in the references given below.

DEFINITION 2.1. Let X be a topological vector space and $f : X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ be a convex function which is finite at $x \in X$. A continuous linear functional $x^* \in X^*$ (X^* stands for the dual space) is said to be a *subgradient* of f at x if

$$f(y) - f(x) \geq \langle x^*, y - x \rangle \quad \text{for every } y \in X.$$

The set of all subgradients of f at x is called the *subdifferential* of f at x and is denoted by $\partial f(x)$.

Convex functions are known to have nice properties relating to minima (see [2, 6, 8]). Below we cite two of them.

THEOREM 2.2. Let X be a topological vector space and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex proper function (i.e. f is finite at least at one point). Then f has a global minimum at $x \in X$ if and only if $0 \in \partial f(x)$.

THEOREM 2.3. Let X be a reflexive Banach space and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. If $M \subset X$ is a nonempty convex closed bounded subset and $M \cap \text{dom } f \neq \emptyset$, where $\text{dom } f := \{x \mid f(x) < \infty\}$, then there is $x_0 \in M$ such that

$$f(m) \geq f(x_0) \quad \text{for every } m \in M.$$

Proof. The method of proof of Theorem 25.1 of [6] works in this case. ■

The next results are counterparts of known classical results for nondifferentiable functions. The first is the obvious part of the Moreau–Rockafellar theorem (see [8]).

THEOREM 2.4. Let X be a topological vector space and $f, g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex functions, finite at $x \in X$. Then

$$\partial f(x) + \partial g(x) \subseteq \partial(f + g)(x).$$

The lemma below relates to the density of the domain of the subdifferentials in the domain of a given function. It can be obtained directly, repeating the method of proof of the Brøndsted–Rockafellar theorem (see e.g. Theorem 3.18 of [9]). However, in order to avoid a long argument, we just indicate that it is a consequence of a mean value theorem.

LEMMA 2.5. Let X be a Banach space and $g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function, finite at $a \in X$. Then there are $(x_k) \subset X$ and $(x_k^*) \subset X^*$ such that

$$\lim_{k \rightarrow \infty} x_k = a, \quad \lim_{k \rightarrow \infty} g(x_k) = g(a) \quad \text{and} \quad x_k^* \in \partial g(x_k) \quad \text{for every } k \in \mathbb{N}.$$

Proof. This is a simple consequence of Theorem 4.3 of [16] and Proposition 6.5.1 of [3] (see also [11, 12]). ■

Lemma 3.6 of [5] can be simplified to the following one when Proposition 6.5.1 of [3] is taken into account.

LEMMA 2.6. Let X be a Banach space and $f, h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex proper lower semicontinuous functions. If the inf-convolution is exact at x , i.e. there is $\bar{y} \in X$ such that the infimum in (1.1) is attained at \bar{y} and $\bar{z} = x - \bar{y}$, then

$$\partial(f \square h)(x) \subseteq \partial f(\bar{y}) \cap \partial h(\bar{z}).$$

It is of interest that the converse inclusion also holds.

LEMMA 2.7. If X be a Banach space and $f, h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions with $f \square h$ finite at $x \in X$. Then for every y such that

$$(2.1) \quad f \square h(x) = f(y) + h(x - y)$$

we have

$$\partial f(y) \cap \partial h(x - y) \subseteq \partial(f \square h)(x).$$

Proof. Let y satisfy (2.1) and $y^* \in \partial f(y) \cap \partial h(x - y)$. We have

$$f(z) - f(y) \geq \langle y^*, z - y \rangle$$

and

$$h(u - z) - h(x - y) \geq \langle y^*, (u - z) - (x - y) \rangle \quad \text{for all } u, z \in X,$$

hence

$$f(z) + h(u - z) \geq f(y) + h(x - y) + \langle y^*, u - x \rangle,$$

which implies

$$f \square h(u) - f \square h(x) \geq \langle y^*, u - x \rangle,$$

so $y^* \in \partial(f \square h)(u)$. ■

In [3] (see Theorem 3.1.1) we find the Ky Fan inequality, which remains true for the weak topology when X is a reflexive Banach space.

THEOREM 2.8. Let K be a nonempty, weakly compact, convex subset of a reflexive Banach space and $\phi : K \times K \rightarrow \mathbb{R}$ be a function satisfying

(1) $\forall y \in K, x \mapsto \phi(x, y)$ is lower semicontinuous with respect to the weak topology,

(2) $\forall x \in K, y \mapsto \phi(x, y)$ is concave,

(3) $\forall y \in K, \phi(y, y) \leq 0$.

Then there exists $\bar{x} \in K$ which is a solution to

$$\forall y \in K, \quad \phi(\bar{x}, y) \leq 0.$$

3. Auxiliary results. Let us recall that a proper lower semicontinuous convex function h on a Banach space $X, h : X \rightarrow \mathbb{R} \cup \{+\infty\}$, is said to be strictly convex if for all $a, b \in \text{dom } h$ with $a \neq b$ we have

$$h(ta + (1 - t)b) < th(a) + (1 - t)h(b)$$

for every $t \in (0, 1)$, and h is said to be uniformly convex if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$\|a - b\| \geq \varepsilon$ implies

$$h\left(\frac{a + b}{2}\right) \leq \frac{h(a) + h(b) - \delta\|a - b\|}{2} \quad \text{for all } a, b \in \text{dom } h.$$

The lemma below tells us that if h is uniformly convex then the set $\text{argmin } h$ consists of only one point which is not “too far” from the origin. Additionally, the operator ∂h is then injective and surjective.

LEMMA 3.1. Let X be a Banach space and $h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous uniformly convex function. Then

(1) $\lim_{\|x\| \rightarrow \infty} h(x)/\|x\| = \infty$,

(2) there is exactly one point, say $x \in X$, such that $h(x) = \inf_X h$,

(3) for every proper convex lower semicontinuous function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and every $x \in X$ there exists $y \in X$ such that

$$f \square h(x) = f(y) + h(x - y),$$

(4) for every $x^* \in X^*$ there is $x \in X$ such that $x^* \in \partial h(x)$,

(5) for all $x, y \in X$ with $x \neq y$ we have $\partial h(x) \cap \partial h(y) = \emptyset$.

Proof. (1) By Lemma 2.5 there are $x_0 \in X$ and $x_0^* \in X^*$ such that $x_0^* \in \partial h(x_0)$, i.e.

$$(3.1) \quad h(x) - h(x_0) \geq \langle x_0^*, x - x_0 \rangle \quad \text{for every } x \in X,$$

hence

$$\liminf_{\|x\| \rightarrow \infty} h(x)/\|x\| \geq -\|x_0^*\| > -\infty.$$

Define

$$a := \liminf_{\|x\| \rightarrow \infty} h(x)/\|x\|$$

and choose a sequence $(x_n) \subset X$ such that $\lim_{n \rightarrow \infty} h(x_n)/\|x_n\| = a$ and $\lim_{n \rightarrow \infty} \|x_n\| = \infty$. By uniform convexity of h , there is $C > 0$ such that

$$\frac{h(2^{-1}(x_n + x_m))}{\|2^{-1}(x_n + x_m)\|} \leq \frac{h(x_n) + h(x_m)}{\|x_n + x_m\|} - C \frac{\|x_n - x_m\|}{\|x_n + x_m\|}$$

for every fixed n and m large enough, which forces, letting $m \rightarrow \infty$, that $a \leq a - C$, thus $a = \infty$.

(2) Now, let $(y_n) \subset X$ be such that $h(y_n) \rightarrow \inf_X h$. It follows from (3.1) and part (1) that $\inf_X h \in \mathbb{R}$. Now assume that for some $\varepsilon > 0$ there is a subsequence $(y_{n_k}) \subseteq (y_n)$ such that $\|y_{n_k} - y_{n_{k+1}}\| \geq \varepsilon$ for every $k \in \mathbb{N}$. Since h is uniformly convex, there is $\delta > 0$ such that

$$\varepsilon \leq \|y_{n_k} - y_{n_{k+1}}\| \leq \delta^{-1} \left(\frac{h(y_{n_k}) + h(y_{n_{k+1}})}{2} - \inf_X h \right) \quad \text{for every } k \in \mathbb{N},$$

which is impossible since the right hand side tends to zero. Hence (y_n) is a Cauchy sequence and converges to some point $x \in \text{argmin } h$. Since h is uniformly convex there can only be one such point.

(3) is an immediate consequence of (2) and the fact that if f is convex then the function $y \mapsto f(y) + h(x - y)$ is uniformly convex for every $x \in X$.

(4) Take $x^* \in X^*$. By (2) and (3) the set $\text{arg min}(h(\cdot) - \langle x^*, \cdot \rangle)$ is nonempty; let x belong to it. We have

$$h(y) - \langle x^*, y \rangle \geq h(x) - \langle x^*, x \rangle \quad \text{for every } y \in X,$$

so $x^* \in \partial h(x)$.

(5) Suppose that there are $x, y \in X$ with $x \neq y$ and $\partial h(x) \cap \partial h(y) \neq \emptyset$. Hence there is $y^* \in \partial h(x) \cap \partial h(y)$ such that

$$\begin{aligned} h(2^{-1}(x+y)) - h(x) &\geq \langle y^*, 2^{-1}(x+y) - x \rangle, \\ h(2^{-1}(x+y)) - h(y) &\geq \langle y^*, 2^{-1}(x+y) - y \rangle. \end{aligned}$$

Thus $h(2^{-1}(x+y)) \geq 2^{-1}(h(x) + h(y))$, which is impossible. ■

Assertion (3) of Lemma 3.1 is nothing but the statement that the infimum in (1.1) is always attained when h is uniformly convex. Below we provide a criterion for that, relaxing the assumption on h .

DEFINITION 3.2. Let X be a Banach space and $f, h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions and $\alpha \geq 0$. The function f is said to be α -subdifferentially dominated by h if for every pair $(y, y^*) \in X \times X^*$ with $y^* \in \partial f(y)$ there is $r \in \mathbb{R}$ such that the set $S(h, r) := \{x \in X \mid h(x) \leq r\}$ is nonempty, bounded and for every $x \notin \text{int } S(h, r)$ with $h(x) = r$ there is $x_0 \in S(h, r)$ such that

$$-Dh(x; -(x - x_0)) > \langle y^*, x - x_0 \rangle + \alpha \|x - x_0\|,$$

where

$$Dh(x; -(x - x_0)) = \inf_{t>0} \frac{h(x - t(x - x_0)) - h(x)}{t}.$$

Remark 3.3. Note that if $h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function such that

$$\lim_{\|x\| \rightarrow \infty} h(x)/\|x\| = \infty$$

then every proper lower semicontinuous convex function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is α -subdifferentially dominated by h for every $\alpha \geq 0$.

Indeed, let $y^* \in \partial f(y)$. Assume that $h(x_0) \in \mathbb{R}$ and choose $R > 1$ such that

$$\frac{h(z) - h(x_0)}{\|z - x_0\|} \geq \|y^*\| + \alpha + 1 \quad \text{for every } z \in X \text{ such that } \|z - x_0\| \geq R.$$

Let $r := (\|y^*\| + \alpha + 1 + |h(x_0)|)R$ and $h(x) = r$. If $\|x - x_0\| \geq R$ then

$$\frac{h(x) - h(x_0)}{\|x - x_0\|} \geq \|y^*\| + \alpha + 1;$$

otherwise

$$\frac{h(x) - h(x_0)}{\|x - x_0\|} \geq \frac{R(\|y^*\| + \alpha + 1) + |h(x_0)| - h(x_0)}{\|x - x_0\|} > \|y^*\| + \alpha + 1.$$

By convexity we get

$$-\inf_{t \in (0,1]} \frac{h(x - t(x - x_0)) - h(x)}{t\|x - x_0\|} \geq -\frac{h(x_0) - h(x)}{\|x - x_0\|},$$

thus

$$-Dh(x; -(x - x_0)) \geq (\|y^*\| + \alpha + 1)\|x - x_0\|.$$

THEOREM 3.4. Let X be a reflexive Banach space and $f, h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions and $\alpha \geq 0$. If f is α -subdifferentially dominated by h then

(1) for every $(y, y^*) \in X \times X^*$ with $y^* \in \partial f(y)$ there is $r \in \mathbb{R}$ for which $S(h, r)$ is bounded, and for every $l^* \in B(0, \alpha)$ there is $x \in S(h, r)$ such that $y^* + l^* \in \partial h(x)$, where $B(0, \alpha) = \{x^* \in X^* \mid \|x^*\| \leq \alpha\}$,

(2) if $\alpha > 0$ then $f \square h(x) > -\infty$ for every $x \in X$, and for every u such that $f \square h(u)$ is finite there exists $y \in X$ such that $f \square h(u) = f(y) + h(u - y)$, i.e. the inf-convolution is exact at every $x \in \text{dom } f \square h$.

Proof. Take any $y^* \in \partial f(y)$. Since f is α -subdifferentially dominated by h , there exists $r \in \mathbb{R}$ such that the set $K := S(h, r)$ is nonempty, bounded, convex and for every $x \notin \text{int } K$ with $h(x) = r$ there is $x_0 \in K$ such that

$$-Dh(x; -(x - x_0)) > \langle y^*, x - x_0 \rangle + \alpha \|x - x_0\|.$$

Take any $l^* \in B(0, \alpha)$ and define $\phi : K \times K \rightarrow \mathbb{R}$ by

$$\phi(u, v) := \langle y^* + l^*, v - u \rangle - h(v) + h(u) \quad \text{for } (u, v) \in K \times K.$$

By Theorem 2.8 we get $\bar{x} \in K$ such that

$$(3.2) \quad \langle y^* + l^*, v - \bar{x} \rangle \leq h(v) - h(\bar{x}) \quad \text{for every } v \in K.$$

If $y^* + l^* \notin \partial h(\bar{x})$ then there is $\bar{v} \in X$ such that

$$\langle y^* + l^*, \bar{v} - \bar{x} \rangle > h(\bar{v}) - h(\bar{x}),$$

hence for $v_t := t\bar{v} + (1 - t)\bar{x}$, $t \in (0, 1]$, we get

$$\langle y^* + l^*, \bar{v} - \bar{x} \rangle > h(\bar{v}) - h(\bar{x}) \geq \frac{h(v_t) - h(\bar{x})}{t}.$$

Thus, if either $h(\bar{x}) < r$ or $\bar{x} \in \text{int } K$, then by (3.2) we get a contradiction for $t > 0$ small enough. If $h(\bar{x}) = r$, $\bar{x} \notin \text{int } K$, then there is $\bar{x}_0 \in K$ such that

$$(3.3) \quad -Dh(\bar{x}; -(\bar{x} - \bar{x}_0)) > \langle y^*, \bar{x} - \bar{x}_0 \rangle + \alpha \|\bar{x} - \bar{x}_0\|.$$

On the other hand, by (3.2) we get

$$\langle y^* + l^*, \bar{x}_0 - \bar{x} \rangle \leq Dh(\bar{x}; -(\bar{x} - \bar{x}_0)),$$

hence

$$\alpha \|\bar{x}_0 - \bar{x}\| + \langle y^*, \bar{x} - \bar{x}_0 \rangle \geq -Dh(\bar{x}; -(\bar{x} - \bar{x}_0)),$$

which contradicts (3.3). Thus $y^* + l^* \in \partial h(\bar{x})$ and (1) is proven.

Now let $\alpha > 0$ and $x \in X$. If there is $R(x) > 0$ such that

$$(3.4) \quad f \square h(x) = \inf_{y \in B(0, R(x))} (f(y) + h(x - y)),$$

then taking into account Theorem 2.3 we infer that the infimum in (1.1) is attained. So consider the case when for some sequence $(y_n) \subset X$ with $\|y_n\| \rightarrow \infty$ the sequence $(f(y_n) + h(x - y_n))$ is bounded from above. It follows from Lemma 2.5 that $\partial f(y) \neq \emptyset$ for some $y \in X$, so let $y^* \in \partial f(y)$. By (1) there is $r \in \mathbb{R}$ such that $S(h, r)$ is weakly compact and for every $l_n^* \in B(0, \alpha)$ we have a point $\bar{x}_n \in S(h, r)$ so that $y^* + l_n^* \in \partial h(\bar{x}_n)$, and moreover,

$$f(y_n) - f(y) \geq \langle y^*, y_n - y \rangle$$

and

$$x - y_n - h(\bar{x}_n) \geq \langle y^* + l_n^*, x - y_n - \bar{x}_n \rangle \quad \text{for every } n \in \mathbb{N},$$

hence

$$f(y_n) + h(x - y_n) \geq f(y) + h(\bar{x}_n) + \langle y^*, x - y - \bar{x}_n \rangle + \langle l_n^*, x - \bar{x}_n \rangle - \langle l_n^*, y_n \rangle.$$

Note that for a properly chosen sequence (l_n^*) (namely $\langle l_n^*, y_n \rangle = -\alpha \|y_n\|$) we get a contradiction, since the left hand side is bounded from above and the right hand side becomes ∞ as $n \rightarrow \infty$. Hence, if $\|y_n\| \rightarrow \infty$ then $f(y_n) + h(x - y_n) \rightarrow \infty$, and thus for every $x \in X$ there is $R(x) > 0$ such that (3.4) is valid. ■

Observe that combining Remark 3.3 and Theorem 3.4 we conclude that if f is a proper lower semicontinuous convex function such that $\lim_{\|x\| \rightarrow \infty} f(x)/\|x\| = \infty$ then its subdifferential operator ∂f is surjective. This property can also be used to distinguish reflexive Banach spaces:

PROPOSITION 3.5. *Let X be a Banach space. The following conditions are equivalent:*

- (1) X is reflexive,
- (2) for every proper lower semicontinuous convex function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$,

$$\lim_{\|x\| \rightarrow \infty} f(x)/\|x\| = \infty \quad \text{implies} \quad X^* = \bigcup_{x \in X} \partial f(x).$$

Proof. (1) \Rightarrow (2) follows from Remark 3.3 and Theorem 3.4.

(2) \Rightarrow (1). Let $K \subset X$ be any closed, convex, bounded and nonempty subset and $x^* \in X^*$. Define

$$f(x) = \begin{cases} -\langle x^*, x \rangle & \text{if } x \in K, \\ +\infty & \text{if } x \notin K. \end{cases}$$

For this proper lower semicontinuous convex function we have $\lim_{\|x\| \rightarrow \infty} f(x)/\|x\| = \infty$, hence there is $\bar{x} \in K$ such that $0 \in \partial f(\bar{x})$, which implies that

$$\langle x^*, \bar{x} \rangle \geq \langle x^*, x \rangle \quad \text{for every } x \in K.$$

Thus every continuous linear functional x^* attains its supremum on K . By the James theorem (see [7]) the set K is weakly compact, which implies the reflexivity. ■

4. Main results. In this section sufficient conditions for the cancellation law to hold are given.

LEMMA 4.1. *Let X be a topological vector space and $f, h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper convex functions and $y \in X$. If*

$$\partial f(y) \cap \partial h(x) \neq \emptyset \quad \text{for some } x \in \text{dom } h$$

then for $\bar{x} := y + x$ the inf-convolution $f \square h(\bar{x})$ is finite and

$$f \square h(\bar{x}) = f(y) + h(\bar{x} - y).$$

Proof. Assume that $\partial f(y) \cap \partial h(x) \neq \emptyset$ for some $x \in X$. Let $\bar{x} := y + x$. It is easy to check that

$$\partial h(\bar{x} - y) = -\partial h(\bar{x} - \cdot)(y),$$

hence, by Theorem 2.4, we get

$$0 \in \partial f(y) - \partial h(x) = \partial f(y) + \partial h(\bar{x} - \cdot)(y) \subseteq \partial(f(\cdot) + h(\bar{x} - \cdot))(y).$$

It follows from Theorem 2.4 that $f \square h(\bar{x}) = f(y) + h(\bar{x} - y)$. ■

THEOREM 4.2. *Let X be a topological vector space and $f, g, h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper convex functions such that*

- (1) $f \square h(x) = g \square h(x)$ for every $x \in X$,
- (2) if there is $z \in X$ such that $f(z) > g(z)$ then there are $x, y \in X$ such that $f(y) > g(y)$ and $\partial f(y) \cap \partial h(x) \neq \emptyset$,
- (3) if there is $z \in X$ such that $g(z) > f(z)$ then there are $x, y \in X$ such that $g(y) > f(y)$ and $\partial g(y) \cap \partial h(x) \neq \emptyset$.

Then $f = g$.

Proof. Let $z \in X$ and $f(z) \neq g(z)$. Suppose $f(z) > g(z)$. By assumption (2) there are $x, y \in X$ such that $f(y) > g(y)$ and $\partial f(y) \cap \partial h(x) \neq \emptyset$. It follows

from Lemma 4.1 that for some $\bar{x} \in X$ we have

$$f \square h(\bar{x}) = f(y) + h(\bar{x} - y),$$

hence

$$f \square h(\bar{x}) > g(y) + h(\bar{x} - y) \geq g \square h(\bar{x}),$$

which contradicts (1); thus $f(z) = g(z)$ for every $z \in X$. The proof of the other case ($g(z) > f(z)$) is similar. ■

Conditions (2) and (3) are sufficient for the cancellation law to hold. In the sequel we provide an analytical condition to ensure that (2) and (3) are valid on a Banach space when the functions involved are lower semicontinuous, namely

$$\bigcup_{y \in X} \partial f(y) \cup \partial g(y) \subseteq \bigcup_{x \in X} \partial h(x) \text{ and } \text{dom } \partial f(y) = \text{dom } \partial g(y).$$

Let us look how it works in particular cases.

THEOREM 4.3. *Let X be a reflexive Banach space and $f, g, h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper convex lower semicontinuous functions and $\alpha > 0$. If f and g are α -subdifferentially dominated by h and*

- (1) $f \square h(x) = g \square h(x)$ for every $x \in X$,
- (2) $\partial f(u) \cap \partial h(x - u) \cap \partial h(x - v) \cap \partial g(v) = \emptyset$ for all $x, u, v \in X$, $u \neq v$,

then $f = g$.

Proof. It follows from Theorem 3.4 that

$$(4.1) \quad \bigcup_{y \in X} \partial f(y) \cup \partial g(y) \subseteq \bigcup_{x \in X} \partial h(x)$$

and for every $x \in X$ such that $f \square h(x) \in \mathbb{R}$ there are $u, v \in X$ for which

$$f \square h(x) = f(u) + h(x - u) = g(v) + h(x - v).$$

It follows from Lemma 4.1 and (4.1) that for every \bar{u} with $\partial f(\bar{u}) \neq \emptyset$ there is \bar{x} such that

$$f \square h(\bar{x}) = f(\bar{u}) + h(\bar{x} - \bar{u}) \text{ and } \partial f(\bar{u}) \cap \partial h(\bar{x} - \bar{u}) \neq \emptyset.$$

By Lemma 2.7 we get

$$\emptyset \neq \partial f(\bar{u}) \cap \partial h(\bar{x} - \bar{u}) \subseteq \partial(g \square h)(\bar{x}).$$

There is $\bar{v} \in X$ such that

$$g \square h(\bar{x}) = g(\bar{v}) + h(\bar{x} - \bar{v}),$$

hence, by Lemma 2.6, we get

$$\emptyset \neq \partial f(\bar{u}) \cap \partial h(\bar{x} - \bar{u}) \subseteq \partial g(\bar{v}) \cap \partial h(\bar{x} - \bar{v}),$$

which, by (2), implies that $\bar{u} = \bar{v}$, thus $\partial f(\bar{u}) \neq \emptyset$ implies $\partial g(\bar{u}) \neq \emptyset$ (analogously $\partial g(\bar{v}) \neq \emptyset \Rightarrow \partial f(\bar{v}) \neq \emptyset$); so we have $\text{dom } \partial f = \text{dom } \partial g$. Now applying Lemma 2.5 and taking into account (4.1) we infer that assumptions (2) and (3) of Theorem 4.2 are satisfied, which implies $f \equiv g$. ■

Remark 4.4. If $h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a strictly convex proper lower semicontinuous function then

$$\partial h(x) \cap \partial h(y) = \emptyset \quad \text{for all } x, y \in X, x \neq y.$$

This can be obtained in the same way as assertion (5) of Lemma 3.1.

Note that if we take into account Remark 3.3 then Theorem 4.3 yields the following

COROLLARY 4.5. *Let X be a reflexive Banach space. If $f, g, h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous convex functions such that h is strictly convex and*

$$\lim_{\|x\| \rightarrow \infty} h(x)/\|x\| = \infty$$

then

$$f \square h \equiv g \square h \text{ implies } f \equiv g.$$

THEOREM 4.6. *Let X be a Banach space and $f, g, h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. Moreover, suppose h is uniformly convex. Then*

$$f \square h \equiv g \square h \text{ implies } f \equiv g.$$

Proof. It follows from Lemma 3.1 that

$$\bigcup_{y \in X} \partial f(y) \cup \partial g(y) \subseteq \bigcup_{x \in X} \partial h(x)$$

and

$$\partial h(u) \cap \partial h(v) = \emptyset \quad \text{for all } u, v \in X, u \neq v,$$

and for every x for which $f \square h(x)$ is finite the inf-convolutions $f \square h$ and $g \square h$ are finite. Repeating the reasoning from Theorem 4.3 we deduce that $\text{dom } \partial f = \text{dom } \partial g$, and again applying Lemma 2.5 we infer that assumptions (2) and (3) of Theorem 4.2 are satisfied. ■

References

- [1] H. Attouch, *Variational Convergence for Functions and Operators*, Pitman Adv. Publ. Program, Boston, 1984.
- [2] J.-P. Aubin, *Optima and Equilibria. An Introduction to Nonlinear Analysis*, Springer, Berlin, 1993.
- [3] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

- [4] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley Interscience, New York, 1983.
- [5] R. Correa, A. Jofré and L. Thibault, *Characterization of lower semicontinuous convex functions*, Proc. Amer. Math. Soc. 116 (1992), 67–72.
- [6] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.
- [7] R. B. Holmes, *Geometric Functional Analysis and its Applications*, Springer, New York, 1975.
- [8] A. D. Ioffe and V. M. Tihomirov, *Theory of Extremal Problems*, North-Holland, Amsterdam, 1979.
- [9] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Springer, Berlin, 1989.
- [10] C. Pontini, *Solving in the affirmative a conjecture about a limit of gradients*, J. Optim. Theory Appl. 70 (1991), 623–629.
- [11] T. Rockafellar, *Directionally Lipschitzian functions and subdifferential calculus*, Proc. London Math. Soc. 39 (1979), 331–355.
- [12] —, *Generalized directional derivatives and subgradients of nonconvex functions*, Canad. J. Math. 32 (1980), 257–280.
- [13] —, *On a special class of convex functions*, J. Optim. Theory Appl. 70 (1991), 619–621.
- [14] —, *Convex Analysis*, Princeton University Press, Princeton, 1970.
- [15] L. Thibault and D. Zagrodny, *Integration of subdifferentials of lower semicontinuous functions on Banach spaces*, J. Math. Anal. Appl., to appear.
- [16] D. Zagrodny, *Approximate mean value theorem for upper subderivatives*, Nonlinear Anal. 12 (1988), 1413–1428.
- [17] —, *An example of bad convex function*, J. Optim. Theory Appl. 70 (1991), 631–637.

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Concerning entire functions in B_0 -algebras

by

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Abstract. We construct a non- m -convex non-commutative B_0 -algebra on which all entire functions operate. Our example is also a Q -algebra and a radical algebra. It follows that some results true in the commutative case fail in general.

A B_0 -algebra (an algebra of type B_0) is a topological algebra whose underlying topological vector space is a completely metrizable locally convex space. The topology of a B_0 -algebra A can be given by means of a sequence $(|\cdot|_i)$ of seminorms such that

$$(1) \quad |x|_1 \leq |x|_2 \leq \dots \quad \text{for all } x \text{ in } A$$

and

$$(2) \quad |xy|_i < C_i |x|_{i+1} |y|_{i+1} \quad \text{for all } x, y \in A, \quad i = 1, 2, \dots,$$

where C_i are positive constants (one can easily have $C_i = 1$ for all i , but here it is more convenient to have inequalities of the form (2)). A B_0 -algebra A is said to be *multiplicatively-convex* (m -convex for short) if the seminorms (1) can be chosen so that instead of (2) we have

$$(3) \quad |xy|_i \leq |x|_i |y|_i \quad \text{for all } x, y \in A, \quad i = 1, 2, \dots$$

Note that (1) implies that if $\|\cdot\|$ is a continuous seminorm on a B_0 -algebra A , then there is an index m and a positive constant C such that

$$(4) \quad \|x\| \leq C|x|_m \quad \text{for all } x \text{ in } A.$$

An element x of an algebra A is said to be *quasi-invertible* if there is an element y in A , called a *quasi-inverse* of x , such that $x \circ y = y \circ x = 0$, where $x \circ y = xy + x + y$. This is equivalent to $(x+e) \cdot (y+e) = (y+e) \cdot (x+e) = e$, if A has a unit element e , or to this relation in the unitization A_1 of A , if there is no unit in A . That means that the quasi-inverse of an element x is uniquely determined by x .