Montrons que $S \notin \overline{S\Phi(X)}$. En effet, $\exists \varepsilon > 0$ tel que, si $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in B(X)$ avec $\|T-S\| < \varepsilon$, alors $A \in \Phi_-(l_p)$ avec $\mathrm{ind}(A) = \mathrm{ind}(U) = \infty$ et $D \in \Phi_+(l_q)$ avec $\mathrm{ind}(D) = \mathrm{ind}(V) = -\infty$.

Encore la proposition précédente montre que $T \notin S\Phi(X)$.

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Illumination bodies and affine surface area

by

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Abstract. We show that the affine surface area $\mathrm{as}(\partial K)$ of a convex body K in $\mathbb{R}^n$ can be computed as

$$\mathrm{as}(\partial K) = \lim_{\delta\to 0} d_n \frac{\mathrm{vol}_n(K^\delta) - \mathrm{vol}_n(K)}{\delta^{2/(n+1)}}$$

where d_n is a constant and K^δ is the illumination body.

For a convex body K in $\mathbb{R}^n$ with sufficiently smooth boundary ∂K the affine surface area is defined as

$$\int_{\partial K} \mathcal{K}(x)^{1/(n+1)} d\mu(x)$$

where $\mathcal{K}(x)$ is the Gaussian curvature at $x \in \partial K$ and μ is the surface measure on ∂K. It has been one of the goals of geometric convexity theory to extend the notions of differential geometry to convex hypersurfaces without differentiability assumptions. For the affine surface area this has only been done recently and then three different ways to extend affine surface area were given. One is due to Lutwak [Lu], the other to Leichtweiss [L2] and the third to Schütt and Werner [SW]. The last two extensions are based on floating bodies (see [L1] or [SW] for more information) and the fact, already known to Blaschke [B] for sufficiently smooth bodies in $\mathbb{R}^3$, that the affine surface area can also be computed as

$$\lim_{\delta\to 0} c_n \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_\delta)}{\delta^{2/(n+1)}} \tag{1}$$

where

$$c_n = 2\left(\frac{\mathrm{vol}_{n-1}(B_2^{n-1}(0,1))}{n+1}\right)^{2/(n+1)}$$

is a constant and K_δ is a floating body.

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Here we consider another class of convex bodies and show that for this class of bodies an expression similar to (1) also leads to the affine surface area, thus answering a question asked by Calabi. In some ways this class of bodies seems to be the most natural one when considering extensions of the affine surface area. Using methods different from ours this question was also solved by R. Howard [H] in the special case of convex bodies with C^4-boundary. R. Howard and E. Lutwak have drawn my attention to this problem.

Throughout the paper we use the following notations. $B_2^n(x,r)$ is the n-dimensional Euclidean ball with center x and radius r. For a point $x\in\mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm of x. For a point $x\in\partial K$, $N(x)$ is the outer normal to ∂K in x. It is well known that $N(x)$ exists uniquely almost everywhere. For a convex set A in $\mathbb{R}^n$ and $x\in\mathbb{R}^n$, $\mathrm{co}[x,A]$ is the convex hull of x and A. Please note that some of the constants that appear in the paper depend on the dimension n.

Let K be a convex body in $\mathbb{R}^n$ and $\delta>0$ be given. Denote by K^δ the convex body in $\mathbb{R}^n$ given as follows:

$$K^\delta=\{x\in\mathbb{R}^n:\mathrm{vol}_n(\mathrm{co}[x,K])-\mathrm{vol}_n(K)\le\delta\}.$$

We call K^δ an *illumination body*.

Then we have the following theorem.

THEOREM.

$$\lim_{\delta\to0} d_n\frac{\mathrm{vol}_n(K^\delta)-\mathrm{vol}_n(K)}{\delta^{2/(n+1)}}=\int_{\partial K}\mathcal{K}(x)^{1/(n+1)}\,d\mu(x)$$

where μ is the surface measure on ∂K and

$$d_n=2\left(\frac{\mathrm{vol}_{n-1}(B_2^{n-1}(0,1))}{n(n+1)}\right)^{2/(n+1)}$$

is a constant.

In the proof of the Theorem we follow the ideas of [SW]. We need several lemmas.

LEMMA 1. *Suppose* 0 *is in the interior of* K. *Then*

$$\mathrm{vol}_n(K^\delta)-\mathrm{vol}_n(K)=\frac1n\int_{\partial K}\langle x,N(x)\rangle\left(\left(\frac{\|x^\delta\|}{\|x\|}\right)^n-1\right)d\mu(x)$$

where x^δ is the point on ∂K^δ where the halfline determined by 0 *and x intersects ∂K^δ.*

The proof of Lemma 1 is standard.

For $x\in\partial K$ denote by $r(x)$ the radius of the biggest Euclidean ball contained in K that touches ∂K at x.

Remark. It was shown in [SW] that

(i) $\mu\{x\in\partial K: r(x)\ge t\}\ge(1-t)^{n-1}\mathrm{vol}_{n-1}(\partial K)$,

(ii) $\int_{\partial K} r(x)^{-\alpha}\,d\mu(x)<\infty\ \forall\alpha,\ 0<\alpha<1$.

LEMMA 2. *Suppose* 0 *is in the interior of* K. *Then for all x with $r(x)>0$ we have*

$$0\le\frac1n\langle x,N(x)\rangle\delta^{-2/(n+1)}\left(\left(\frac{\|x^\delta\|}{\|x\|}\right)^n-1\right)\le d\,r(x)^{-(n-1)/(n+1)}$$

where d is a constant that does not depend on x and δ.

LEMMA 3. *The limit*

$$\lim_{\delta\to0}\frac1n\langle x,N(x)\rangle\delta^{-2/(n+1)}\left(\left(\frac{\|x^\delta\|}{\|x\|}\right)^n-1\right)\quad\text{exists a.e.}$$

and equals

(i) $\frac{1}{d_n}\varrho(x)^{-(n-1)/(n+1)}$ *if the indicatrix of Dupin at $x\in\partial K$ is an $(n-1)$-dimensional sphere with radius $\sqrt{\varrho}$,*

(ii) *zero if the indicatrix of Dupin at x is an elliptic cylinder.*

Remark. (i) Since the indicatrix of Dupin exists a.e. [L2] and $r(x)>0$ a.e. [SW], the indicatrix of Dupin exists a.e. and is an ellipsoid or an elliptic cylinder.

(ii) If the indicatrix is an ellipsoid, we can reduce this case to the case of a sphere by an affine transformation with determinant 1 (see e.g. [SW]).

Proof of the Theorem. We may assume that 0 is in the interior of K. By Lemma 1 we have

$$\frac{\mathrm{vol}_n(K^\delta)-\mathrm{vol}(K)}{\delta^{2/(n+1)}}=\frac1n\int_{\partial K}\delta^{-2/(n+1)}\langle x,N(x)\rangle\left(\left(\frac{\|x^\delta\|}{\|x\|}\right)^n-1\right)d\mu(x).$$

By Lemma 2 and the remark preceding it, the integrand is bounded uniformly in δ by an L^1-function and by Lemma 3 it converges pointwise a.e. We apply Lebesgue's convergence theorem.

Proof of Lemma 2. Let $x\in\partial K$ such that $r(x)>0$. Denote by H the tangent hyperplane to x and by H^+ and H^- the two halfspaces generated by H. To prove Lemma 2, we distinguish two cases:

(i) $\|x^\delta-x\|/r(x)\le1$. Consider then the shaded cone C of Figure 1. Clearly we have

$$\begin{aligned}\delta&=\mathrm{vol}_n(\mathrm{co}[x^\delta,K])\\&\ge\mathrm{vol}_n(\mathrm{co}[x^\delta,B_2^n(x-r(x)N(x),r(x))]\cap H^+)\ge\mathrm{vol}_n(C).\end{aligned}$$

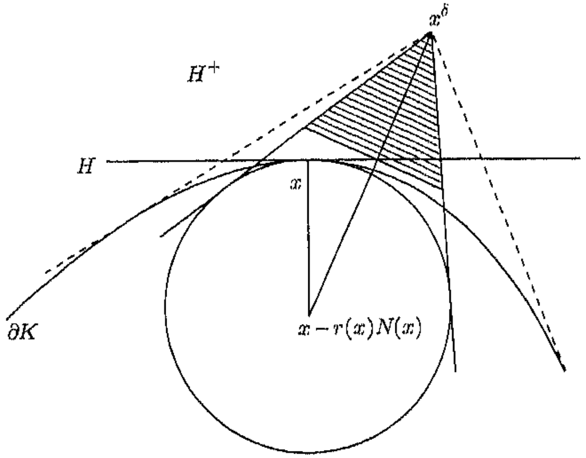

Fig. 1

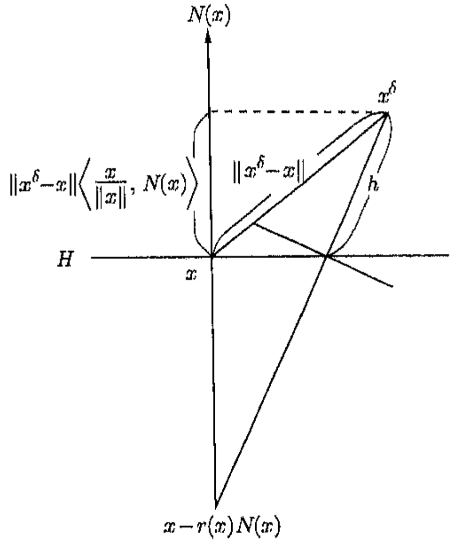

Fig. 2

We compute the radius ϱ and the height h of the cone C. We get (see Figure 2)

$$h=\frac{\left(1+\frac{2\|x^\delta-x\|}{r(x)}\left\langle\frac{x}{\|x\|},N(x)\right\rangle+\frac{\|x^\delta-x\|^2}{r(x)^2}\right)^{1/2}\|x^\delta-x\|\left\langle\frac{x}{\|x\|},N(x)\right\rangle}{1+\frac{\|x^\delta-x\|}{r(x)}\left\langle\frac{x}{\|x\|},N(x)\right\rangle}$$

and

$$\varrho=\frac{r(x)^{1/2}}{\|x^\delta-x\|^{1/2}}\cdot\frac{h}{\left(2\left\langle\frac{x}{\|x\|},N(x)\right\rangle+\frac{\|x^\delta-x\|}{r(x)}\right)^{1/2}},$$

hence

$$\begin{aligned}\delta&\geq\frac{1}{n}\operatorname{vol}_{n-1}(B_2^{n-1}(0,1))\varrho^{n-1}h\\&=\frac{1}{n}\operatorname{vol}_{n-1}(B_2^{n-1}(0,1))r(x)^{(n-1)/2}\\&\quad\times\frac{\left(1+2\frac{\|x^\delta-x\|}{r(x)}\left\langle\frac{x}{\|x\|},N(x)\right\rangle+\frac{\|x^\delta-x\|^2}{r(x)^2}\right)^{n/2}}{\left(1+\frac{\|x^\delta-x\|}{r(x)}\left\langle\frac{x}{\|x\|},N(x)\right\rangle\right)^n}\\&\quad\times\frac{\|x^\delta-x\|^{(n+1)/2}\left\langle\frac{x}{\|x\|},N(x)\right\rangle^n}{\left(2\left\langle\frac{x}{\|x\|},N(x)\right\rangle+\frac{\|x^\delta-x\|}{r(x)}\right)^{(n-1)/2}}\\&\geq\frac{1}{n}\operatorname{vol}_{n-1}(B_2^{n-1}(0,1))r(x)^{(n-1)/2}\\&\quad\times\frac{1}{\left(1+2\frac{\|x^\delta-x\|}{r(x)}\left\langle\frac{x}{\|x\|},N(x)\right\rangle+\frac{\|x^\delta-x\|^2}{r(x)^2}\right)^{n/2}}\\&\quad\times\frac{\|x^\delta-x\|^{(n+1)/2}\left\langle\frac{x}{\|x\|},N(x)\right\rangle^n}{\left(2\left\langle\frac{x}{\|x\|},N(x)\right\rangle+\frac{\|x^\delta-x\|}{r(x)}\right)^{(n-1)/2}}\\&\geq\frac{1}{n}\operatorname{vol}_{n-1}(B_2^{n-1}(0,1))r(x)^{(n-1)/2}\\&\quad\times\frac{\|x^\delta-x\|^{(n+1)/2}\left\langle\frac{x}{\|x\|},N(x)\right\rangle^n}{\left(1+\frac{\|x^\delta-x\|}{r(x)}\right)^n\left(2+\frac{\|x^\delta-x\|}{r(x)}\right)^{(n-1)/2}}.\end{aligned}$$

As x and x^δ are colinear, we have $\|x^\delta\|=\|x\|+\|x^\delta-x\|$ and hence

$$\frac{1}{n}\langle x,N(x)\rangle\left(\left(\frac{\|x^\delta\|}{\|x\|}\right)^n-1\right)=\frac{1}{n}\langle x,N(x)\rangle\left(\left(1+\frac{\|x^\delta-x\|}{\|x\|}\right)^n-1\right)$$

$$= \left\langle \frac{x}{\|x\|}, N(x) \right\rangle \|x^\delta - x\| \times \left[1 + \frac{1}{n}\binom{n}{2} \frac{\|x^\delta - x\|}{\|x\|} + \ldots + \frac{1}{n}\left(\frac{\|x^\delta - x\|}{\|x\|}\right)^{n-1} \right]$$

$$\le d \left\langle \frac{x}{\|x\|}, N(x) \right\rangle \|x^\delta - x\|$$

where d is a constant. Hence altogether we get

$$\frac{1}{n} \cdot \frac{\langle x, N(x)\rangle}{\delta^{2/(n+1)}} \left(\left(\frac{\|x^\delta\|}{\|x\|} \right)^n - 1 \right) \le d \frac{n^{2/(n+1)} r(x)^{-(n-1)/(n+1)}}{(\mathrm{vol}_{n-1}(B_2^{n-1}(0,1)))^{2/(n+1)}} \times \frac{\left(1 + \frac{\|x^\delta - x\|}{r(x)}\right)^{2n/(n+1)} \left(2 + \frac{\|x^\delta - x\|}{r(x)}\right)^{(n-1)/(n+1)}}{\left\langle \frac{x}{\|x\|}, N(x) \right\rangle^{(n-1)/(n+1)}}$$

As 0 is in the interior of K, we can choose $\alpha > 0$ such that $B_2^n(0, 1/\alpha) \subseteq \mathring{K} \subseteq K \subseteq B_2^n(0, \alpha)$. Then $\langle x/\|x\|, N(x)\rangle \ge 1/\alpha^2$ and the above expression can be estimated with some new constant d' by

$$d' r(x)^{-(n-1)/(n+1)} \left(1 + \frac{\|x^\delta - x\|}{r(x)}\right)^{2n/(n+1)} \left(2 + \frac{\|x^\delta - x\|}{r(x)}\right)^{(n-1)/(n+1)}.$$

As by assumption $\|x^\delta - x\|/r(x) \le 1$, Lemma 2 is proved in this case.

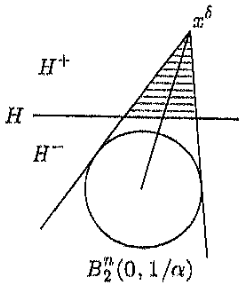

Fig. 3

(ii) $\|x^\delta - x\|/r(x) > 1$. We then proceed as follows (see [SW]). Consider now the cone $C = \mathrm{co}[x^\delta, B_2^n(0,1/\alpha)] \cap H^+$ (see Figure 3). Clearly $\delta \ge \mathrm{vol}_n(C)$ and $\mathrm{vol}_n(C)$ is minimal if H is parallel to the base of the cone.

Hence

$$\delta \ge \frac{1}{n} \cdot \frac{\|x^\delta - x\|^n}{\|x^\delta\|^{n-1}\alpha^{n-1}} \mathrm{vol}_{n-1}(B_2^{n-1}(0,1))$$

and consequently

$$\frac{1}{n} \cdot \frac{\langle x, N(x)\rangle}{\delta^{2/(n+1)}} \left(\left(\frac{\|x^\delta\|}{\|x\|} \right)^n - 1 \right) \le d \frac{1}{\|x^\delta - x\|^{(n-1)/(n+1)}} \le d r(x)^{-(n-1)/(n+1)}$$

where d is a constant.

Proof of Lemma 3. We can suppose without loss of generality that $0 \in \mathring{K}$. As x and x^δ are colinear, we have $\|x^\delta\| = \|x\| + \|x^\delta - x\|$. Consequently,

$$\frac{1}{n}\langle x, N(x)\rangle \left(\left(\frac{\|x^\delta\|}{\|x\|} \right)^n - 1 \right) \ge \left\langle \frac{x}{\|x\|}, N(x) \right\rangle \|x^\delta - x\| \tag{2}$$

and

$$\frac{1}{n}\langle x, N(x)\rangle \left(\left(\frac{\|x^\delta\|}{\|x\|} \right)^n - 1 \right) \le \left\langle \frac{x}{\|x\|}, N(x) \right\rangle \|x^\delta - x\| \left(1 + d \frac{\|x^\delta - x\|}{\|x\|} \right) \tag{3}$$

for some constant d, if we choose δ sufficiently small. Put

$$a = \frac{(1 - \langle x/\|x\|, N(x)\rangle^2)^{1/2}}{\langle x/\|x\|, N(x)\rangle}.$$

As $0 \in \mathring{K}$, $\langle x/\|x\|, N(x)\rangle > 0$ and hence a is finite and $a \ge 0$.

Suppose first the indicatrix of Dupin at $x \in \partial K$ is a Euclidean sphere with radius $\sqrt{\varrho}$. If the indicatrix is an ellipsoid, we may reduce that to the case of a sphere as in [SW]. Introduce a coordinate system such that $x = 0$ and $N(x) = (0, \ldots, 0, -1)$. H_0 is the tangent hyperplane to ∂K at $x = 0$. Denote by $\{H_t : t \ge 0\}$ the family of hyperplanes parallel to H_0 that have non-empty intersection with K and are at distance t from H_0. For $t > 0$, H_t^+ is the halfspace generated by H_t that contains $x = 0$. By Lemma 9 of [SW] there is a monotone function f on $\mathbb{R}^+$ with $\lim_{t \to 0} f(t) = 1$ such that for sufficiently small t,

$$\begin{aligned} &\{(f(t)^{-1}x_1, \ldots, f(t)^{-1}x_{n-1}, t) : \\ &\qquad x = (x_1, \ldots, x_{n-1}, t) \in B_2^n((0, \ldots, 0, \varrho), \varrho)\} \\ &\subseteq K \cap H_t \subseteq \{(f(t)x_1, \ldots, f(t)x_{n-1}, t) : \\ &\qquad x = (x_1, \ldots, x_{n-1}, t) \in B_2^n((0, \ldots, 0, \varrho), \varrho)\}. \end{aligned} \tag{4}$$

Let $\varepsilon > 0$ be given. By (4) we can choose t_0 so small that for all $t \le t_0$,

$$\begin{aligned} &B_2^n((0, \ldots, 0, \varrho - \varepsilon), \varrho - \varepsilon) \cap H_t^+ \\ &\subseteq K \cap H_t^+ \subseteq B_2^n((0, \ldots, 0, \varrho + \varepsilon), \varrho + \varepsilon) \cap H_t^+. \end{aligned} \tag{5}$$

Let $t_1 = \min(t_0, \varepsilon)$. Now we choose δ so small that

(i) estimate (3) above holds,

(ii) $(2/(\varrho-\varepsilon))^{1/2} a \|x^\delta - x\|^{1/2} \langle x/\|x\|, N(x)\rangle^{1/2} < \varepsilon$,

(iii) $2\|x^\delta - x\| \langle x/\|x\|, N(x)\rangle$
$\times(1 + 4(2/\varrho)^{1/2} a \|x^\delta - x\|^{1/2} \langle x/\|x\|, N(x)\rangle^{1/2}) < t_1$.

Moreover, we can choose the coordinate system such that the $x_1, \ldots \ldots, x_{n-2}$-coordinates of x^δ are 0. The main step in proving assertion (i) of the lemma consists in estimating $\delta = \mathrm{vol}_n(\mathrm{co}[x^\delta, K]) - \mathrm{vol}_n(K)$ from above and from below. We first show that δ can be estimated from below by

$$\delta \ge \frac{2^{(n+1)/2}}{n(n+1)} \mathrm{vol}_{n-1}(B_2^{n-1}(0,1)) \varrho^{(n-1)/2} \left(\|x^\delta - x\| \left\langle \frac{x}{\|x\|}, N(x) \right\rangle \right)^{(n+1)/2}$$
$$\times \left\{ (n+1)(1 - d'\varepsilon) - n \left(\frac{g(\Delta/(\varrho+\varepsilon))}{\sqrt{2}} \right)^{n+1} (1 + d''\varepsilon) \right\}$$

where d' and d'' are constants, Δ is the height of the cap of Figure 5 and g is a continuous function such that $\lim_{t\to 0} g(t) = \sqrt{2}$. Using this estimate for δ and (3) we get

$$\frac{1}{n} \cdot \frac{\langle x, N(x)\rangle \left(\left(\frac{\|x^\delta\|}{\|x\|} \right)^n - 1 \right)}{\delta^{2/(n+1)}}$$
$$\le \frac{1}{2} \left(\frac{n(n+1)}{\mathrm{vol}_{n-1} B_2^{n-1}(0,1)} \right)^{2/(n+1)} \varrho^{-(n-1)/(n+1)} \left(1 + d \frac{\|x^\delta - x\|}{\|x\|} \right)$$
$$\times \left\{ (n+1)(1 - d'\varepsilon) - n \left(\frac{g(\Delta/(\varrho+\varepsilon))}{\sqrt{2}} \right)^{n+1} (1 + d''\varepsilon) \right\}^{-2/(n+1)}.$$

Using the estimate from above for δ and (2) we get a similar estimate from below for the left-hand side above. Figures 4 and 5 illustrate the idea of the estimate for δ from below. Consider the shaded cone C of Figure 4 and the shaded cap of Figure 5. By (5) we clearly have

$$\delta \ge \mathrm{vol}_n(C) - \mathrm{vol}_n(\mathrm{cap}).$$

We start by computing the radius r of the cone C. Let H be the hyperplane containing all the points where a tangent line from x^δ to $B_2^n((0, \ldots, 0, \varrho-\varepsilon), \varrho-\varepsilon)$ touches $B_2^n((0, \ldots, 0, \varrho-\varepsilon), \varrho-\varepsilon)$ and let A_1 resp. A_2 be these points of $H \cap B_2^n((0, \ldots, 0, \varrho-\varepsilon), \varrho-\varepsilon)$ with biggest resp. smallest x_n-coordinate a_1 resp. a_2. We get

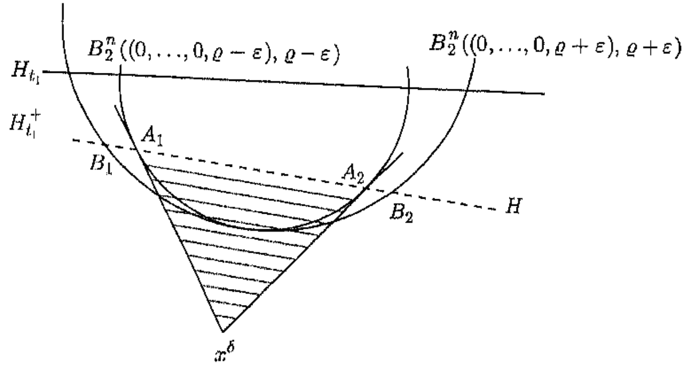

Fig. 4

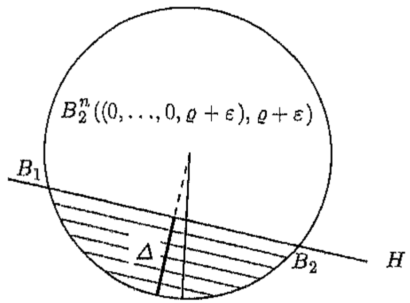

Fig. 5

$$a_1 = \|x^\delta - x\| \left\langle \frac{x}{\|x\|}, N(x) \right\rangle \left(1 + \sqrt{\frac{2}{\varrho-\varepsilon}} a \|x^\delta - x\|^{1/2} \left\langle \frac{x}{\|x\|}, N(x) \right\rangle^{1/2} \right.$$
$$\left. + \frac{2}{\varrho-\varepsilon} \|x^\delta - x\| \left\langle \frac{x}{\|x\|}, N(x) \right\rangle \left(\frac{1}{2\langle x/\|x\|, N(x)\rangle^2} - 1 \right) \right)$$
$$+ \text{higher order terms in } \|x^\delta - x\|,$$

$$a_2 = \|x^\delta - x\| \left\langle \frac{x}{\|x\|}, N(x) \right\rangle \left(1 - \sqrt{\frac{2}{\varrho-\varepsilon}} a \|x^\delta - x\|^{1/2} \left\langle \frac{x}{\|x\|}, N(x) \right\rangle^{1/2} \right.$$
$$\left. + \frac{2}{\varrho-\varepsilon} \|x^\delta - x\| \left\langle \frac{x}{\|x\|}, N(x) \right\rangle \left(\frac{1}{2\langle x/\|x\|, N(x)\rangle^2} - 1 \right) \right)$$
$$+ \text{higher order terms in } \|x^\delta - x\|.$$

By assumption (iii), $a_2 \le a_1 < t_1$. Moreover, by assumption (ii),

$$a_1 = \|x^\delta - x\| \langle x/\|x\|, N(x)\rangle (1 + d_1\varepsilon), \quad a_2 = \|x^\delta - x\| \langle x/\|x\|, N(x)\rangle (1 - d_2\varepsilon)$$

for some positive constants d_1 and d_2. Hence we get for the radius r of the cone C,

$$\begin{aligned}
r &= \tfrac{1}{2}\{(a_1-a_2)^2 + [(2(\varrho-\varepsilon)a_1 - a_1^2)^{1/2} + (2(\varrho-\varepsilon)a_2 - a_2^2)^{1/2}]^2\}^{1/2} \\
&= \Big\{\|x^\delta - x\|^3 \Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle^3 \frac{2a^2}{\varrho-\varepsilon} + \text{higher order terms in } \|x^\delta - x\| \\
&\quad + 2(\varrho-\varepsilon)\|x^\delta - x\| \Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle \\
&\quad \times \Big[\frac{1}{2}\Big((1+d_1\varepsilon)^{1/2}\Big(1-\frac{a_1}{2(\varrho-\varepsilon)}\Big)^{1/2} \\
&\quad + (1-d_2\varepsilon)^{1/2}\Big(1-\frac{a_2}{2(\varrho-\varepsilon)}\Big)^{1/2}\Big)\Big]^2\Big\}^{1/2} \\
&\ge \Big[2(\varrho-\varepsilon)\|x^\delta - x\| \Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle\Big]^{1/2} \\
&\quad \times \Big\{\Big(1-\frac{t_1}{2(\varrho-\varepsilon)}\Big)\Big[\frac{(1+d_1\varepsilon)^{1/2} + (1-d_2\varepsilon)^{1/2}}{2}\Big]^2\Big\} \\
&\quad + \text{higher order terms in } \|x^\delta - x\| \\
&\ge \Big[2(\varrho-\varepsilon)\|x^\delta - x\| \Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle\Big]^{1/2}(1-d'\varepsilon)
\end{aligned}$$

for some new constant d'. Now we compute the height h of the cone C. We get

$$\begin{aligned}
h &= \Big[2(\varrho-\varepsilon)\|x^\delta - x\| \Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle + (\varrho-\varepsilon)^2 + \|x^\delta - x\|^2\Big]^{1/2} \\
&\quad - [(\varrho-\varepsilon)^2 - r^2]^{1/2} \\
&= (\varrho-\varepsilon)\Big[\Big(1 + \frac{2\|x^\delta - x\|\langle x/\|x\|, N(x)\rangle}{\varrho-\varepsilon} + \Big(\frac{\|x^\delta - x\|}{\varrho-\varepsilon}\Big)^2\Big)^{1/2} \\
&\quad - \Big(1-\Big(\frac{r}{\varrho-\varepsilon}\Big)^2\Big)^{1/2}\Big] \\
&= (\varrho-\varepsilon)\Big[1 + \frac{\|x^\delta - x\|\langle x/\|x\|, N(x)\rangle}{\varrho-\varepsilon} \\
&\quad + \frac{1}{2}\Big(\frac{\|x^\delta - x\|}{\varrho-\varepsilon}\Big)^2\Big(1-\Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle^2\Big) \\
&\quad - \Big(1-\frac{1}{2}\Big(\frac{r}{\varrho-\varepsilon}\Big)^2 - \frac{1}{8}\Big(\frac{r}{\varrho-\varepsilon}\Big)^4\Big)\Big] \\
&\quad + \text{higher order terms in } \|x^\delta - x\|
\end{aligned}$$

$$\begin{aligned}
&\ge (\varrho-\varepsilon)\Big[\frac{\|x^\delta - x\|\langle x/\|x\|, N(x)\rangle}{\varrho-\varepsilon}(1+(1-d'\varepsilon)^2) \\
&\quad + \frac{1}{2}\Big(\frac{\|x^\delta - x\|}{\varrho-\varepsilon}\Big)^2\Big(1-\Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle^2(1-(1-d'\varepsilon)^4)\Big)\Big] \\
&\quad + \text{higher order terms in } \|x^\delta - x\| \\
&\ge 2\|x^\delta - x\|\langle x/\|x\|, N(x)\rangle(1-d''\varepsilon)
\end{aligned}$$

for a constant d''. Hence we get

$$\begin{aligned}
\mathrm{vol}_n(C) &= \frac{1}{n}\,\mathrm{vol}_{n-1}(B_2^{n-1}(0,1))hr^{n-1} \\
&\ge \frac{1}{n}\,\mathrm{vol}_{n-1}(B_2^{n-1}(0,1)) \\
&\quad \times 2^{(n+1)/2}\|x^\delta - x\|\langle x/\|x\|, N(x)\rangle^{(n+1)/2}(\varrho-\varepsilon)^{(n-1)/2}(1-d'\varepsilon)
\end{aligned}$$

for a new constant d'.

We now compute $\mathrm{vol}_n(\mathrm{cap})$. By [L2], p. 459, we have

$$\mathrm{vol}_n(\mathrm{cap}) = \frac{1}{n+1}g\Big(\frac{\Delta}{\varrho+\varepsilon}\Big)^{n+1}\mathrm{vol}_{n-1}(B_2^{n-1}(0,1))(\varrho+\varepsilon)^{(n-1)/2}\Delta^{(n+1)/2}$$

where Δ is the height of the cap of Figure 5 and g is a continuous function such that $\lim_{t\to 0} g(t) = \sqrt{2}$.

To compute Δ, let B_1 resp. B_2 be those points of $H \cap B_2^n((0,\dots,0,\varrho+\varepsilon),\varrho+\varepsilon)$ with biggest resp. smallest x_n-coordinate b_1 resp. b_2. We get

$$\begin{aligned}
b_1 &= \|x^\delta - x\|\Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle \\
&\quad \times \frac{B}{A}\Big[1 + \frac{2^{3/2}a(\varrho+\varepsilon)^{1/2}}{(AB)^{1/2}(\varrho-\varepsilon)}\|x^\delta - x\|^{1/2}\Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle^{1/2} \\
&\quad + \frac{4a^2(\varrho+\varepsilon)}{AB(\varrho-\varepsilon)^2}\|x^\delta - x\|\Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle\Big] \\
&\quad + \text{higher order terms in } \|x^\delta - x\|, \\
b_2 &= \|x^\delta - x\|\Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle \\
&\quad \times \frac{B}{A}\Big[1 - \frac{2^{3/2}a(\varrho+\varepsilon)^{1/2}}{(AB)^{1/2}(\varrho-\varepsilon)}\|x^\delta - x\|^{1/2}\Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle^{1/2} \\
&\quad + \frac{4a^2(\varrho+\varepsilon)}{AB(\varrho-\varepsilon)^2}\|x^\delta - x\|\Big\langle \frac{x}{\|x\|}, N(x)\Big\rangle\Big] \\
&\quad + \text{higher order terms in } \|x^\delta - x\|
\end{aligned}$$

where

$$A=(1+d_1\varepsilon)^{1/2}\left(1-\frac{a_1}{2(\varrho-\varepsilon)}\right)^{1/2}+(1-d_2\varepsilon)^{1/2}\left(1-\frac{a_2}{2(\varrho-\varepsilon)}\right)^{1/2},$$

$$B=(1+d_1\varepsilon)^{1/2}(1-d_2\varepsilon)^{1/2}\times\left[(1-d_2\varepsilon)^{1/2}\left(1-\frac{a_1}{2(\varrho-\varepsilon)}\right)^{1/2}+(1+d_1\varepsilon)^{1/2}\left(1-\frac{a_2}{2(\varrho-\varepsilon)}\right)^{1/2}\right].$$

By assumption (iii), $b_2\le b_1<t_1$. Hence for some constants d_3 and d_4 we get

$$b_1=\|x^\delta-x\|\langle x/\|x\|,N(x)\rangle(1+d_3\varepsilon),$$
$$b_2=\|x^\delta-x\|\langle x/\|x\|,N(x)\rangle(1-d_4\varepsilon).$$

Finally—similar to the computation of h—we get for the height Δ of the cap

$$\begin{aligned}\Delta&=\varrho+\varepsilon-\{(\varrho+\varepsilon)^2-\tfrac14(b_1-b_2)^2\\&\quad-\tfrac14[(2(\varrho+\varepsilon)b_1-b_1^2)^{1/2}+(2(\varrho+\varepsilon)b_2-b_2^2)^{1/2}]^2\}^{1/2}\\&=(\varrho+\varepsilon)\left[1-\left\{1-\left(\frac{b_1-b_2}{2(\varrho+\varepsilon)}\right)^2\right.\right.\\&\quad\left.\left.-\left[\frac{(2(\varrho+\varepsilon)b_1-b_1^2)^{1/2}+(2(\varrho+\varepsilon)b_2-b_2^2)^{1/2}}{2(\varrho+\varepsilon)}\right]^2\right\}^{1/2}\right]\\&=\|x^\delta-x\|\left\langle\frac{x}{\|x\|},N(x)\right\rangle\\&\quad\times\left[\frac12\left((1+d_3\varepsilon)^{1/2}\left(1-\frac{b_1}{2(\varrho+\varepsilon)}\right)^{1/2}\right.\right.\\&\quad\left.\left.+(1-d_4\varepsilon)^{1/2}\left(1-\frac{b_2}{2(\varrho+\varepsilon)}\right)^{1/2}\right)\right]^2\\&\quad+\text{higher order terms in }\|x^\delta-x\|\end{aligned}$$

or

$$\Delta\le\|x^\delta-x\|\langle x/\|x\|,N(x)\rangle(1+d''\varepsilon)\qquad\text{for some constant } d''.$$

Hence we get

$$\begin{aligned}\mathrm{vol}_n(\mathrm{cap})&\le\frac{1}{n+1}g\left(\frac{\Delta}{\varrho+\varepsilon}\right)^{n+1}\mathrm{vol}_{n-1}(B_2^{n-1}(0,1))\\&\quad\times(\varrho+\varepsilon)^{(n-1)/2}(\|x^\delta-x\|\langle x/\|x\|,N(x)\rangle)^{(n+1)/2}(1+d''\varepsilon)\end{aligned}$$

for some new constant d''. Consequently,

$$\begin{aligned}\delta&\ge\frac{2^{(n+1)/2}}{n(n+1)}\mathrm{vol}_{n-1}(B_2^{n-1}(0,1))(\|x^\delta-x\|\langle x/\|x\|,N(x)\rangle)^{(n+1)/2}\varrho^{(n-1)/2}\\&\quad\times\left\{(n+1)\left(1-\frac{\varepsilon}{\varrho}\right)^{(n-1)/2}(1-d'\varepsilon)\right.\\&\quad\left.-n\left(1+\frac{\varepsilon}{\varrho}\right)^{(n-1)/2}\left(\frac{g(\Delta/(\varrho+\varepsilon))}{\sqrt2}\right)^{n+1}(1+d''\varepsilon)\right\}.\end{aligned}$$

The estimate for δ from above is done in a similar way.

Suppose next the indicatrix of Dupin at x is an elliptic cylinder. We can again assume (see [SW]) that it is a spherical cylinder, i.e. the product of a k-dimensional plane and an $(n-k-1)$-dimensional Euclidean sphere of radius ϱ. We can moreover assume that ϱ is arbitrarily large (see also [SW]).

By Lemma 9 of [SW] we then have for sufficiently small t and some $\varepsilon>0$,

$$B_2^n((0,\dots,0,\varrho-\varepsilon),\varrho-\varepsilon)\cap H_t\subseteq K\cap H_t.$$

Using similar methods, this implies assertion (ii) of Lemma 3.

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